(SOME REMARKS ON THE)

GEOMETRY of SUPERSYMMETRIC pp-WAVES

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Based on:

hep-th/0105308 (GP)
0110242 (MB, CMH, GP)
0201081 (MB, CMH, GP)
0202111 (MB, GP)

and WORK in PROGRESS with GP

math.DG/0109162

GP = George PAPADOPOULOS
MB = Matthias BLAU
CMH = Chris HULL

MAXIMAL SUSY & PP-WAVES

eg: D=11 SUGRA (g,F)

$$\delta_\varepsilon \psi_\mu \big|_{\psi=0} =: \mathcal{D}_\mu \varepsilon$$

$$= \underbrace{\nabla_\mu \varepsilon}_{\text{SPIN CONNECTION}} + \underbrace{\Omega_\mu \varepsilon}_{F\text{-dependent algebraic term}}$$

MAXIMAL SUSY $\Rightarrow$ $\mathcal{D}$ is <u>flat</u>

$\Rightarrow$ 1) F is PARALLEL and DECOMPOSABLE

2) Riemann $= T(g,F)$ (algebraic)

1) + 2) $\Rightarrow$ (M,g) LOCALLY SYMMETRIC

Tracing 2) $\Rightarrow$ $\underbrace{R}_{\text{SCALAR CURVATURE}} \propto |F|^2$

TWO CASES:

$R \neq 0 \Rightarrow$ FREUND-RUBIN (using deRham-Wu dec. theorem)

$\therefore \quad AdS_4 \times S^7 \qquad AdS_7 \times S^4$

$R = 0 \Rightarrow$ (i) $F = 0 \Rightarrow$ FLAT

(ii) F NULL $\Rightarrow$ CW pp-WAVE (Kowalski-Glikman '84)

Theorem (Cahen-Wallach '69)

Indecomposable Lorentzian symmetric spaces of dim ≥ 3 are:

AdS $\qquad$ dS $\qquad$ CW pp-WAVE

CW pp-WAVES

$M^D = G/K$ depending on $A \in S^2\mathbb{R}^{D-2}$

$$\mathfrak{g} = \langle e_i, e_i^*, e_+, e_- \rangle \qquad i = 1, \ldots, D-2$$

$$\left.\begin{aligned} [e_i, e_j^*] &= A_{ij}\, e_+ \\ [e_-, e_i] &= e_i^* \\ [e_-, e_i^*] &= \sum_j A_{ij}\, e_j \end{aligned}\right\} \text{SOLVABLE !}$$

3.

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \qquad \mathfrak{k} = \langle e_i^* \rangle$$
$$\mathfrak{p} = \langle e_i, e_+, e_- \rangle$$

SYMMETRIC
$$\begin{cases} [\mathfrak{k},\mathfrak{k}] \subset \mathfrak{k} & (\text{abelian}) \\ [\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p} \\ [\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k} \end{cases}$$

$$\eta \in S^2\mathfrak{p}^*$$

$$\left.\begin{array}{l} \eta(e_i,e_j) = \delta_{ij} \\ \eta(e_+,e_-) = 1 \end{array}\right\} \ \mathfrak{k}\text{-INVARIANT}$$

$\therefore$ $(G/K, \eta)$ is a LORENTZIAN SYMMETRIC SPACE

FURTHER,

INDECOMPOSABLE $\iff$ A_{ij} NONDEGENERATE

In natural coordinates $(x^i, x^\pm)$

CW PP-WAVE METRIC
$$\boxed{g = 2\,dx^+dx^- + \sum_{ij} A_{ij}x^ix^j(dx^-)^2 + \sum_i (dx^i)^2}$$

4. supplemented by:

D=11
$$\begin{cases} F = dx^- \wedge C \\ \\ C = \frac{1}{3!}\sum_{i,j,k} C_{ijk}\, dx^i \wedge dx^j \wedge dx^k \in \Lambda^3 \mathbb{R}^9 \end{cases}$$

CONSTANT

or

IIB
$$\begin{cases} F = dx^- \wedge B \\ \\ B \in \Lambda^4_+ \mathbb{R}^8 \end{cases}$$

or ...

Symmetries

Isometries of g: $G \rtimes SO_A$

isotropy of A

subgroup of $SO(D-2)$ preserving A

symmetries of SUGRA solution:

$$G \rtimes (SO_A \cap SO_F)$$

$\subset SO(D-2)$ preserving F

CW pp-WAVES are natural Ansätze for SUGRA solutions preserving $\nu \geq \frac{1}{2}$ SUSY.

e.g. KOWALSKI-GLIKMAN SOLUTION of D=11

$$A_{ij} = \left(\begin{array}{c|c} -\mathbb{1}_3 & \\ \hline & -\frac{1}{4}\mathbb{1}_6 \end{array}\right)$$

$\boxed{\nu = 1}$

$$C = 3\, dx^1 \wedge dx^2 \wedge dx^3$$

Symmetries : $G \rtimes (SO_3 \times SO_6)$

$$\dim = 38$$

eg: IIB pp-WAVE

$\boxed{\nu = 1}$

$$A_{ij} = -\mathbb{1}_8$$

$$B = \frac{1}{2}\left(dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 + dx^5 \wedge dx^6 \wedge dx^7 \wedge dx^8\right)$$

Symmetries : $G \rtimes (SO_4 \times SO_4)$

$$\dim = 30$$

Other $\frac{1}{2} < \nu < 1$ pp-waves have been found by

Michelson (hep-th/0203140)
Cvetic, Lü, Pope (hep-th/0203229)
Gauntlett, Hull (hep-th/0202255)
⋮

Other $\nu = 1$ pp-WAVES in lower dimension have been found by

Meessen (hep-th/0111031)

Similarities between $\nu = 1$ solutions

D=11

$$\left.\begin{array}{l} AdS_4 \times S^7 \\ AdS_7 \times S^4 \\ \text{CW pp-wave} \end{array}\right\}$$

All are LSS & dimension of symmetry groups agrees;

$$\dim\left(SO(3,2) \times SO(8)\right) = \dim\left(SO(2,6) \times SO(5)\right) = 38$$

Closer inspection $\Rightarrow$ CW pp-WAVE isometry algebra is a <u>contraction</u> of $SO(3,2) \times SO(8)$ & $SO(2,6) \times SO(5)$.

"LIMIT"

7.

<u>IIB</u>

$\left.\begin{array}{l} AdS_5 \times S^5 \\ \text{PP-WAVE} \end{array}\right\}$ Also conformally flat.

ALSO:

CW PP-WAVES embed isometrically in $\mathbb{R}^{2,D}$:

$$\begin{cases} U_1^2 + U_2^2 = 1 \\ U_1 V_1 + U_2 V_2 = \sum_{i,j=1}^{D-2} A_{ij} X^i X^j \end{cases}$$

$(\underbrace{U_1, V_1}_{\text{null}}, \underbrace{U_2, V_2}, X^i)$ coords. of $\mathbb{R}^{2,D}$

(cf. $AdS \times S$)

PENROSE - GÜVEN LIMITS

$\gamma \subset M$ null geodesic

Choose coordinates : $(\underbrace{U}_{\text{affine parameter}}, \underbrace{V, y^i}_{\text{label geodesic congruence}})$ twist-free

$$g = dV\left(dU + \alpha dV + \sum_i \beta_i dy^i\right) + \sum C_{ij} dy^i dy^j$$

Boost in (U,V)-plane + rescale (y^i)

$$U = u$$
$$V = \Omega^2 v$$
$$y^i = \Omega y^i$$

Define

$$\bar{g} := \lim_{\Omega \to 0} \Omega^{-2} g(\Omega)$$

Changing coordinates...

$$\bar{g} = 2dx^+dx^- + \sum \boxed{A_{ij}(x^-)} x^i x^j (dx^-)^2 + \sum (dx^i)^2$$

$(\bar{g}, \bar{F}, \ldots)$ inherits the following properties from $(g, F, \ldots)$:

- CURVATURE EQUATIONS
  - eg: CONFORMAL FLATNESS (Weyl = 0)
  - LOCAL SYMMETRY (∇Riemann = 0)
  - EINSTEIN $\rightarrow$ RICCI-FLAT
- ISOMETRIES
- CONFORMAL ISOMETRIES
- SUPERSYMMETRIES

A COVARIANCE PROPERTY

$$\gamma \xrightarrow{\text{isometry}} \gamma'$$

$$\Downarrow$$

$$\mathrm{PGL}_{\gamma} \underset{(\text{isometric})}{\cong} \mathrm{PGL}_{\gamma'}$$

ALLOWS CLASSIFICATION OF PGL's OF SOLUTIONS WITH ENOUGH SYMMETRY:

γ determined from

$$\gamma(0) = p \in M$$

$$[\dot{\gamma}(0)] \in \text{Celestial sphere in } T_pM$$

(a null direction at p

Different PGL's $\overset{1-1}{\longleftrightarrow}$ orbits of isometry group of M in "celestial sphere bundle"

eg: $AdS_p \times S^q$

- HOMOGENEOUS $\Rightarrow$ p arbitrary
- ISOTROPY at p has 2 orbits on celestial sphere in T_pM
  - BIG (generic) $\Rightarrow$ CW pp-WAVE
  - SMALL ($\dot{\gamma}(0) \cap T_pS = 0$) $\Rightarrow$ flat

eg: Brane solutions, ... (see paper!)

Finally ...

$\exists$ PGL's for multi-time spacetimes

'null planes' replace γ's