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A Matrix Model for the 2d Black Hole

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Two-dimensional bosonic string theory:

$$S = \frac{1}{4\pi}\int_\Sigma d^2\sigma \left[G_{\mu\nu}(X)\nabla_a X^\mu \nabla^a X^\nu + T(X) + \hat{R}^{(2)}\Phi(X)\right]$$

where $X^1 = x(\sigma_1,\sigma_2)$, $X^2 = \varphi(\sigma_1,\sigma_2)$,

has two classical solutions:

① Flat space: $G_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

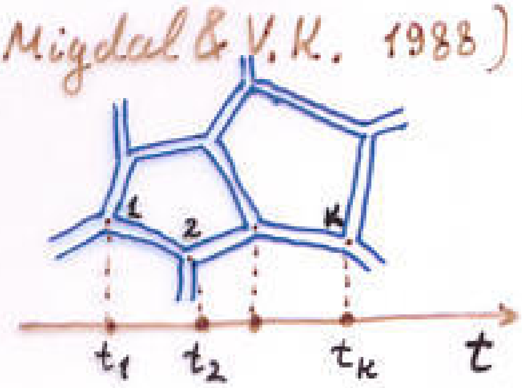
$\Phi = 2(\varphi_0 - \varphi)$, $T = 2(\varphi_0-\varphi)e^{-2(\varphi-\varphi_0)}$

Well described by the „old" Matrix (singlet) Quantum Mechanics (A. Migdal & V.K. 1988)

$$\mathcal{L} = \mathrm{tr}\left[\dot{M}^2 + V(M)\right]$$

World sheets $\longleftrightarrow$ Planar Graphs

(F. David, V.K. 1985)

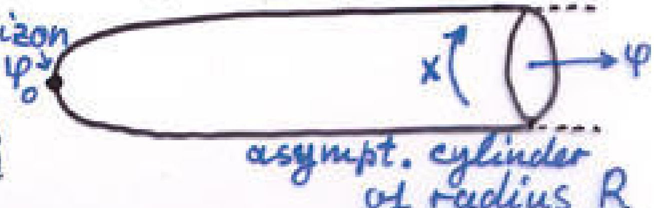


② „Cigar" space: $ds^2 = g(\varphi)\,dt^2 + \frac{1}{g(\varphi)}dx^2$

$\Phi = 2(\varphi_0 - \varphi)$, $T = 0$, $g(\varphi) = 1 - \exp\left[2(\varphi_0-\varphi)\frac{1}{R}\right]$

(G. Mandal et al., S. Elitzur et al. 1991)

What is the „holographic" Matrix Model description?



Our Model: condensation of Kosterlitz-Thouless-Berezinski vortices on the world sheet (described by non-singlet states in MQM)

②

Cigar Model as 2D string perturbed by vortices

V. Fateev
A. & Al. Zamolodchikov

Based on the equivalence between:

$\left[\frac{SL(2,C)}{SU(2)\cdot U(1)}\right]_k$ - coset

„Sine-Liouville"

$$\mathcal{L} = (\partial X)^2 + (\partial \varphi)^2 + Q\hat{R}\varphi + \lambda e^{b\varphi} \cos R(X_L - X_R)$$

Central Charges

$c = \frac{3k}{k-2} - 1$

$c = 2 + 6Q^2 \; ; \; b = -\frac{1}{Q}$

Compactification radius of asymptotic states:

$$R = \sqrt{k} := \sqrt{2 + \frac{1}{Q^2}} \iff c_{cigar} = c_{SL}$$

Dimensions of operators:

$$\Delta^{\pm}_{j,n_1,n_2} = -\frac{j(j+1)}{R^2 - 2} + \frac{1}{4}\left(\frac{n_1}{R} \pm n_2 R\right)^2$$

Same for

$$V_{j\,n_1 n_2} \sim e^{ip_L X_L + ip_R X_R + 2Qj\varphi}$$

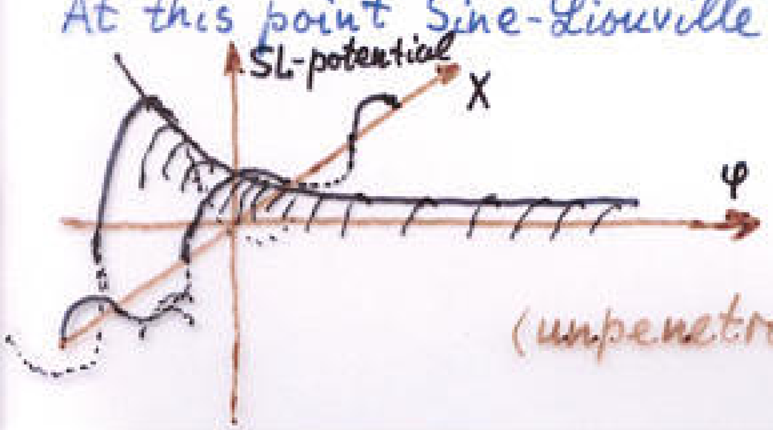
with $p_{L,R} = \frac{n_1}{R} \pm n_2 R$

Coincidence of a large class of correlators: $\langle V_{j,n_1,n_2} V_{j,-n_1,-n_2} \rangle$, $\langle V \; V \; V \rangle$

This equivalence is an example of strong-weak duality.

The Black Hole point: $c = 26$, $k = \frac{9}{4}$, $R = 3/2$
(E. Witten 1991; E. & H. Verlinde, R. Dijkgraaf 1992)

At this point Sine-Liouville = Black Hole = 2D string with world-sheet vortices



Cigar „wall" is substituted by Liouville „wall" (unpenetrable only quantum mechanically)

3 Black Hole as a Perturbation of c = 1 string

$$\mathcal{L}_{c=1} = \frac{1}{4\pi}\left[(\partial x)^2 + (\partial\varphi)^2 + 2\hat{R}\varphi + \lambda e^{(R-2)\varphi}\cos R(x_L - x_R)\right] + \mu\varphi e^{-2\varphi}$$

Parameters: $\left.\begin{matrix} c_{tot} = 26 \\ Q = -2 \end{matrix}\right\}$ fixed

R – compactif. radius

λ – fugacity of vortices

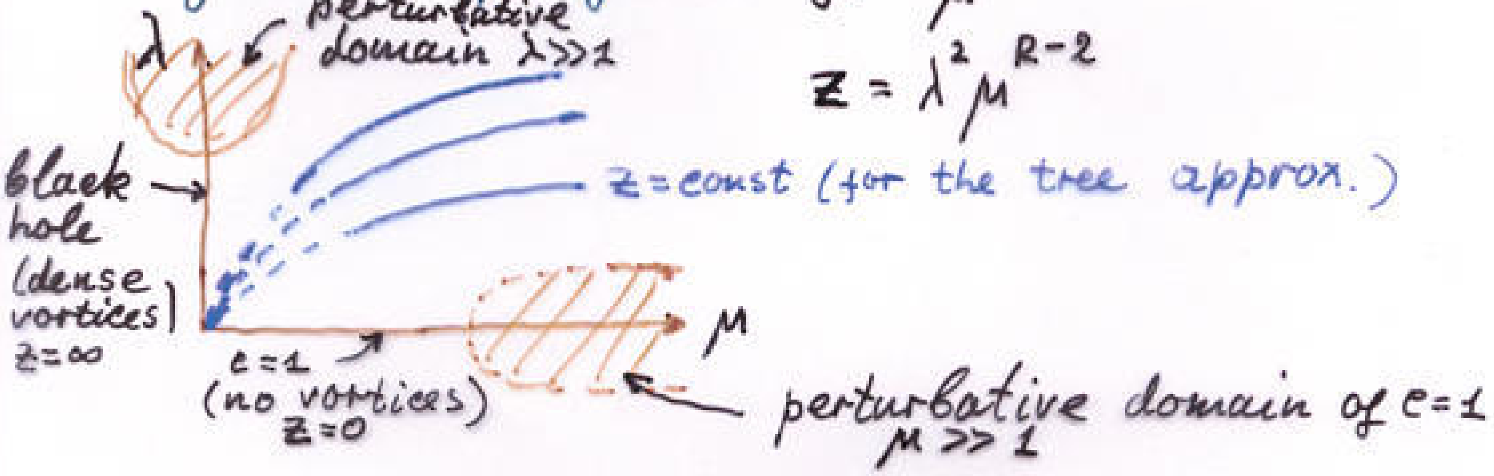
μ – cosmological const.

KPZ-DDK scaling: $\lambda \sim \mu^{\frac{2-R}{2}}$

Physical couplings:

$$g_s = \frac{1}{\mu}$$

$$z = \lambda^2 \mu^{R-2}$$

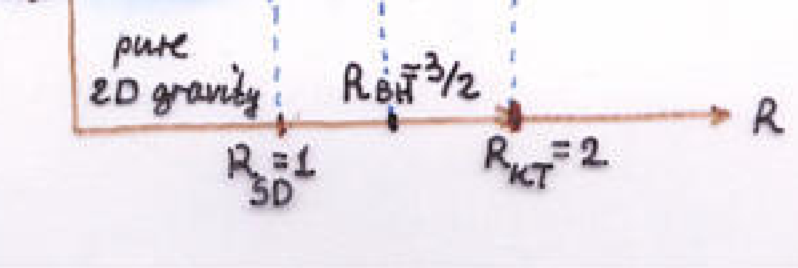


String Partition Function:

$$F(\lambda,\mu) \simeq -\frac{R}{2}\mu^2\left[\log\mu + A_0(z)\right] + \frac{R + \frac{1}{R}}{24}\log\mu + A_1(z) + \sum_{h=2}^{\infty}\mu^{2-2h}A_h(z)$$

Our Matrix Model ~~ex~~trapolates between the known $c = 1$ string ($z = 0$) and the unknown black hole ($z = \infty$) physics

(c_{tot}, R) phase diagram:



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D. Gross, I. Klebanov 1990

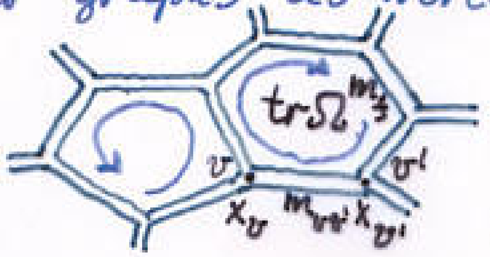
D. Boulatov, V. K. 1991

Matrix Quantum Mechanics and Vortices on Planar Graphs

$$Z_N[\Omega] = \int \mathcal{D}^{N^2} M(x)\, \exp - \mathrm{tr} \int_0^{2\pi R} dx \left(M'^2_x + M^2 + \frac{g}{\sqrt{N}} M^3 \right)$$

$$M(2\pi R) = \Omega^+ M(0)\, \Omega$$ - twisted periodic boundary cond. $\Omega \in SU(N)$

Planar graphs as worldsheets in periodic 1D space



x_v - coordinate of vertex v

$m_v \in \mathbb{Z}$ - gauge field

$m_f = \sum_{\langle vv' \rangle \in f} m_{vv'}$ - vorticity through the face f

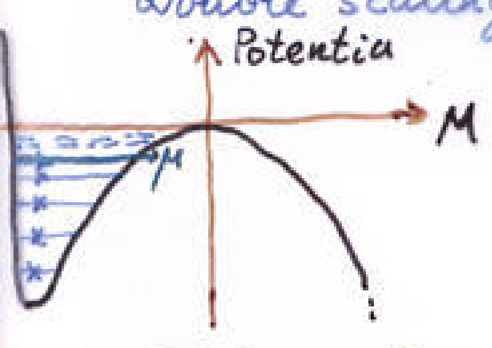
String partition function:

$$F_N(\Omega) = \sum_{h=0}^{\infty} N^{2-2h} \sum_k g^k \sum_{\text{graphs } G_k^h} \sum_{\{m\}} \prod_f \frac{\mathrm{tr}}{N} \Omega^{m_f} \times$$

$$\times \int \prod_v dx_v\, \exp\Big[- \sum_{\langle vv' \rangle} | x_v - x_{v'} + 2\pi R\, m_{vv'} | \Big]$$

Villain action for BKT vortices

Double scaling limit and inverted matrix oscillator



$$Z \to \int \mathcal{D}^{N^2} M\, \exp - \mathrm{tr} \int_0^{2\pi R} dx \left(M'^2 - M^2 \right)$$

$$M_{ij}(2\pi R) = \frac{z_i}{z_j} M_{ij}(0)$$

$$\Omega = \mathrm{diag}(z_1, z_2, \dots, z_N)$$

Integrating over $M_{ij}(x)$ and Ω ......

$$\exp F(\lambda, \mu) = \sum_{N=1}^{\infty} \frac{e^{2\pi R \mu N}}{N!} \oint \prod_{k=1}^{N} \frac{dz_k}{2\pi i}\, e^{\lambda (z_k + z_k^{-1})} \prod_{i \neq j} \frac{z_i - z_j}{e^{i\pi R} z_i - e^{-i\pi R} z_j}$$

Grand Canonical Partition function =

„projector" onto the states with charge ±1

= τ-function of Toda hierarchy!

⑤ # Toda equation and genus expansion

$$\frac{\partial}{\partial\lambda_+}\frac{\partial}{\partial\lambda_-}F(\lambda,\mu) = \exp\left[2F(\lambda,\mu) - F(\lambda,\mu+i) - F(\lambda,\mu-i)\right]$$

$\boxed{\lambda_\pm = \lambda e^{\pm i\varphi}}$ Written directly in physical space of parameters: $\mu = g_s^{-1}$, λ – coupling to winding mode

Boundary condition: unperturbed $c=1$ string

D. Gross, I. Klebanov 1990

$$F(0,\mu) = \int_{-\infty}^{\infty}\frac{dE}{\pi}\sum_{n=0}^{\infty}\frac{n+\frac{1}{2}}{E^2+(n+\frac{1}{2})^2}\ln\left[1+e^{2\pi R(-\mu+E)}\right]$$

$$= -\frac{R}{2}\mu^2\log\mu - \frac{1}{24}\left(R+\frac{1}{R}\right)\ln\mu + O(1/\mu^2)$$

Toda equation(s) define in principle the whole 2D string theory.

Perturbative calculation of string partition function from Toda eq. and KPZ-DDK scaling

$$\partial_\mu^2 F(\lambda,\mu) \xrightarrow[g_s^{-1}=\mu\sim\lambda^{\frac{2}{2-R}}\to\infty]{} \frac{2R}{2-R}\log\lambda + X_0(y) + \sum_{h\geq 1}\lambda^{-\frac{4h}{2-R}}X_h(y)$$

$h=0$: sphere: $\frac{1}{4}\lambda^{-1}\partial_\lambda\lambda\partial_\lambda F_0 = e^{\partial_\mu^2 F_0}$ $\quad\boxed{y = \frac{\mu}{\lambda^{2/(2-R)}}}$

solution: $\boxed{\exp[R^{-1}X_0] - \exp[(1-R^{-1})X_0] = y}$

$$F_0(\lambda,\mu) = -\frac{R}{2}\mu^2\log\mu + R\mu^2\sum_{n=1}^{\infty}\left(\lambda^2\mu^{R-2}\right)^n\frac{\Gamma(n(2-R)-2)}{n!\,\Gamma(n(1-R)+1)}$$

conjectured by G. Moore (1992)

$h=1$: torus

$$F_1(\lambda,\mu) = -\frac{R+R^{-1}}{24}\log\lambda^{\frac{2}{2-R}} - \frac{1+R^{-2}}{24}X_0(y) + \frac{1}{24}\log\left[1-(R-1)e^{(2R^{-1}-1)X_0(y)}\right]$$

Any genus h

$$(y\partial_y + 2h-2)^2 F_h - e^{-X_0(y)}\partial_y^2 F_h = \mathcal{H}(F_0, F_1, \dots, F_{h-1})$$

known function

II order ODE (with polynomial coeff.)

Black Hole limit: $z = \lambda^2 \mu^{R-2} \to \infty$

$0 < R < 1$ $F_0(\mu, z)$ has a singularity at finite real $z_c > 0$: obstacle to reach the Black hole limit

E. Hsu D. Kutasov 1992 The system flows to $c=0$ string in I.R.

$1 < R < 2$ $F_0(\mu, z)$ has no singularity for $0 < z < \infty$

The system flows to the „black hole" phase

$$F(\lambda) = B_0 \lambda^{\frac{2}{2-R}} - \frac{R + R^{-1}}{24} \log \lambda^{\frac{2}{2-R}} + \sum_{h=2}^{\infty} B_h \lambda^{\frac{4(1-h)}{2-R}}$$

B_h - universal functions of R

The space-time metric is unknown for general R. New black hole solutions?

$R = 3/2$ cigar point 

$$F_{BH}(\lambda) = B_0 \lambda^8 - \frac{13}{288} \log \lambda^8 + \sum_{h=2}^{\infty} B_h \lambda^{8-8h}$$

First term is non-singular (non-universal) and may be subtracted.

$$F_{BH}^{\text{sphere}} = 0$$ (G. Gibbons & M. Perry 1992)

From FZZ conjecture the mass of black hole $M \sim g_s^{-2} \sim \lambda^8$

$$F_{BH} = -\frac{13}{288} \log M + \sum_{h=2}^{\infty} B_h M^{-h}$$

Thermodynamical consequences - in the talk of D. Kutasov

Comments and Problems

1. Toda hierarchy and commuting flows

$$\mathcal{L} = \frac{1}{4\pi}\left[(\partial x)^2 + (\partial\varphi)^2 + 2\hat{R}\varphi + \mu\varphi e^{\varphi} + \sum_n t_n e^{(|n|R-2)\varphi + inR(x_L - x_R)}\right]$$

$$\updownarrow$$

$$Z(\mu, t_1, t_2, t_3, \dots) = \int d\Omega\, e^{\mathrm{tr} \sum_{n\in\mathbb{Z}} t_n \Omega^n} Z(\mu, \Omega)$$

$t_1, t_2, \dots$ – Toda „times"

Correlators of winding modes can be calculated by Hirota equations

Momentum modes ?

2. Minkowski continuation: by use of eigenvalue hamiltonians in fixed irreps r:

$$\hat{H}_r = P_r \sum_{k=1}^{N}\left[-\frac{1}{2}\frac{\partial^2}{\partial x_k^2} - \frac{1}{2}x_k^2\right] + \frac{1}{2}\sum_{i\neq j}\frac{\hat{t}^r_{ij}\hat{t}^r_{ij}}{(x_i - x_j)^2}$$

$$Z_N(R,\lambda) = \sum_r g_r(\lambda)\, \mathrm{Tr}_r\, e^{-2\pi R \hat{H}}$$

Direct calculation of the entropy of states of the black hole ?

Black hole formation ?

3. Das-Jevicki-Polchinski collective field action in presence of vortices and the target space picture ?