CLASSICAL & PLANAR LIMITS OF NONCOMMUTATIVE VACUA & SOLITONS

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based on works with

GAUTAM MANDAL
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(to appear)

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MOMBAI

EMERGENT THEME

FIND OUT SOLVABLE LIMITS OF STRING THEORIES / M~ THEORY

- OPEN STRING THEORY IN B_{MN}~ BACKGROUND

+

- TACHYON CONDENSATION & D~BRANES AS SOLITONS

⇓ EXTERNAL FIELD PROBLEMS

STARTING SET~UP :

NONCOMMUTATIVE FIELD THEORY

ORDER

BETTER UNDERSTANDING OF VACUA, SOLITONS & LOW~ENERGY SPECTRA

INITIATED BY

GOPAKUMAR, MINWALLA, STROMINGER

QUESTION

- WHAT ARE QUANTUM ASPECTS OF NONCOMMUTATIVE VACUA & SOLITONS ?
- ANY SYSTEMATIC SEMICLASSICAL LIMITS ?

APPROACH

$\Theta \to \infty$ LIMIT OF NONCOMMUTATIVE D-DIM. FIELD THEORY $\parallel$ $N \to \infty$ LIMIT OF (D-2)-DIM MATRIX MODEL

(CF. TWISTED EGUCHI-KAWAI)

FINDINGS

- QUANTUM DEFORMATION OF SOLITONS
- UNDERSTAND CLASSICAL LIMIT TO GMS SOLITONS
- (SELF-CONSISTENT) DYNAMICAL BREAKING OF TRANSLATIONAL SYMMETRY
- NOVEL HIGGS MECHANISM

$N \to \infty$ SCALING LIMITS

MATRIX INTEGRAL

$$\boxed{Z = \int DM \cdot e^{-\beta \, \mathrm{Tr}_N V(M)}}$$

$M =$ $(N \times N)$ Hermitian matrix

AFTER DIAGONALIZATION, $M = \Omega \cdot \Lambda \cdot \Omega^{+}$ $\Lambda = \mathrm{diag}(\lambda_1, \ldots)$

$$Z = \exp[- N\beta \cdot F]$$

WITH

'VANDERMONDE' MEASURE

$$F = \int d\lambda \, \rho(\lambda) + \frac{N}{\beta} \int d\lambda \int d\lambda' \, \rho(\lambda) \cdot \log|\lambda - \lambda'| \cdot \rho(\lambda')$$

$$\rho(\lambda) \equiv \frac{1}{N} \sum_{i=1}^{N} \delta(\lambda - \lambda_i)$$

THREE POSSIBLE SADDLE-POINT BEHAVIOR

① $\beta \propto N^{\nu}$ $(\nu > 1)$

FIXED-ORDER PERTURBATION SERIES

$\uparrow$ $\tilde{\beta} \to \infty$

② $\beta \propto N$; $\tilde{\beta} = \beta/N$ = FIXED

PLANAR LIMIT

③ $\beta \propto N^{\nu}$ $(\nu < 1)$

D=2 NONCOMMUTATIVE FIELD THEORY

- $$S = \int d^2\vec{y}\ \left[(\vec{\partial}\phi)^2 + V(\phi) \right]_*$$

  ON NONCOMMUTATIVE PLANE
  $$[y^1, y^2] = i\theta$$
  AND $*$- PRODUCT

- TAKE $\theta \to \infty$ LIMIT

  $$\vec{y} \to \vec{x} = \frac{1}{\sqrt{\theta}}\vec{y} \qquad \text{YIELDS}$$

  $$S = \theta \int d^2\vec{x}\ \left[V_*(\phi) + \frac{1}{\theta}(\vec{\partial}\phi)^2 \right]$$

  WITH $[x^1, x^2] = i$.

- WEYL-MOYAL MAP

  $\mathcal{H}_\theta$ = 1-PTC. HILBERT SPACE CARRYING REPS. OF HEISENBERG ALGEBRA

  $$\phi(x) \leftrightarrow \hat{\phi}$$
  $$\int d^2\vec{x} \cdots \leftrightarrow \mathrm{Tr}_{\mathcal{H}_\theta} \cdots$$

  GIVES

  $$S = \theta\, \mathrm{Tr}_{\mathcal{H}_\theta} \left[V(\hat{\phi}) + \underbrace{\frac{1}{\theta}[a,\hat{\phi}][a^\dagger,\hat{\phi}]}_{\text{sub-leading}} \right]$$

- CLASSICAL SOLUTIONS

$$\hat{\Phi} = \sum \lambda_i \hat{P}_i = \begin{pmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \ddots & & \\ & & & \lambda_k & \\ & & & & \ddots \end{pmatrix}$$

WITH

$$\hat{P}_i \cdot \hat{P}_j = \delta_{ij} \hat{P}_j$$

$$\mathrm{Tr}_{\mathcal{H}_\theta} \hat{P}_i = 1$$

ONE-DIMENSIONAL PROJECTION OPERATOR

$$\sum \hat{P}_i = \mathbb{1}$$

AND

$$V'(\lambda_i) = 0 .$$

- At $\theta = \infty$, $\exists$ $U(\infty)$ SYMMETRY

$$\hat{\Phi} \sim U \cdot \hat{\Phi} \cdot U^+$$

WHERE

$$U \cdot U^+ = \mathbb{1}, \quad U^+ U = \mathbb{1}.$$

- VACUUM $\hat{\Phi} = \bar{\lambda} \mathbb{1}$

$\lambda_1 = \lambda_2 = \cdots = \bar{\lambda}$ WITH $V'(\bar{\lambda}) = 0$

GMS SOLITON

$\lambda_1 = \bar{\lambda}$; $\lambda_2 = \lambda_3 = \cdots = \bar{\lambda}'$ ETC.

- EXAMPLE

$$V(\lambda) = -\frac{1}{2}\lambda^2 + \frac{1}{4}\lambda^4$$

$$\hat{\Phi}_{N_1, N_2} = (-1)\cdot \mathbb{P}_{N_1} + (+1)\cdot \mathbb{P}_{N_2}$$

$N_1 + N_2 = N$

INVARIANT INFORMATION CARRIED BY DENSITY VARIABLE

$$\rho(\lambda) = \left(\frac{N_1}{N}\right)\cdot \delta(\lambda+1) + \left(\frac{N_2}{N}\right)\cdot \delta(\lambda-1)$$

$$\equiv \frac{1}{N}\sum_i \delta(\lambda - \lambda_i)$$

$N = \dim \mathcal{H}_\Theta$

SO FAR, CLASSICAL.

WHAT ABOUT QUANTUM-MECHANICAL ?

$$\boxed{Z_{NC} = \int D\hat{\Phi}\cdot e^{-\Theta \,\mathrm{Tr}\, V(\hat{\Phi})} \quad \text{AT } \Theta \to \infty}$$

LARGE Θ - EFFECTS

$$Z_{NC} = \int D\hat{\Phi} \cdot e^{-\Theta \, \mathrm{Tr}_{\mathcal{H}} V(\hat{\Phi})}$$

DIFFERENCE

- $\dim \mathcal{H}_\Theta = \infty$

  AS OPPOSED TO $N \to \infty$, $\beta \to \infty$; $\bar{\beta} = \beta / N$ = FIX'D

→ DEFINE NONCOMMUTATIVE FIELD THEORY WITH UV AND IR CUTOFFS

(cf. TWISTED EGUCHI - KAWAI)

(cf. $\mathbb{T}_\Theta^N$ - Rieffel)

- THE PRESCRIPTION YIELDS PLANAR LIMIT WITH

$$\boxed{\bar{\Theta} = (\Theta / \dim \mathcal{H}_\Theta) = \text{FIXED}}$$

QUANTUM 'GMS' SOLITON (INSTANTON) IN $\Theta \to \infty$ IS GIVEN BY SADDLE-POINT OF MATRIX MODEL

$\xrightarrow[\bar{\Theta} \to \infty]{}$ CLASSICAL 'GMS' SOLITON

ZERO-DIM. MATRIX MODEL

$$Z_0 = \int DM \cdot e^{-\beta \, Tr \, V(M)}$$

$M = (N \times N)$ Hermitian matrices

- LARGE-N LIMIT

$\beta \to \infty$, $N \to \infty$, $\bar{\beta} = \beta/N$ = FIXED

YIELDS

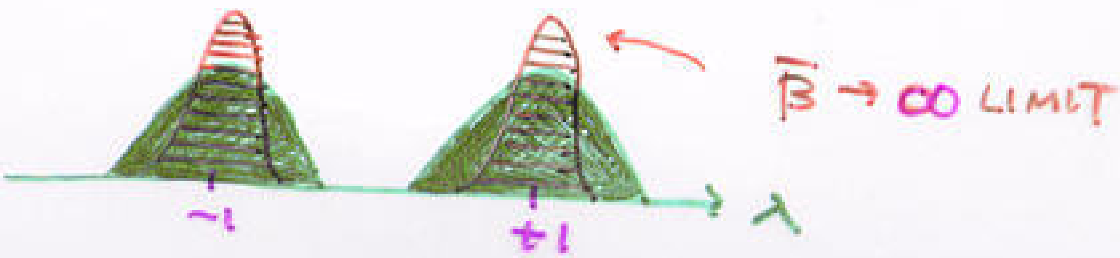
$$Z_0 = \exp[-F(\bar{\beta})]$$

$$F(\bar{\beta}) = N^2 \bar{\beta} \Big[\int d\lambda \cdot \rho(\lambda) V(\lambda) - \frac{1}{\bar{\beta}} \int d\lambda \int d\lambda' \rho(\lambda) \cdot \log|\lambda - \lambda'| \cdot \rho(\lambda') \Big]$$

- $N = \infty$ SADDLE-POINT SOLUTION

$$\rho(\lambda) = \frac{\bar{\beta}}{2} \sqrt{\lambda^2 (\lambda^2 - a^2)(b^2 - \lambda^2)}$$

$$a = \left(1 - \frac{2}{\sqrt{\bar{\beta}}}\right)^{1/2} ; \quad b = \left(1 + \frac{2}{\sqrt{\bar{\beta}}}\right)^{1/2}$$



ORIGIN OF VANDERMONDE

- FUNCTIONAL INTEGRAL MEASURE

$$\int D\phi(\vec{x})\Big|_{NC}$$

IS DEFINED VIA NORM

$$\| \delta\phi \|^2_{NC} = \int d\vec{x}\ \delta\phi(\vec{x}) \underset{\sim}{*} \delta\phi(\vec{x})$$

$$= \int d\vec{x}\ \delta\phi(\vec{x}) \cdot \delta\phi(\vec{x}) \quad = \| \delta\phi \|^2_{COMM.}$$

BOTH COMMUTATIVE AND NONCOMM. MEASURES ARE THE SAME.

HOW DOES THE VANDERMONDE JACOBIAN ARISE ?

- WAY TO DERIVE IT

  = WEYL–MOYAL MAP

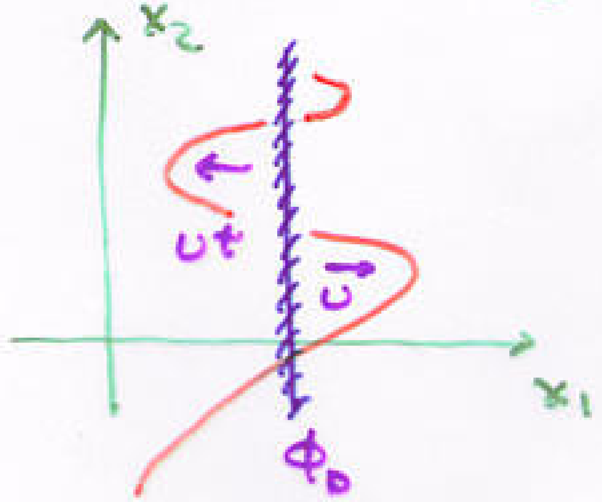
TURNS OUT MOST CONVENIENT TO CHOOSE 'PLANE-WAVE' BASIS

$$\hat{x}_1 |p\rangle = p|p\rangle \ ; \ \hat{x}_2 |q\rangle = q|q\rangle$$

- DIAGONALIZATION THEN FOLLOWS AS

$$\boxed{\phi(\vec{x}) = U(\vec{x}) * \phi_D(x_2) * U^{-1}(\vec{x})}$$

FUNCTION OF x_2-ONLY



- $[D\phi]_{NC} = D\phi_D(x_2) \cdot DU(\vec{x}) \cdot J[\phi_D]$

WE FIND

$$\boxed{J[\phi_D] = \prod_{x_2, p} \left[\phi_D\left(x_2 + \frac{\theta}{2} p\right) - \phi_D\left(x_2 - \frac{\theta}{2} p\right) \right]^2}$$

RECALL

$\hat{x}_1 |p\rangle = p|p\rangle$

Moyal $*$-product representation

→ NONCOMMUTATIVE FIELD THEORY IN 'DIAGONAL' FIELD $\phi_D(x_2)$

$$\boxed{Z = \int D\phi_D \cdot \exp\left[-\theta L_{x_1} \int dx_2\, V(\phi_D) + \int dp \int dx_2 \log\left(\phi_D(x_2 + \tfrac{\theta}{2}p) - \phi_D(x_2 - \tfrac{\theta}{2}p)\right)\right]}$$

$L_{x_1} = \int dx_1$, $L_{x_2} = \int dx_2$

- PLANAR LIMIT

$$\boxed{\theta \cdot L_{x_1} \cdot L_{x_2} \quad \propto \quad L_p \cdot L_{x_2}}$$

- IF ADOPT TWISTED EGUCHI-KAWAI PRESCRIPTION

$L_p \sim L_{x_2} \sim N \cdot a$

$L_{x_1} \sim \pi / a$

$\Rightarrow \boxed{\theta \propto N}$

- CONTINUUM LIMIT

$a \propto N^{-\nu}$; $\quad \nu = \frac{1}{2} \quad T_\theta^2$

$0 < \nu < \frac{1}{2} \quad R_\theta^2$

(2+1)-DIM. NONCOMMUTATIVE FIELD THEORY

$$S_{NC} = \int dt\, d^2\vec{y} \left[\frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}(\vec{\partial}\phi)^2 - V_*(\phi) \right]$$

WITH $[Y^1, Y^2] = i\Theta$

$\Theta \to \infty$ LIMIT

$$S_{NC} = \Theta \int dt\, d^2\vec{x} \left[\frac{1}{2}(\partial_t \phi)^2 - V_*(\phi) - \frac{1}{\Theta}\frac{1}{2}(\vec{\partial}\phi)^2 \right]$$

sub-leading

AT $\Theta = \infty$, IT REDUCES TO (0+1)-DIM MATRIX MODEL OF FORM

$$\boxed{S_M = \beta \int dt \cdot \mathrm{Tr} \left[\dot{M}^2(t) - V(M) \right]}$$

AFTER DIAGONALIZATION

$$M(t) = \Omega(t) \cdot \Lambda(t) \cdot \Omega^+(t)$$

$$\Lambda(t) = \begin{pmatrix} \lambda_1(t) & & \\ & \lambda_2(t) & \\ & & \ddots \end{pmatrix}$$

VANDERMONDE DETERMINANT TURNS EIGENVALUES $\{\lambda_i(t)\}$ INTO POSITIONS OF 'FERMIONS'.

- 2ND-QUANTIZATION DESCRIPTION

$$H = \sum_{i=1}^{N} \left[-\frac{1}{\beta} \frac{\partial^2}{\partial \lambda_i^2} + \beta V(\lambda_i) \right]$$

$$= \int dx\, \Psi^\dagger(x,t) \left(-\frac{1}{\beta} \partial_x^2 + \beta V(x) \right) \Psi(x,t)$$

EIGENVALUE DISTRIBUTION

$$\rho(x,t) = \int dp\, F_W(x,p;t)$$

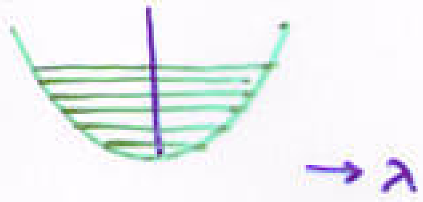
Wigner distribution function

$$F_W(x,p;t) = \int dq\, \Psi^\dagger(x) * e^{ipq/\hbar} * \Psi(x)$$

• $\boxed{M_{soliton}\Big|_{PLANAR} > M_{soliton}\Big|_{CLASSICAL}}$

OBVIOUS , SINCE

$$M_{SOLITON} = \int d\lambda \, \rho(\lambda) \, \nabla(\lambda)$$



AT EACH CUT.

TO BE CONTRASTED TO

2ND-ORDER PERTURBATION THEORY

DYNAMICAL SYMMETRY BREAKING OF TRANSLATIONAL INVARIANCE

CONSIDER $V(\phi)$

CLASSICALLY

$$\hat{\Phi} = \begin{pmatrix} \phi_0 & & & \\ & \phi_0 & & \\ & & \phi_0 & \\ & & & \ddots \end{pmatrix} = \phi_0 \cdot \mathbb{1}$$

Weyl – Moyal map

$$\langle \phi(\vec{x}) \rangle = \int \frac{d\vec{k}}{(2\pi)^2} e^{i\vec{k}\cdot\vec{x}} \cdot \phi_0 \delta(\vec{k})(2\pi)^2$$

$$= \phi_0 \cdot$$ homogeneous condensate

PLANAR LIMIT

$$\hat{\Phi} = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \ddots \end{pmatrix}$$ WHERE

$$\neq \phi_0 \mathbb{1}$$

$$\langle \phi(\vec{x}) \rangle = \int \frac{d\vec{k}}{(2\pi)^2} e^{i\vec{k}\cdot\vec{x}} \tilde{\phi}(\vec{k})$$ INHOMOGENEOUS !

UPSHOT

IN PLANAR LIMIT
TRANSLATIONAL SYMMETRY is BROKEN
DYNAMICALLY VIA QUANTUM EFFECT.

SELF-CONSISTENT AS ALL PLANAR DIAGRAMS
ARE RESUMMED

REMARK

$\frac{1}{\theta}(\vec{\partial}\phi)^2$ term seems to drive $\langle \phi(\vec{x}) \rangle$
into 'stripe' phase.

$\hookrightarrow$ What to make out of this?

SPONTANEOUS BREAKING OF NONCOMMUTATIVE GAUGE INVARIANCE

Recall

(PART OF) GAUGE ORBIT IN NONCOMMUTATIVE GAUGE THEORY IS EQUIVALENT TO TRANSLATION IN NONCOMMUTATIVE DIRECTION

ADJOINT REPRESENTATION IN $U(1)_{NC}$

$$\delta_\varepsilon \phi(\vec{x}) = \int \frac{d^2\vec{k}}{(2\pi)^2} e^{i\vec{k}\cdot\vec{x}} \tilde{\varepsilon}(\vec{k}) \times \left[\phi(\vec{x} \underset{\sim\sim}{+ \theta\wedge\vec{k}}) - \phi(\vec{x} \underset{\sim\sim}{- \theta\wedge\vec{k}}) \right]$$

RECALL

- $\theta \to 0$ LIMIT: ADJOINT REP. SCALAR AND PHOTON DECOUPLES
- $U(N)$ COMMUTATIVE GAUGE THEORY: ADJOINT REP. SCALAR LEADS TO $U(N) \to [U(1)]^N$ GENERICALLY

→ $U_{NC}(1)$ GAUGE THEORY WITH 'ADJOINT' REPRESENTATION SCALAR

RESULT

$\boxed{\text{IN UNITARY GAUGE, SPONTANEOUSLY BROKEN } U_{NC}(1) \text{ GAUGE THEORY w/ SPATIALLY VARYING MASS GAP.}}$

$$S_{eff} = \int dx \, [\, \tfrac{1}{4} F_{\mu\nu} * F_{\mu\nu}$$

$$- \tfrac{1}{2} A_\mu(x) * \underset{\sim\sim\sim}{M^2(x)} * A_\mu(x) \,]$$

↑ $\langle \Phi(\vec{x}) \rangle^2$ IN PLANAR LIMIT

$$= \sum_{\vec{p}} \left(v_{\vec{p}}^2 \, e^{i\vec{p}\cdot\vec{x}} + c.c. \right)$$

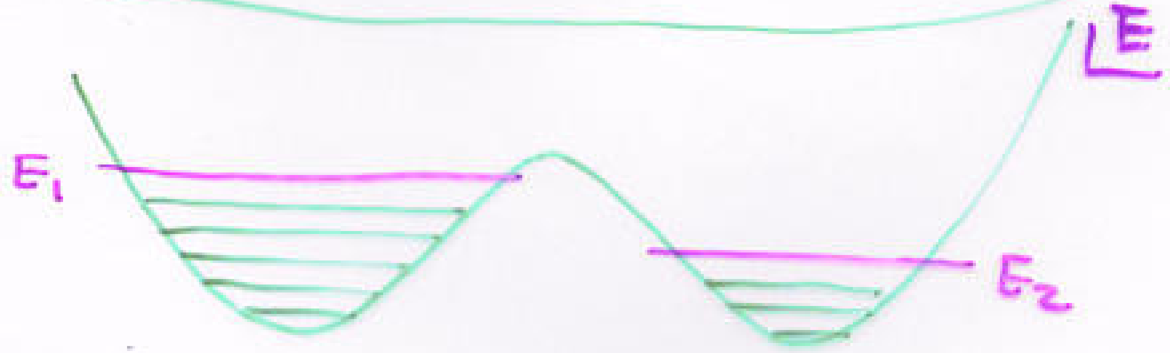
CLASSICAL SOLUTION

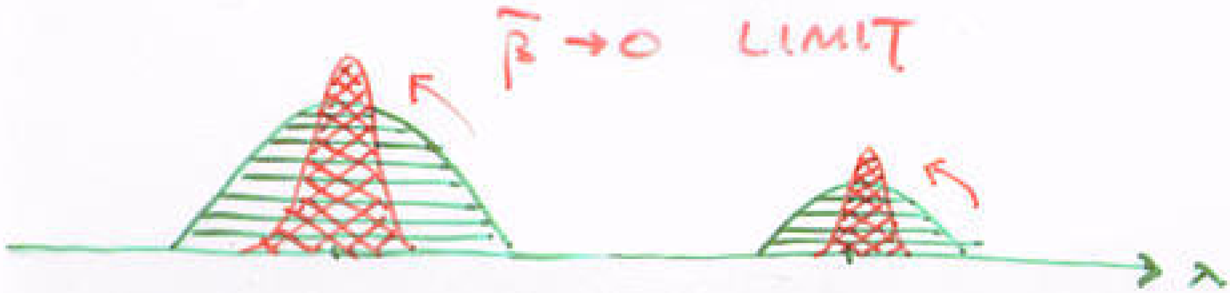
GIVEN BY PHASE-SPACE DENSITY OF N-FERMIONS OCCUPYING THE LOWEST N LEVELS OF 1-PARTICLE HAMILTONIAN.

IN WKB APPROX.

$$p(x) = \sqrt{2(E - \bar{\beta} V(x))}$$

FOR $V(x) < E/\bar{\beta}$





* TUNNELLING SUPPRESSED

→ SMOOTH CLASSICAL 'GMS' LIMIT