$\mathcal{N}=4$ (Self-Dual) Yang-Mills as a String Theory

Based on work in progress with Edward Witten

Outline of talk

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II. String theory for $\mathcal{N}=4$ $d=4$ self-dual super-Yang-Mills

III. Conjecture for string theory for ordinary $\mathcal{N}=4$ $d=4$ super-Yang-Mills

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Motivation: Study Maldacena conjecture at small radius.

At present, only quantizable formalism for superstring on $AdS_5 \times S^5$ is pure spinor formalism

Review of formalism in flat background:

$$S = \int d^2z \, [\partial x^\mu \bar\partial x_\mu + p_\alpha \bar\partial \theta^\alpha + w_\alpha \bar\partial \lambda^\alpha + \hat p_{\hat\alpha} \partial \hat\theta^{\hat\alpha} + \hat w_{\hat\alpha} \partial \hat\lambda^{\hat\alpha}]$$

$$Q = \int dz \, \lambda^\alpha d_\alpha$$

$\mu = 0, \ldots, 9 \quad \alpha = 1, \ldots, 16 \quad \hat\alpha = 1, \ldots, 16$

$$d_\alpha = p_\alpha + \gamma^\mu_{\alpha\beta} \partial x_\mu \theta^\beta + (\theta \gamma^\mu \partial\theta)(\gamma_\mu \theta)_\alpha$$

$$\lambda^\alpha \gamma^\mu_{\alpha\beta} \lambda^\beta = 0$$

$\lambda \gamma^\mu \lambda = 0$ defines a "pure spinor" and implies $Q^2 = 0$.

Can show that cohomology of Q describes physical superstring spectrum.

Massless vertex op: $V = \lambda^\alpha A_\alpha(x, \theta)$

$QV = 0 \Rightarrow (\gamma^{\mu_1 \ldots \mu_5})^{\alpha\beta} D_\alpha A_\beta = 0 \Rightarrow$ $d=10$ super-YM equations of motion

Tree amplitudes: $\mathcal{A} = \langle V_1 V_2 V_3 \int W_4 \ldots \int W_N \rangle$

where $\langle (\lambda\gamma^\mu\theta)(\lambda\gamma^\nu\theta)(\lambda\gamma^\rho\theta)(\theta\gamma_{\mu\nu\rho}\theta) \rangle = 1.$

This formalism can be generalized to any consistent $d=10$ supergravity background.

For $AdS_5 \times S^5$ background with R-R flux N,

$$S = (N g_s)^{\frac{1}{2}} \int d^2 z \Big[J^\mu \bar{J}_\mu + (\gamma^{01234})_{\alpha\hat{\beta}} (3 J^{\hat{\beta}} \bar{J}^\alpha - J^\alpha \bar{J}^{\hat{\beta}})$$

$$+ w_\alpha \bar{\nabla} \lambda^\alpha + \hat{w}_{\hat{\alpha}} \nabla \hat{\lambda}^{\hat{\alpha}} + (w \gamma^{[\mu\nu]} \lambda)(\hat{w} \gamma_{[\mu\nu]} \hat{\lambda}) \Big]$$

$\mu = 0, \ldots, 9$ $\quad \alpha = 1, \ldots, 16$ $\quad \hat{\alpha} = 1, \ldots, 16$ $\quad (\mu\nu) \in SO(4,1) \times SO(5)$

$$\bar{\nabla} \lambda^\alpha = \bar{\partial} \lambda^\alpha + \bar{J}^{[\mu\nu]} (\gamma_{[\mu\nu]} \lambda)^\alpha$$

$$J^A(x, \theta, \hat{\theta}) = (g^{-1} \partial g)^A \qquad A = (\mu, \alpha, \hat{\alpha}, [\mu\nu])$$

$$\bar{J}^A(x, \theta, \hat{\theta}) = (g^{-1} \bar{\partial} g)^A \qquad g(x, \theta, \hat{\theta}) \in \frac{PSU(2,2|4)}{SO(4,1) \times SO(5)}$$

$$Q = \int dz \, \lambda^\alpha (\gamma^{01234})_{\alpha\hat{\beta}} J^{\hat{\beta}}$$

$$\hat{Q} = \int d\bar{z} \, \hat{\lambda}^{\hat{\alpha}} (\gamma^{01234})_{\beta\hat{\alpha}} \bar{J}^\beta$$

where $\lambda \gamma^\mu \lambda = \hat{\lambda} \gamma^\mu \hat{\lambda} = 0$

Unlike Metsaev-Tseytlin GS action, this action is quantizable since κ-symmetry is replaced by BRST invariance. Proven to have correct massless spectrum (NB, O. Chandia), to be one-loop conformally invariant (B.C. Vallilo), and to have infinite number of classically conserved currents (B.C. Vallilo).

Although this action can be used to study the Maldacena conjecture at large AdS radius, the model becomes strongly coupled at small AdS radius. Furthermore, it is expected to receive $\frac{1}{\text{radius}}$ corrections. For example, the chiral current $(w\gamma^{[cd]}\lambda)$ transforms anomalously under the local $SO(4,1)\times SO(5)$, which implies $\frac{1}{\text{radius}}$ corrections to the action. So the action at small radius could look very different from the action at large radius.

One clue may come from the Chern-Simons/conifold duality of Gopakumar-Vafa. This duality can be proven (Ooguri, Vafa) by comparing closed strings on the S^2 conifold with Witten's open string theory on S^3 for Chern-Simons. So perhaps to understand the $AdS_5\times S^5$ duality, one should look for an open string theory which describes $\mathcal{N}=4$ $d=4$ super-Yang-Mills.

String theory for $\mathcal{N}=4$ self-dual super-Yang-Mills:

Although no string theory was known for $\mathcal{N}=4$ $d=4$ super-YM, there is one which (almost) describes $\mathcal{N}=0$ $d=(2,2)$ self-dual Yang-Mills

$$S = \int d^2z \left(\partial X^{\alpha\dot\alpha} \bar\partial X_{\alpha\dot\alpha} + \eta^{\dot\alpha} \bar\partial \psi_{\dot\alpha} + \hat\eta^{\dot\alpha} \partial \hat\psi_{\dot\alpha} \right) \qquad \begin{matrix} \alpha=1,2 \\ \dot\alpha=1,2 \end{matrix}$$

N=4 generators: $\psi^{\dot\alpha} \partial X_{\alpha\dot\alpha} = (G^+, \tilde G^+)$ (Ademollo et al)

$\eta^{\dot\alpha} \partial X_{\alpha\dot\alpha} = (G^-, \tilde G^-)$ (Vafa, Ooguri) (NB, Vafa)

In N=4 topological string, twist $(\psi^{\dot\alpha}, \eta_{\dot\alpha})$ to have spin $(0,1)$ and define $Q = \int dz (\lambda^1 G^+ + \lambda^2 \tilde G^+) = \int dz \left(\lambda^\alpha \psi^{\dot\alpha} \partial X_{\alpha\dot\alpha} \right)$

where (λ^1, λ^2) are projective constants.

Cohomology of Q contains a single state whose equation of motion is self-dual Yang-Mills. But since λ^α is not a worldsheet field, Lorentz invariance is not manifest. Also, string loops do not compute self-dual YM loop diagrams. (Siegel)

Suppose one tries to make λ^α a worldsheet field without affecting the cohomology. So fermions need to cancel the new λ^α bosonic fields.

Consider

$$S = \frac{1}{\alpha'}\int d^2z \left(\partial x^{\alpha\dot\alpha}\bar\partial x_{\alpha\dot\alpha} + \eta^{\dot\alpha}\bar\partial\psi_{\dot\alpha} + w^\alpha\bar\partial\lambda_\alpha + e\bar\partial f + g\bar\partial h + \hat\eta^{\dot\alpha}\partial\hat\psi_{\dot\alpha} + \ldots\right.$$

$$Q = \int dz \Big(\lambda^\alpha\psi^{\dot\alpha}\partial x_{\alpha\dot\alpha} + e\psi^{\dot\alpha}\psi_{\dot\alpha} + f\lambda^\alpha\partial\lambda_\alpha$$
$$+ h\left(\lambda^\alpha w_\alpha - \psi^{\dot\alpha}\eta_{\dot\alpha} - 2fe\right)\Big)$$

$e\psi^{\dot\alpha}\psi_{\dot\alpha}$ term is needed since $G^+(y)\,\tilde G^+(z) \to \frac{\psi^{\dot\alpha}\psi_{\dot\alpha}}{(y-z)^2}$.

h ghost comes from gauging projective invariance of λ^α.

Can classically use Q to gauge λ^α to be constant,

but $Q^2 = 4(\alpha')^2\int dz\, h\partial h \Rightarrow$ theory is anomalous.

To remove anomaly, consider similar model with

$(x^{\alpha\dot\alpha}, \theta^{\alpha j})$ superspace variables for $j=1$ to $\mathcal{N}$.

$$S = \frac{1}{\alpha'}\int d^2z \Big(\partial x^{\alpha\dot\alpha}\bar\partial x_{\alpha\dot\alpha} + \partial\theta^{\alpha j}\bar\partial\theta_{\alpha j} + \eta^{\dot\alpha}\bar\partial\psi_{\dot\alpha} + b^j\bar\partial c_j$$
$$+ w^\alpha\bar\partial\lambda_\alpha + e\bar\partial f + g\bar\partial h + \hat\eta^{\dot\alpha}\partial\hat\psi_{\dot\alpha} + \ldots\Big)$$

$$Q = \int dz \Big(\lambda^\alpha\left(\psi^{\dot\alpha}\partial x_{\alpha\dot\alpha} + c_j\partial\theta^j_\alpha\right) + e\left(\psi^{\dot\alpha}\psi_{\dot\alpha} + c^j c_j\right)$$
$$+ f\lambda^\alpha\partial\lambda_\alpha + h\left(\lambda^\alpha w_\alpha - \psi^{\dot\alpha}\eta_{\dot\alpha} - c^j b_j - 2fe\right)\Big]$$

$Q^2 = (4-\mathcal{N})(\alpha')^2\int dz\, h\partial h \Rightarrow \mathcal{N}=4$ is not anomalous!

Cohomology of Q describes $\mathcal{N}=4$ selfdual YM.

Review of $\mathcal{N}=4$ self-dual super-Yang-Mills:

For $\mathcal{N}<4$, self-dual super-YM contains ½ the states of ordinary super-YM.

$$\underbrace{A_-,\ \chi^j_\alpha,\ \varphi^{jk}}_{\text{self-dual}} \qquad \underbrace{A_+,\ \bar{\chi}_{\dot{\alpha}j},\ \bar{\varphi}_{jk}}_{\text{anti-self-dual}}$$

But for $\mathcal{N}=4$, $\varphi^{jk} = \epsilon^{jklm}\bar{\varphi}_{lm}$ and states of self-dual YM are equal to states of ordinary super-YM. However, the interactions are different.

$$S_{\text{self-dual}} = \int d^4x\, \mathrm{Tr}\Big[G^{\alpha\beta}\sigma^{\mu\nu}_{\alpha\beta}(\partial_\mu A_\nu - \partial_\nu A_\mu + g[A_\mu, A_\nu]) + \bar{\chi}^{\dot{\alpha}}_j \nabla_{\alpha\dot{\alpha}}\chi^{\alpha j} + \nabla_{\alpha\dot{\alpha}}\varphi^{jk}\nabla^{\alpha\dot{\alpha}}\varphi^{lm}\epsilon_{jklm} + \bar{\chi}^{\dot{\alpha}}_j\bar{\chi}_{\dot{\alpha}k}\varphi^{jk}\Big]$$

$\nabla_{\alpha\dot{\alpha}} = \sigma^\mu_{\alpha\dot{\alpha}}(\partial_\mu + gA_\mu)$, $G^{\alpha\beta}$ is independent field which describes on-shell an anti-self-dual field strength.

Coupling constant g can be removed by scaling $A_\mu \to \frac{1}{g}A_\mu$ and $G^{\alpha\beta}\to g\,G^{\alpha\beta}$. Four-point and higher-point tree amplitudes all vanish.

Can treat $\mathcal{N}=4$ super-YM as perturbation of $\mathcal{N}=4$ self-dual YM by defining (Siegel)

$$S_{sYM} = S_{\text{self-dual}} + \lambda^2_{YM}\int d^4x\, \mathrm{Tr}\Big[G^{\alpha\beta}G_{\alpha\beta} + \chi^j_\alpha\chi^{\alpha k}\varphi^{lm}\epsilon_{jklm} + [\varphi,\varphi]^2\Big]$$

Using $Q = \int dz\, [\, \lambda^\alpha (\psi^{\dot\alpha} \partial x_{\alpha\dot\alpha} + c_j \partial \theta_\alpha^j) + \ldots]$, one can easily show that only states in cohomology are

$$V = \lambda^\alpha \psi^{\dot\alpha} e^{ik\cdot x} \Big(A_{\alpha\dot\alpha} + \theta_\alpha^j \bar\chi_{\dot\alpha j} + \theta_\alpha^j \theta^{\beta k} k_{\beta\dot\alpha} \varphi^{lm} \epsilon_{jklm}$$
$$+ \theta_\alpha^j \theta^{\beta k} \theta^{\gamma l} k_{\beta\dot\alpha} \chi_\gamma^m \epsilon_{jklm} + (\theta^4)_\alpha^{\beta\gamma\delta} k_{\beta\dot\alpha} G_{\gamma\delta} \Big)$$
$$+ \lambda^\alpha c^j e^{ik\cdot x} \left(\theta_\alpha^k \varphi^{lm} + \theta_\alpha^k \theta_\beta^l \chi^{\beta m} + \theta_\alpha^k \theta_\beta^l \theta_\gamma^m G^{\beta\gamma} \right) \epsilon_{jklm}$$
$$= \lambda^\alpha \psi^{\dot\alpha} A_{\alpha\dot\alpha}(x,\theta) + \lambda^\alpha c^j A_{\alpha j}(x,\theta)$$

where $[A_{\alpha\dot\alpha}, A_{\alpha j}]$ are $\mathcal{N}=4$ self-dual YM gauge superfields.

Integrated vertex operators $\int dz\, W$ are defined (as usual) by $QW = \partial V$.

Tree-level scattering amplitudes are defined by

$$\mathcal{A} = \langle V_1\, V_2\, V_3 \int dz_4\, W_4 \ldots \int dz_N\, W_N \rangle \quad \text{where}$$

$$\langle \lambda^\alpha \lambda^\beta \lambda^\gamma\, \psi^{\dot\alpha} \psi_{\dot\alpha}\, \theta_\alpha^j \theta_\beta^k \theta_\gamma^l c^m \epsilon_{jklm} \rangle = 1.$$

Normalization is related to $\langle (\lambda\gamma^\mu\theta)(\lambda\gamma^\nu\theta)(\lambda\gamma^\rho\theta)(\theta\gamma_{\mu\nu\rho}\theta) \rangle = 1$ by dimensional reduction where $\lambda^{\alpha j} \equiv \lambda^\alpha c^j$, $\psi^{\dot\alpha} = c^j \theta_j^{\dot\alpha}$.

This formalism can be interpreted as covariant quantization of a "self-dual Green-Schwarz superstring" proposed by Siegel in 1992. The target-space theory has $\mathcal{N}=4$ $d=4$ superconf. inv. described by generators

$$(P_{\alpha\dot{\alpha}}, M_{\alpha\beta}, M_{\dot{\alpha}\dot{\beta}}, K_{\alpha\dot{\alpha}}, D, R^j_k \,;\, q_{\alpha j}, \bar{q}^j_{\dot{\alpha}}, s^j_\alpha, \bar{s}_{\dot{\alpha} j}).$$

Of these 30 + 32 symmetries, 16 + 16 are manifest:

$$P_{\alpha\dot{\alpha}} = \int \partial x_{\alpha\dot{\alpha}} \quad , q_{\alpha j} = \int \partial \theta^j_\alpha \quad \to P_{\alpha J} = \int \partial Y_{\alpha J}$$

$$\left.\begin{aligned} \bar{q}^j_{\dot{\alpha}} + \bar{s}_{\dot{\alpha} j} &= \int (\theta^{\alpha j} \partial x_{\alpha\dot{\alpha}} + c^j \eta_{\dot{\alpha}} + b^j \psi_{\dot{\alpha}}) \\ M_{\dot{\alpha}\dot{\beta}} &= \int (x_{\alpha\dot{\alpha}} \partial x_{\beta\dot{\beta}} \varepsilon^{\alpha\beta} + \eta_{(\dot{\alpha}} \psi_{\dot{\beta})}) \\ R_{jk} &= \int (\theta_{\alpha j} \partial \theta_{\beta k} \varepsilon^{\alpha\beta} + b_{[j} c_{k]}) \end{aligned}\right\} R_{JK} = \int (Y_{\alpha J} \partial Y_{\beta K} \varepsilon^{\alpha\beta} + B_{[J} C_{K]})$$

$$M_{\alpha\beta} = \int (x_{\alpha\dot{\alpha}} \partial x_{\beta\dot{\beta}} \varepsilon^{\dot{\alpha}\dot{\beta}} + \theta^j_\alpha \partial \theta^j_\beta + \omega_{(\alpha} \lambda_{\beta)}) = \int (Y_{\alpha J} \partial Y^J_\beta + \omega_{(\alpha} \lambda_{\beta)})$$

$J = (j, \dot{\alpha})$ is an $OSp(4|2)$ index

$Y_{\alpha J} = (\theta_{\alpha j}, x_{\alpha\dot{\alpha}})$, $C_J = (c_j, \psi_{\dot{\alpha}})$, $B^J = (b^j, \eta^{\dot{\alpha}})$

$$S = \frac{1}{\alpha'} \int d^2 z \, (\partial Y_{\alpha J} \bar{\partial} Y^{\alpha J} + B^J \bar{\partial} C_J + \omega^\alpha \bar{\partial} \lambda_\alpha + \hat{B}^J \partial \hat{C}_J + \hat{\omega}^\alpha \partial \hat{\lambda}_\alpha)$$

To relate to pure spinor formalism, define $\lambda^{\alpha J} = \lambda^\alpha C^J$ where λ^α is projective and $C^J C^J = c^j c^j + \psi^{\dot{\alpha}} \psi_{\dot{\alpha}} \equiv 0$.

Then $Q = \int [\lambda^\alpha (\psi^{\dot{\alpha}} \partial x_{\alpha\dot{\alpha}} + c_j \partial \theta^j_\alpha) + \ldots] \to \boxed{Q = \int \lambda^{\alpha J} \partial Y_{\alpha J}}$

String theory conjecture for ordinary $\mathcal{N}=4$ super-YM

One can deform $\mathcal{N}=4$ self-dual YM into ordinary $\mathcal{N}=4$ sYM where α and $\dot\alpha$ indices are treated equally. Is there some deformation of the open string theory that treats α and $\dot\alpha$ indices on equal footing? Yes!

In this deformed model, the 16+16 manifest symmetries get deformed to $OSp(4|4)$ with generators $(P_{\alpha\dot\alpha}+K_{\alpha\dot\alpha}, M_{\alpha\beta}, M_{\dot\alpha\dot\beta}, R_{jk}; q_{\alpha j}+s_\alpha^j, \bar q_{\dot\alpha}^j+\bar s_{\dot\alpha j})$.
Different contractions of this group lead to either the previous model for $\mathcal{N}=4$ self-dual YM or an analogous model for $\mathcal{N}=4$ anti-self-dual YM.

Since $[P^\mu+K^\mu, P^\nu+K^\nu]\neq 0$, this model is no longer free and describes a string theory on AdS_4. But if states in the BRST cohomology preserve spacetime conformal invariance, the target-space theory will be independent of the AdS_4 radius and one can perform any calculation at infinitely large radius.

Action is an $\dfrac{OSp(4|4)}{OSp(4|2)\times SU(2)}$ sigma model:

$$S = \int d^2z \Big[J^{\alpha J}\bar{J}_{\alpha J} + B^J \bar{\nabla} C_J + w^\alpha \bar{\nabla}\lambda_\alpha + \hat{B}^J \nabla \hat{C}_J + \hat{w}^\alpha \nabla \hat{\lambda}_\alpha + (B^{[J}C^{K]})(\hat{B}_{[J}\hat{C}_{K]}) + (w^{(\alpha}\lambda^{\beta)})(\hat{w}_{(\alpha}\hat{\lambda}_{\beta)}) \Big]$$

$J = (j, \dot{\alpha}) = OSp(4|2)$ index, $\alpha = SU(2)$ index

$$\bar{\nabla} C_J = \bar{\partial} C_J + \bar{J}_{[JK]} C^K$$

$$\bar{\nabla}\lambda^\alpha = \bar{\partial}\lambda^\alpha + \bar{J}^{(\alpha\beta)}\lambda_\beta$$

$$J^A(x,\theta) = (g^{-1}\partial g)^A \qquad A = (\alpha J, [JK], (\alpha\beta))$$

$$\bar{J}^A(x,\theta) = (g^{-1}\bar{\partial} g)^A \qquad g(x,\theta) \in \frac{OSp(4|4)}{OSp(4|2)\times SU(2)}$$

$$Q = \int dz\, \lambda^\alpha C^J J_{\alpha J}(x,\theta)$$

$$\hat{Q} = \int d\bar{z}\, \hat{\lambda}^\alpha \bar{C}^J \bar{J}_{\alpha J}(x,\theta)$$

where $\lambda^\alpha, \hat{\lambda}^\alpha$ are projective and $C^J C^J = \hat{C}^J \hat{C}^J = 0$

Model closely resembles pure spinor $AdS_5\times S^5$ model.

Preliminary computations at large radius suggests this model has correct spectrum and interactions of ordinary $\mathcal{N}=4$ super-Yang-Mills.

Conclusions and speculations

- Have constructed a free open string theory which describes $\mathcal{N}=4$ $d=4$ self-dual YM.
- Have conjectured an open string theory on AdS_4 which describes ordinary $\mathcal{N}=4$ $d=4$ super-YM.

Could this open string theory on AdS_4 be related to pure spinor closed string theory on $AdS_5 \times S^5$?

Compare with Chern-Simons / conifold duality:

Chern-Simons on $S^3 \rightarrow \mathcal{N}=4$ super-YM on AdS_4

S^2 of conifold $\rightarrow S^5$ of Maldacena conjecture

In Gopakumar-Vafa duality, S^2 of conifold comes from T^*S^3 of closed string which vanishes on D-brane

Where could S^5 come from in AdS_4 model?

In AdS_4 model, scalars φ^{jk} couple to ghost currents $c_{[j} b_{k]}$ which act like six internal momenta.

Perhaps S^5 could come from the six bosonic modes $c_{[j} b_{k]} - \hat{c}_{[j} \hat{b}_{k]}$ which vanish on the D-brane?