Wilson Lines in Noncommutative Gauge Theories *

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Contents of the talk

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Introduction

In matrix model formulation of superstring theory, not only matter but also space* or even space-time† may emerge out of matrices. Noncommutative gauge theory (NCYM) may be regarded as a concrete realization of such a possibility. ‡

The gauge invariant observables in NCYM involve not only closed Wilson loops but also open Wilson loops (Wilson lines). It has also been pointed out that the operators which couple to graviton multiplets may be constructed through them.§

In NCYM, the graviton exchange process arises at one loop level through non-planar diagrams. In matrix model picture, it represents the interaction between two diagonal blocks caused by off-diagonal long 'open strings'. It is in fact found that such a coupling involves the Wilson lines. ¶

See Fig. 1

*Banks, Fischler, Shenker and Susskind,

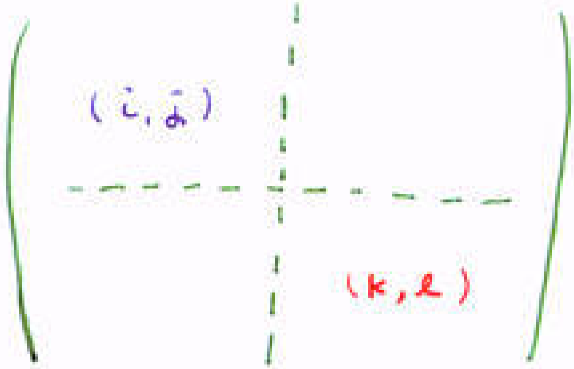
†Ishibashi, Kawai, Kitazawa and Tsuchiya,

‡Connes, Douglas and Schwarz,

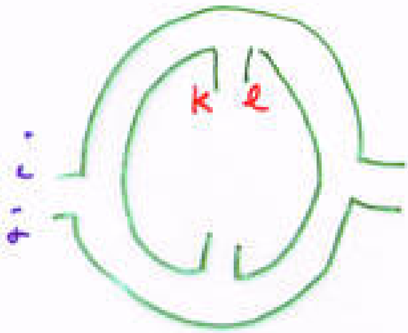
§Ishibashi, Iso, Kawai and Kitazawa,

¶Liu and Michelson; Zanon; Das and Trivedi,

Matrix model picture

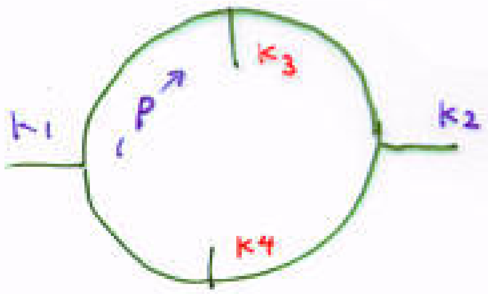


$\leftrightarrow \quad X^\mu$



$$\sim \frac{1}{(x_i - x_k)^8} T^{\mu\nu}(x_i)\, T_{\mu\nu}(x_k)$$

NCYM



$$\sim \int d^4p \, \exp(i\, C^{\mu\nu} k_\mu p_\nu) \frac{1}{p^8}$$

$$\sim C^4 \int d^4x \, \exp(i\, k_\mu x^\mu) \frac{1}{x^8}$$

where $x^\mu = C^{\mu\nu} p_\nu$

Fig. 1

Wilson Lines

We recall IIB matrix model action:

$$S = -\frac{1}{g^2} Tr(\frac{1}{4}[A_\mu, A_\nu][A^\mu, A^\nu] + \frac{1}{2}\bar{\psi}\Gamma^\mu[A_\mu, \psi]). \qquad (1)$$

The NCYM is obtained by expanding the theory around the following background:

$$[\hat{p}_\mu, \hat{p}_\nu] = iB_{\mu\nu}, \qquad (2)$$

where $B_{\mu\nu}$ are c-numbers. We assume the rank of $B_{\mu\nu}$ to be $\tilde{d}$ and define its inverse $C^{\mu\nu}$ in $\tilde{d}$ dimensional subspace. We expand $A_\mu = \hat{p}_\mu + \hat{a}_\mu$.

NCYM can be realized through matrix models by the the following map from matrices onto functions *

$$\begin{aligned}
\hat{a} &\rightarrow a(x), \\
\hat{a}\hat{b} &\rightarrow a(x) \star b(x), \\
Tr &\rightarrow \sqrt{detB}(\frac{1}{2\pi})^{\frac{d}{2}} \int d^d x, \\
g^2 &\rightarrow \sqrt{detB}(\frac{1}{2\pi})^{\frac{d}{2}} g_{NC}^2. \qquad (3)
\end{aligned}$$

*Aoki, Ishibashi, Iso, Kawai, Kitazawa and Tada

The gauge invariant observables in NCYM are the Wilson loops:

$$W(C) = Tr[Pexp(i\int_C dx^\mu(\sigma)A_\mu)] \tag{4}$$

where C denotes a contour parametrized by $x^\mu(\sigma)$. The Wilson loops in the matrix model model are mapped into those in NCYM by eq.(3):

$$\begin{aligned} & Tr[Pexp(i\int_C dx^\mu(\sigma)A_\mu)] \\ = & \sqrt{detB}(\frac{1}{2\pi})^{\frac{d}{2}}\int d^dx \\ & \times Pexp(i\int_C dy^\mu(\sigma)a_\mu(x+y(\sigma)))exp(ik^\mu x_\mu))_\star \end{aligned} \tag{5}$$

where the total momentum k_μ is related to the vector d^μ which connects the two ends of the contour as $k_\mu = B_{\mu\nu}d^\nu$. The symbol $\star$ in the above expression reminds us that all products of fields must be understood as $\star$ products. *

*I.I.K.K.; Ambjorn, Makeenko, Nishimura and Szabo; Gross, Hashimoto and Itzaki; Rey and Das; Dhar and Wadia

The Wilson loops obey the Schwinger-Dyson equations such as

$$
\begin{aligned}
& < \frac{1}{g^2} Tr[\{[A_\mu,[A_\mu,A_\nu]] + \frac{1}{2}\Gamma_\nu[\psi,\bar{\psi}]_+\} \\
& \times Pexp(i\int_{C_1} dx^\mu(\sigma_1)A_\mu)]W(C_2) > \\
= \;& i\int_0^1 dt\dot{x}^\nu < Tr[Pexp(i\int_0^t d\sigma_1\dot{x}^\mu A_\mu)] \\
& Tr[Pexp(i\int_t^1 d\sigma_1\dot{x}^\mu A_\mu)]W(C_2) > \\
+ \;& i\int_0^1 ds\dot{x}^\nu < Tr[Pexp(i\int_0^s d\sigma_2\dot{x}^\mu A_\mu) \\
& \times exp(i\int_{C_1} dx^\mu(\sigma_1)A_\mu)exp(i\int_s^1 d\sigma_2\dot{x}^\mu A_\mu)] > .
\end{aligned}
\tag{6}
$$

The first and second term on the right-hand side of the above equation represents the splitting and joining of the loops respectively. * The loop equations may be useful to investigate the behavior of the Wilson lines in the strong coupling regime. †

*Fukuma, Kawai, Kitazawa and Tsuchiya; Abou-Zeid and Dorn; Dhar and Wadia

†Polyakov and Rychkov

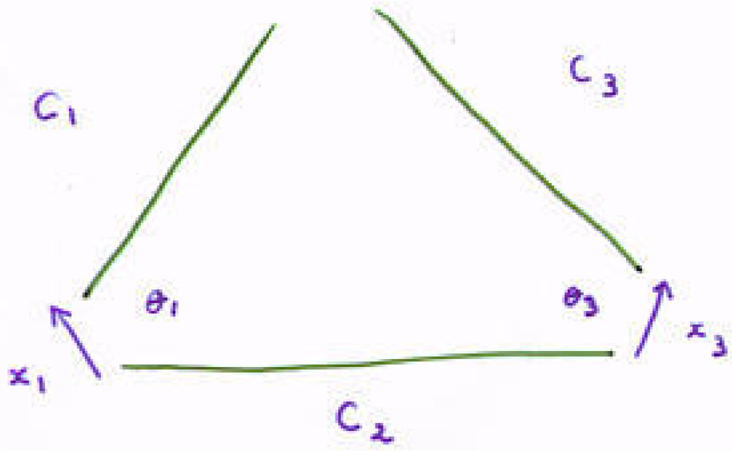
Wilson Lines as Wilson Loops

In NCYM, generic gauge invariant operators (Wilson Lines) are specified by the open contours. Let us consider a three point function.

$$
\begin{aligned}
& Tr[Pexp(i\int_{C_1} dx^\mu(\sigma)A_\mu)]Tr[Pexp(i\int_{C_2} dx^\mu(\sigma)A_\mu)] \\
& Tr[Pexp(i\int_{C_3} dx^\mu(\sigma)A_\mu)] \\
= \; & \frac{1}{N}\sum_{\vec{x}_2}\frac{1}{N}\sum_{\vec{x}_3} Tr[Pexp(i\int_{C_1} dx^\mu(\sigma)A_\mu)exp(i\vec{x}_2\cdot\hat{p}) \\
& Pexp(i\int_{C_2} dx^\mu(\sigma)A_\mu)exp(i\vec{x}_3\cdot\hat{p}) \\
& \times Pexp(i\int_{C_3} dx^\mu(\sigma)A_\mu)exp(-i\vec{x}_3\cdot\hat{p})exp(-i\vec{x}_2\cdot\hat{p})].
\end{aligned} \tag{7}
$$

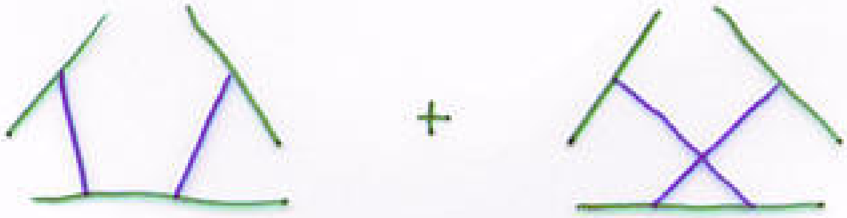
It can be translated into 4 dimensional NCYM as

$$
\begin{aligned}
& exp(i\Phi)\int d^4x\int d^4x_2\int d^4x_3 exp(i(k_2+k_3)\cdot x_2 + ik_3\cdot x_3) \\
& [Pexp(i\int_{C_1} dy^\mu(\sigma)a_\mu(x+y(\sigma))) \\
& exp(i\int_{C_2} dy^\mu(\sigma)a_\mu(x+x_2+y(\sigma))) \\
& \times exp(i\int_{C_3} dy^\mu(\sigma)a_\mu(x+x_2+x_3+y(\sigma)))]_\star
\end{aligned} \tag{8}
$$



In the weak coupling region, we may retain

$$exp(i\Phi)\int d^4x\int d^4x_1\int d^4x_3 exp(ik_1\cdot x_1+ik_3\cdot x_3)$$
$$<P\int_{C_1}da\cdot A(x+x_1+a)\star\int_{C_2}db_1\cdot A(x+b_1)$$
$$\star\int_{C_2}db_2\cdot A(x+b_2)\star\int_{C_3}dc\cdot A(x+x_3+c)> \qquad (9)$$



$$\begin{aligned} & exp(ik_3\cdot(x+b_2))\star exp(ik_1\cdot(x+b_1)) \\ = \; & exp(ik_1\cdot(x+b_1))\star exp(ik_3\cdot(x+b_2+Ck_2)). \end{aligned} \qquad (10)$$

In this expression $\tilde{b}_2=b_2+Ck_2$ is such that $b_1<\tilde{b}_2<b_1+Ck_2$. It can be shown to be independent of b_1 after changing the variable from b_2 to $\tilde{b}_2$.

In this way, we obtain

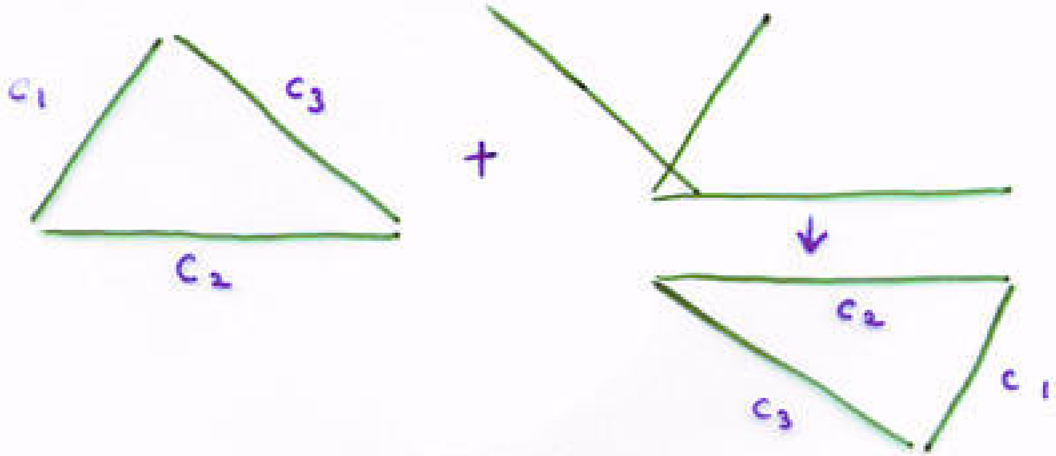
$$
\begin{aligned}
& V g_{NC}^4 exp(\frac{i}{2} k_1 C k_2 sin(\theta_1)) cos(\theta_1) cos(\theta_3) \\
& C k_1 C k_2 C k_3 \frac{1}{{k_1}^2} \frac{1}{{k_3}^2} \int_0^{Ck_2} d|b_2| exp(-i k_1 sin(\theta_1)|b_2|) \\
= \; & V g_{NC}^4 i C \frac{1}{{k_1}^2} \frac{1}{{k_3}^2} C k_2 C k_3 cot(\theta_1) cos(\theta_3) \\
& \{exp(-\frac{i}{2} k_1 C k_2 sin(\theta_1)) - exp(\frac{i}{2} k_1 C k_2 sin(\theta_1))\} \quad (11)
\end{aligned}
$$

where θ_1 and θ_3 are the angles of the two corners of the triangle formed by (C_1, C_2) and (C_2, C_3) respectively.

We emphasize here that straight Wilson lines possess the cyclic symmetry. In fact the Wilson lines can be written as $Tr(U^n)$ in the matrix model construction where $U = exp(i\Delta x \cdot (\hat{p} + \hat{a}))$ and $\Delta x = Ck/n$. So in general the correlator is proportional to the length of the Wilson line and we can fix the location of the one of the propagators.

The propagators are very short in the large k limit since their lengths are of $O(1/k)$. Furthermore the $|b_2|$ integration is only supported by the two infinitesimal segments of the width $O(1/k)$ at the boundaries of the integration region.

Therefore in the large k limit, the first term on the right-hand side of eq.(11) corresponds to a closed triangle configuration made of (C_1, C_2, C_3) in that order. The second term appears to represent the configuration where three Wilson lines share the same end point. By the cyclic symmetry, we can interpret it as the closed triangle made of (C_2, C_1, C_3) in that order.



Therefore we can indeed see that the Wilson lines are bound to form close Wilson loops in the large k limit in the lowest order of perturbation theory. We can also understand the phase in each term $\pm k_1 C k_2 sin(\theta_1)/2$ as the magnetic flux passing through the respective triangle.

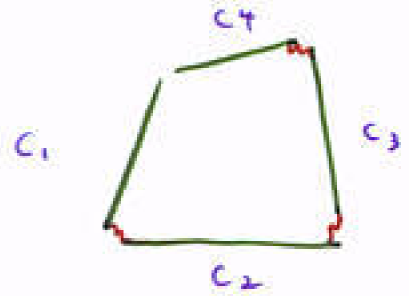
We can extend similar analysis to n point functions of the Wilson lines. By such an analysis we can easily convince ourselves that the n point functions of Wilson lines are effectively described by a group of Wilson loops which can be formed from the Wilson lines. It is certainly clear that the correlation function is saturated by a finite numbers of configurations for n point functions just like the three point function case. We argue that they could only be Wilson loops due to the gauge invariance. See Fig. 2

In the weak coupling region, we might argue that the investigation of the tree diagrams suffices. However it is certainly not so for two point functions. * It is found that the leading contribution at the n-th order is $(\lambda|k||Ck|/4\pi)^n/(n!)^2$. The summation over n can be estimated by the saddle point method as $exp(\sqrt{\lambda|k||Ck|/\pi})$. The average separation of the two Wilson lines is found to be $<n>/k \sim \sqrt{\lambda|Ck|/|k|}$. Although it is much larger than the tree level estimate $1/k$, it is still much smaller than the noncommutativity scale in the weak coupling regime.

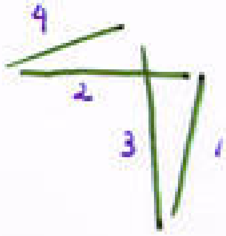
*Gross, Hashimoto and Itzhaki

4 Point functions

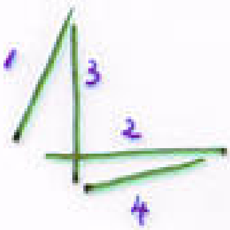
Phases



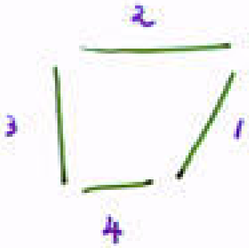
$$-\frac{1}{2}k_1\wedge k_2 - \frac{1}{2}k_3\wedge k_4$$



$$\frac{1}{2}k_1\wedge k_2 - \frac{1}{2}k_3\wedge k_4$$



$$-\frac{1}{2}k_1\wedge k_2 + \frac{1}{2}k_3\wedge k_4$$

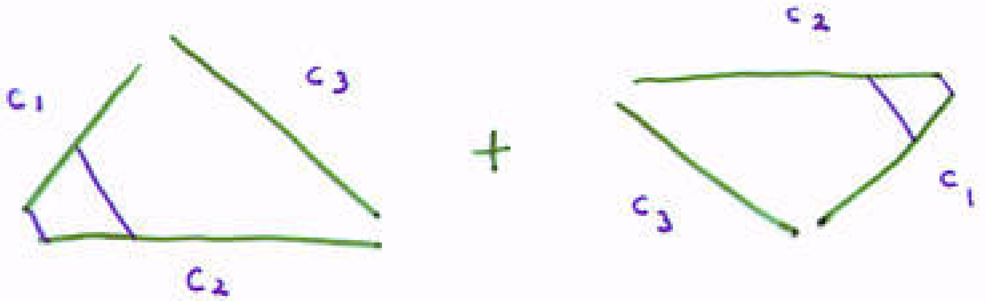


$$\frac{1}{2}k_1\wedge k_2 + \frac{1}{2}k_3\wedge k_4$$

where $k_1\wedge k_2 \equiv C^{\mu\nu}k_{1\mu}k_{2\nu}$

Fig. 2

In the case of three point functions, we also find logarithmic divergences in association of the corners. The first example occurs at the next order.



$$
\begin{aligned}
& exp(i\Phi) \int d^4x \int d^4x_1 \int d^4x_3 exp(ik_1 \cdot x_1 + ik_3 \cdot x_3) \\
& P \int_{C_1} da \cdot A(x + x_1 + a_2) \star \int_{C_1} da \cdot A(x + x_1 + a_1) \\
& \star \int_{C_2} db_1 \cdot A(x + b_1) \star \int_{C_2} db_2 \cdot A(x + b_2) \\
& \star \int_{C_2} db_3 \cdot A(x + b_3) \star \int_{C_3} dc \cdot A(x + x_3 + c) \\
= \; & V \frac{\lambda^3}{m^2} exp(i\Phi) \int_{C_1} da_1 \cdot \int_{C_2} db_1 \int_{C_1} da_2 \cdot \int_{C_2} db_2 \int_{C_2} db_3 \cdot \int_{C_3} dc \\
& \int \frac{d^4p}{(2\pi)^4} \frac{1}{(k_1 - p)^2} \frac{1}{p^2} \frac{1}{{k_3}^2} \\
& exp(-i(k_1 - p) \cdot (a_1 - b_1) - ip \cdot (a_2 - b_2) - ik_3 \cdot (c - b_3)).
\end{aligned}
\tag{12}
$$

We can now fix $a_1 = b_1 = c = 0$ due to the cyclic symmetry of the Wilson lines. In this way we find

$$
\begin{aligned}
&V\frac{\lambda^3}{im^2}cos^2(\theta_1)cot(\theta_3)Ck_1Ck_2C\int\frac{d^4p}{(2\pi)^4}\frac{1}{(k_1-p)^2}\frac{1}{p^2}\frac{1}{k_3{}^2}\\
&\{exp(-\frac{i}{2}k_1\cdot Ck_2)\frac{Ck_1Ck_2}{p\cdot Ck_1p\cdot Ck_2}\\
&\times(1-exp(-ip\cdot Ck_1))(1-exp(ip\cdot Ck_2))\\
&-exp(\frac{i}{2}k_1\cdot Ck_2)\frac{Ck_1Ck_2}{(k_1-p)\cdot Ck_1(k_1-p)\cdot Ck_2}\\
&\times(1-exp(i(k_1-p)\cdot Ck_1))(1-exp(-i(k_1-p)\cdot Ck_2))\}.
\end{aligned}
\tag{13}
$$

In the first term of the above expression, we notice the following factor

$$
\int\frac{d^4p}{(2\pi)^4}\frac{Ck_1Ck_2}{p^2p\cdot Ck_1p\cdot Ck_2}. \tag{14}
$$

Due to the additional propagator $1/(k_1-p)^2$ in the full expression, the large momentum cut-off scale is k_1. The small momentum cut-off can also be seen to be $O(1/Ck)$. Therefore it could give rise to a large factor of $O(\lambda log(Ck^2))$ in the large k limit. It can be regarded as the correction to the first term of the tree amplitude in eq.(11). It can be associated with a corner of the first triangle. The second term is just analogous.

Since each loop gives rise to an additional logarithmic factor in the ladder diagrams, we need to sum them to all orders in the leading log approximation.

For this purpose we consider a generic ladder diagram with n propagators around a corner of a triangle. After summing over n , we obtain the power enhancement factor in association with a corner *

$$exp(\frac{\lambda cot(\theta_1)(\pi - \theta_1)}{4\pi^2} log(Ck_1^2)). \tag{15}$$

It is clear that such a power law can be associated with each corner of the Wilson loops formed by the Wilson lines. Such corner divergences are well known to occur in the Wilson loop expectation values. It is also studied in ordinary gauge theory through AdS/CFT correspondence †. The appearance of such a power law enhancement in the Wilson line correlators in the high energy limit which is characteristic to the Wilson loops is consistent with our assertion that they are indeed equivalent.

*see also Rozali and Raamsdonk

†Drukker, Gross and Ooguri

In the high energy limit in the weak coupling regime, the multi-point correlation functions of the normalized Wilson lines behave as

$$
\begin{aligned}
&< W(k_1)W(k_2)\cdots W(k_n) > \\
\sim \quad & exp(-\sum_{i=1}^{i=n}\sqrt{\frac{\lambda}{4\pi}|Ck_i||k_i|} + \sum_{i=1}^{i=n}\frac{\lambda cot(\theta_i)(\pi-\theta_i)}{4\pi^2}log(|Ck||k|))
\end{aligned}
\tag{16}
$$

where θ_i is the i-th angle of the relevant closed Wilson loop and we assume that k_i are all of the same order k. In this expression, the exponential suppression of the normalized multi-point function is caused by the exponential enhancement of the two point function.

It is conceivable that the equivalence of the Wilson lines and Wilson loops at high energy continues to hold at strong coupling. If so, the expectation value of large Wilson loops in NCYM may be estimated semiclassically by Nambu-Goto action. For such a purpose, we consider superstring theory in a particular background.

See Fig. 3

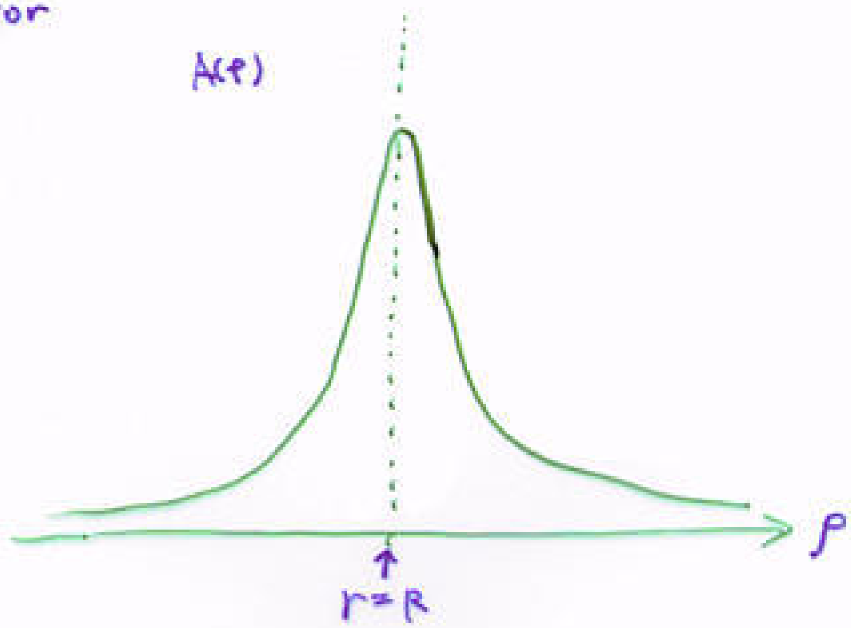
String frame metric

$$\frac{1}{\alpha'} ds^2 = \frac{R^2}{r^2}\left(\frac{d\vec{x}^{\,2}}{1 + R^4/r^4} + d r^2 \right) + R^2 d\Omega_5^2$$

$$= R^2\left(A(\rho)\,(d\vec{x}^{\,2} + d\rho^2) + d\Omega_5^2 \right)$$

Maldacena & Russo
Hashimoto & Itzhaki

Warp factor



We put NC scale $l_{NC} = 1$ here.

Fig. 3

The string frame metric possess the maximum at the scale $R \sim (\lambda)^{1/4}$ in the fifth radial coordinate r. We have proposed to put the Wilson loops at $r = R$. In such a construction, we obtain analogous expression with AdS/CFT correspondence since the relevant Wilson loops are large. The only novelty is to identify the short-distance cut-off with R. With this prescription we predict the strong coupling behavior of the Wilson lines as follows:

$$
\begin{aligned}
& < W(k_1)W(k_2)\cdots W(k_n) > \\
\sim \quad & exp(-R\sum_{i=1}^{i=n}\sqrt{|Ck_i||k_i|} \\
& +R^2\sum_{i=1}^{i=n}\frac{cot(\theta_i)(\pi-\theta_i)}{\pi}log(|Ck||k|/R^2)).
\end{aligned}
\tag{17}
$$

It is because $\sum_{i=1}^{i=n}\sqrt{|Ck_i||k_i|}$ is proportional to the perimeter length of the Wilson loop in fully noncommutative gauge theory or uniformly large momentum region.

We recall here that the average distance of the two point function is $O(R^2)$ with respect to the noncommutativity scale in the weak coupling limit. The standard prescription $R^2 \to R$ may imply that the minimal length scale which can be probed by the high energy limit of the two point function is indeed R in the strong coupling limit. It is hence likely that our prescription to put the Wilson loops at $r = R$ is relevant in such a limit.

The $log(|Ck||k|)$ behavior in the weak coupling expression in eq.(16) is already remarkable in that the short distance cut-off $1/k$ and the long distance cut-off Ck are related through the noncommutativity scale C. It is reminiscent of T duality in string theory. However eq.(17) predicts the appearance of R^2 factor in the logarithm in the strong coupling limit.

Conclusions and discussions

We have shown that the open Wilson lines are bound into a group of closed Wilson loops in the high energy limit in the weak coupling regime.

We have thus found that the stringy suppression factor of normalized multi-point functions can be associated with the world sheet bounded by the Wilson loop. Hence it is natural to expect that the correlation functions can be described through Nambu-Goto action in the strong coupling limit. It is also likely that such a geometric solution emerges from loop equations.

We are still just beginning to understand nonperturbative aspects of NCYM. For example it is very interesting to investigate the possible relevance of Randall-Sundrum mechanism to NCYM. * We hope loop equations are useful to elucidate nonperturbative properties of NCYM.

*Ishibashi, Iso, Kawai and Kitazawa