TWO - LOOP SUPERSTRINGS

joint work with D.H. Phong

Kyoto - Strings 2003

- **Tree level and one-loop well known**
  (Type II 1982, heterotic 1985)

- **Multiloops**
  - odd supermoduli play non-trivial & delicate role
  - formal solution via supermoduli space

- **Two-Loops** (even spin structures)
  - concrete solution of the supermoduli problem
  - concrete formulas for measure and amplitudes

- **Further motivations**
  - M-theory Corrections to 11-D SUGRA (cfr Green Gutperle Vanhove)
  - Non-Abelian orbifold compactifications (cfr Kachru & Silverstein)

RNS SUPERSPACE GAUGE FIXING

- **Worldsheet Data**

$$X^\mu \leftrightarrow (x^\mu, \psi^\mu_\pm)$$
$$(B,\ C) \leftrightarrow \text{Diff}\ (b,\ c) \times \text{SUSY}\ (\beta,\ \gamma)$$
$$({E_M}^A, \Omega_M) \leftrightarrow (g_{mn}, {\chi_m}^\alpha)$$

- **Gauge fixed amplitudes** (E & H Verlinde '87, ED & Phong '88)

$$\mathcal{A}_\mathcal{O} = \int_{s\mathcal{M}} |dm^A|^2 \int D(XBC)\ \Big|\prod_A \delta(\langle H_A|B\rangle)\Big|^2\ \mathcal{O}\ e^{-I}$$

  - Matter + superghost action

$$I \equiv \frac{1}{2\pi}\int dE\ (\frac{1}{2}\mathcal{D}_+X^\mu\mathcal{D}_-X_\mu + B\mathcal{D}_-C + \bar{B}\mathcal{D}_+\bar{C})$$

  - Super-Beltrami differential $\quad H_A = E^{-1}\partial E/\partial m^A$

- **Supermoduli Space**

$$s\mathcal{M}_h \equiv \{{E_M}^A, \Omega_M\}/\text{local symmetries}$$

$$\dim(s\mathcal{M}_h) = \begin{cases} (0|0) & h=0 \\ (1|0)_e \text{ or } (1|1)_o & h=1 \\ (3h-3|2h-2) & h\geq 2 \end{cases}$$

CHIRAL SPLITTING (ED & Phong '86 - '89)

- **Required to separate left and right movers**
  - Type II & heterotic : independent left/right movers
  - Obstructions eg $\chi\bar{\chi}\psi_+^\mu\psi_-^\mu$ term; x^μ - zero mode
  - Criterion : Superholomorphicity on supermoduli space

- **Full Amplitude = integral over internal loop momenta**

$$\mathbf{A}_{\mathcal{O}}[\delta] = \int |dm^A|^2 \int dp_I \left| \mathcal{A}_{\mathcal{O}}[\delta] \exp\{i\pi p_I^\mu \hat{\Omega}_{IJ} p_J^\mu\} \right|^2$$

  - $\hat{\Omega}_{IJ}$ is the superperiod matrix (more later)

- **Chiral $\mathcal{A}_{\mathcal{O}}[\delta]$ is superholomorphic in supermoduli**

$$\mathcal{A}_{\mathcal{O}}[\delta] = \langle \prod_A \delta(\langle H_A | B \rangle)\ \mathcal{O}_+\ \exp\{\int \chi S\} \rangle$$

  - $\mathcal{O}_+$ is the chirally split version of $\mathcal{O}$
  - $S = -\frac{1}{2}\psi_+^\mu \partial x_+^\mu + \frac{1}{2}b\gamma - \frac{3}{2}\beta\partial c - (\partial\beta)c$

INTEGRATING OUT ODD SUPERMODULI

- **Key problem of superstring perturbation theory**
  - Well-defined measure on supermoduli space available
  - Can one pass to consistent measure on moduli space ?

- **A naive derivation of the chiral measure** (cfr E & H Verlinde '87)
  - Simple assumption on gauge slice $m^A = (m^a, \zeta^\alpha)$

$$g_{mn}(m^a) \qquad \chi(z) = \sum_{\alpha=1,2} \zeta^\alpha \delta(z - z_\alpha)$$

  - Integrate odd supermoduli ζ^α

- **Yields the Picture Changing Ansatz** (Friedan Martinec Shenker '85)

$$\text{chiral measure} \sim \langle \mathcal{O} \prod_{a=1}^{3h-3} (\mu_a | b) \prod_{\alpha=1}^{2h-2} Y(z_\alpha) \rangle \prod_{a=1}^{3h-3} dm^a$$

  - Y = picture changing op, z_α supposed arbitrary
  - BUT : calculation exhibits dependence (E & H Verlinde '87)
    - * Interpreted as ambiguity in measure (Attick Rabin Sen 87)
    - * Possible remedy via Cech cohomology (H. Verlinde 87)

MAIN PROPOSAL (ED & Phong '88 & '01)

- **Integrating out odd supermoduli** = PROJECTION
- **Inconsistent Projection** produces ambiguities

$$\begin{array}{ccccl} (g_{mn}, \chi_m) & \sim & (g'_{mn}, \chi'_m) & & \text{under SUSY} \\ \downarrow & & \downarrow & & \\ g_{mn} & \not\sim & g'_{mn} & & \text{under Diff} \times \text{Weyl} \end{array}$$

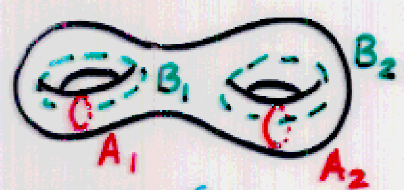
- **Consistent Projection** compatible with local SUSY

$$\begin{array}{ccccl} (g_{mn}, \chi_m) & \sim & (g'_{mn}, \chi'_m) & & \text{under SUSY} \\ \downarrow & & \downarrow & & \\ \hat g_{mn}(m^a) & \sim & \hat g'_{mn}(m^a) & & \text{under Diff} \times \text{Weyl} \end{array}$$

  - moduli m^a must be invariant under local SUSY

- **The Super-Period Matrix** $\hat\Omega$ (even spin structure & genus 2)
  - superholó 1/2-forms $\hat\omega_I$, $\mathcal{D}_-\hat\omega_I = 0$, $I = 1, 2$

$$\oint_{A_I} \hat\omega_J = \delta_{IJ} \qquad \oint_{B_I} \hat\omega_J = \hat\Omega_{IJ}$$



  - Every symmetric Ω_{IJ}, $\mathrm{Im}\Omega > 0 \sim$ Riemann surface
  - Projection via $\hat\Omega$ supersymmetric and well-defined

CALCULATION OF CHIRAL MEASURE

- **Supersymmetric supermoduli** $m^A = (\hat{\Omega}_{IJ}, \zeta^\alpha)$
- **Deformations of complex structures**

$$\Omega \to \hat{\Omega} \quad \left\{ \begin{array}{ccc} g & \to & \hat{g} = g + \hat{\mu} \\ \partial_{\bar{z}} & \to & \hat{\partial}_{\bar{z}} = \partial_{\bar{z}} + \hat{\mu}\partial_z \\ \langle \cdots \rangle(g) & = & \langle \cdots \rangle(\hat{g}) + \int \hat{\mu} \langle T \cdots \rangle(\hat{g}) \end{array} \right.$$

- **Assembling all contributions**

$$\mathcal{A}[\delta] = \frac{\langle \prod_a b(p_a) \prod_\alpha \delta(\beta(q_\alpha)) \rangle}{\det \Phi_{IJ+}(p_a) \det \langle H_\alpha | \Phi^*_\beta \rangle} \left\{ 1 + \int \hat{\mu} \langle T \rangle + \iint \chi\chi \langle SS \rangle \right\}$$

  - Φ_C normalized basis of even/odd superholó 3/2 forms
  - Familiar contribution $\int\int \chi\chi\langle SS\rangle$
  - Novel $\int \hat{\mu}\langle T\rangle$ term due to $\Omega(g) \to \hat{\Omega}(\hat{g})$
  - Novel finite-dimensional determinants

- **Consistency check by explicit calculation** : $\mathcal{A}[\delta]$
  - independent of choices of $\hat{\mu}$, χ_α and points p_a, q_α

MEROMORPHIC FORM OF THE CHIRAL MEASURE

– Choose χ_α supported at points $x_\alpha, \quad \alpha = 1, 2$

$$\mathcal{A}[\delta] = \frac{\langle \prod_a b(p_a) \prod_\alpha \delta(\beta(q_\alpha))\rangle}{\det\omega_I\omega_J(p_a)\cdot \det\psi^*_\beta(x_\alpha)}\left\{1 + \frac{\zeta^1\zeta^2}{16\pi^2}\sum_{i=1}^{6}\mathcal{X}_i\right\}$$

$$\begin{aligned}
\mathcal{X}_1 = \; & -10 S_\delta(x_1,x_2)\partial_{x_1}\partial_{x_2}\ln E(x_1,x_2) \\
& -3\partial_{x_2}G_2(x_1,x_2)G_{3/2}(x_2,x_1) - (1\leftrightarrow 2) \\
& -2G_2(x_1,x_2)\partial_{x_2}G_{3/2}(x_2,x_1) - (1\leftrightarrow 2)
\end{aligned}$$

$$\mathcal{X}_2 = S_\delta(x_1,x_2)\omega_I(x_1)\omega_J(x_2)\partial_I\partial_J \ln\frac{\vartheta[\delta](0)^5\vartheta(D_b)}{\vartheta[\delta](D_\beta)}$$

$$\mathcal{X}_3 = 2S_\delta(x_1,x_2)\sum_a \varpi_a(x_1,x_2)\Big[B_2(p_a)+B_{3/2}(p_a)\Big]$$

$$\mathcal{X}_4 = 2S_\delta(x_1,x_2)\sum_a \partial_{p_a}\partial_{x_1}\ln E(p_a,x_1)\varpi_a(p_a,x_2) - (1\leftrightarrow 2)$$

$$\mathcal{X}_5 = \sum_a S_\delta(p_a,x_1)\partial_{p_a}S_\delta(p_a,x_2)\varpi_a(x_1,x_2) - (1\leftrightarrow 2)$$

$$\begin{aligned}
\mathcal{X}_6 = \; & 3\partial_{x_2}G_2(x_1,x_2)G_{3/2}(x_2,x_1) - (1\leftrightarrow 2) \\
& +2G_{3/2}(x_2,x_1)G_2(x_1,x_2)\partial\bar{\psi}_2(x_2) - (1\leftrightarrow 2) \\
& +2f_{3/2}(x_1)G_2(x_1,x_2)\partial\bar{\psi}_1(x_2) - (1\leftrightarrow 2) \\
& +\partial_{x_2}G_2(x_2,x_1)\partial\bar{\psi}_2(x_1) - (1\leftrightarrow 2)
\end{aligned}$$

EXPLICIT FORMULA VIA ϑ-CONSTANTS

- **Final formula for chiral measure** ($\hat\Omega$ henceforth denoted Ω)

$$d\mu[\delta](\Omega) = \frac{\Xi_6[\delta](\Omega)\ \vartheta[\delta]^4(0,\Omega)}{16\pi^6\ \Psi_{10}(\Omega)}\ d^3\Omega_{IJ}$$

- Unique modular form of weight 10 (cfr Igusa)

$$\Psi_{10}(\Omega) \equiv \prod_{\delta \text{ even}} \vartheta[\delta]^2(0,\Omega)$$

- **A new modular quantity**

$$\Xi_6[\delta](\Omega) \equiv \sum_{1\leq i<j\leq 3} \langle \nu_i|\nu_j\rangle \prod_{k=4,5,6} \vartheta[\nu_i+\nu_j+\nu_k]^4(0,\Omega)$$

  - $\delta = \nu_1+\nu_2+\nu_3,\ \nu_i$ distinct odd spin structures
  - *Signature* $\langle\kappa|\lambda\rangle \equiv \exp 4\pi i(\kappa'\lambda'' - \kappa''\lambda') \ \in \{\pm 1\}$

- **All degeneration limits agree with physical factorization**

MODULAR PROPERTIES – GSO PHASES

$$\begin{aligned} \tilde{\Omega} &= (A\Omega + B)(C\Omega + D)^{-1} \\ d\mu[\tilde{\delta}](\tilde{\Omega}) &= \det(C\Omega + D)^{-5} d\mu[\delta](\Omega), \end{aligned}$$

- Weight 5 matches transf of 10 internal loop momenta
- Unit phases $\Rightarrow$ Unique GSO projection phase factors

$$\sum_\delta d\mu[\delta](\Omega)$$

VANISHING OF COSMOLOGICAL CONSTANT

and 1-, 2- and 3-point functions

$$\sum_\delta \Xi_6[\delta](\Omega)\vartheta[\delta]^4(0,\Omega) = 0$$

$$\sum_\delta \Xi_6[\delta](\Omega)\vartheta[\delta]^4(0,\Omega)\ S_\delta(z_1,z_2)^2 = 0$$

$$\sum_\delta \Xi_6[\delta](\Omega)\vartheta[\delta]^4(0,\Omega)\ S_\delta(z_1,z_2)S_\delta(z_2,z_3)S_\delta(z_3,z_1) = 0$$

- Not a result of Riemann identities alone
- Instead shown using Fay's trisecant identity
- Hyperelliptic formulation of genus 2 (see also Zheng, Wu, Zhu 2002)

THE 4-POINT FUNCTION

- **Involves** $\langle S(z)S(w)\prod_i V_i\rangle$ & $\langle T(z)\prod_i V_i\rangle$

- **Disconnected parts** combine with gauge fixing + ghost

$$= g_s^2\delta(k)\int\frac{|d^3\Omega|^2}{(\det\,\mathrm{Im}\Omega)^5}\iiiint\mathcal{F}_L\overline{\mathcal{F}_R}\exp\left\{-\sum_{i<j}k_i\cdot k_j G(z_i,z_j)\right\}$$

– Chiral partial amplitude (Heterotic : $\overline{\mathcal{F}_R}$ chirally split bosonic)

$$\mathcal{F} = C_S\ S(ijkl) + C_T\ T(ij|kl)$$

– Both S and T holomorphic 1-forms in each z_i, $i=1,\cdots,4$

– Kinematical factors (using $f_i^{\mu\nu}\equiv\epsilon_i^\mu k_i^\nu-\epsilon_i^\nu k_i^\mu$)

  * OLD : $C_S = t_8 f_i f_j f_k f_l$ (cfr 0 & 1 loop)

  * NEW : $C_T = f_{[i}^{\mu\nu}f_{j]}^{\rho\sigma}f_k^{\nu\mu}f_l^{\sigma\rho} + 2f_{[i}^{\mu\nu}f_{j]}^{\nu\sigma}f_k^{\sigma\rho}f_l^{\rho\mu}$

- **Connected parts** are being completed

- **Convergence** (Disconnected part only)

  The z_i and Ω integrations are finite for suitable k_i.

- **Hyperelliptic formulation** Zheng, Wu, Zhu (2002); Wu, Zhu (2003)

COMPACTIFICATIONS

- **Simple Assumptions**

  (1) Compactification modifies matter, not ghosts
  (2) Worldsheet N=1 local supersymmetry maintained
  (3) Z_M, Z_C chiral partition functions on M, C

- **Chiral measure independent of gauge choices**

$$d\mu_C[\delta] = \frac{Z_C}{Z_M}\left\{\frac{\Xi_6[\delta]\vartheta[\delta]^4}{4\Psi_{10}} - \mathcal{Z}\langle S_C(q_1)S_C(q_2)\rangle\right\}\frac{d^3\Omega}{4\pi^6}$$

  - S_C = supercurrent of compactified theory
  - $\mathcal{Z} = Z_M \times$ ghost partition function
  - Presented here for $S_\delta(q_1, q_2) = 0$ gauge

- **Role of genus ≥ 2 in orbifold compactification**

$$\begin{array}{rcl} \gamma \in \pi_1(\Sigma) & \longrightarrow & \rho(\gamma) \in G \quad \text{orbifold group} \\ x(z+\gamma) & = & \rho(\gamma)x(z) \end{array}$$

  - Σ = torus : only commuting pairs $\rho(A)$, $\rho(B)$ contribute
  - Σ = genus ≥ 2 : non-commuting pairs also contribute (since $\pi_1(\Sigma)$ is then non-Abelian)

KACHRU SILVERSTEIN ORBIFOLDS

- Broken SUSY & Vanishing Cosmological Constant ?
- **Asymmetric $Z_2 \times Z_2$ orbifold of Type II**
  - on $\mathbf{R}^4 \times \mathbf{T}^6/\mathbf{G}$, self-dual radius; square torus
  - $\mathbf{G}$ generated by f, g $\qquad (\Lambda_L; \Lambda_R; v_L; v_R; \theta)$

$$
\begin{aligned}
f &= \left[(-1^{1-4}, 1^{56}); I; (0^{1-4}, v_L^{56}); (s^{1-4}, v_R^{56}); (-)^{F_R}\right] \\
g &= \left[I; (-1^{1-4}, 1^{56}); (s^{1-4}, w_L^{56}); (0^{1-4}, w_R^{56}); (-)^{F_L}\right]
\end{aligned}
$$

  - $s =$ shift by πR; $v_{L,R}$ and $w_{L,R}$ obey level matching
- **KKS conjecture**
  - vanishing cosmological constant in perturbation theory

    a) automatic to 1-loop order
    b) KKS arguments FOR 2-loop vanishing
    c) arguments AGAINST 2-loop vanishing (Iengo 1999)

  - Tools now available to settle the issue

$\mathbf{Z}_2$ TWISTING AT 2 LOOPS (Aoki, ED, Phong 2003)

- **$\mathbf{Z}_2$ twist described by 1/2 characteristic ε,**

$$x(z+2\varepsilon) = -x(z)+v \qquad v \in T^6$$
$$x(z+\gamma) = +x(z)+v \qquad \gamma \perp 2\epsilon,\ \gamma \in H^1(\Sigma, \mathbf{Z}),$$

- **Twisting freezes 1 internal loop momentum**

$$p^\mu_\gamma = \oint_\gamma \frac{dz}{2\pi}\partial_z x^\mu$$

  - $\partial_z x^\mu$ involves Prym diff $\nu_\varepsilon(z+2\varepsilon) = -\nu_\varepsilon(z)$
  - $\oint_{A_\varepsilon} \nu_\varepsilon = 1$, and $\oint_{B_\varepsilon} \nu_\varepsilon = \tau$, Prym period

- **Twisted Propagators**

fermion $\quad \langle \psi_+(z)\psi_+(w)\rangle = S_{\delta+\varepsilon}(z,w)$

boson $\quad \langle \partial x_+(z)\partial x_+(w)\rangle \sim S_{\delta_i^+}(z,w)S_{\delta_i^-}(z,w)$

  - $\delta_i^+ + \delta_i^- = \varepsilon$ indexed by genus 1 even μ_i $i = 1,2,3$

- **Partition functions** (4 twisted dims) (Dijkgraaf, E & H Verlinde 1988)

$$\frac{Z_C}{Z_M} = \frac{\vartheta[\delta+\varepsilon](0,\Omega)^2}{\vartheta[\delta](0,\Omega)^2}\,\frac{\vartheta[\delta_i^+]^2\vartheta[\delta_i^-](0,\Omega)^2}{\vartheta[\mu_i](0,\tau)^4}$$

- **Chiral Measure for twisted chiral blocks**

$$d\mu[\delta,\varepsilon,p] = \frac{Z_C}{Z_M}\left\{\frac{\Xi_6[\delta]\vartheta[\delta]^4}{16\pi^2\Psi_{10}} + \Gamma[\delta,\varepsilon]\Big(i\pi p^2 - 4\partial_\tau \ln\vartheta[\mu_i](0,\tau)\Big)\right\}$$

- **Chiral Splitting and Z_2 twistings**
  - yields the non-trivial correction proportional to p^2
    (due to precise relation between $\hat\tau$ and $\hat\Omega$)
  - Explicit calculation yields

$$\frac{Z_C}{Z_M}\Gamma[\delta,\varepsilon] = \pm i\langle\mu_i|\nu_0\rangle\frac{\vartheta[\mu_i](0,\tau)^4}{(2\pi)^7\eta(\tau)^{12}}$$

- **Application to KKS model**
  - Assume chiral GSO sum over δ, for fixed $p,\varepsilon_f,\varepsilon_g$

$$d\mu_{\rm GSO}[\varepsilon_f,\varepsilon_g,p] = \sum_\delta \eta_\delta(\varepsilon_g)d\mu[\delta,\varepsilon_f,p]$$

  - Modular covariant choices

    (1) $\eta_\delta(\varepsilon_g)=1$ Type II again : $d\mu_{\rm GSO}=0$
    (2) $\eta_\delta(\varepsilon_g)=\langle\varepsilon_g|\delta\rangle$ KKS model : $d\mu_{\rm GSO}\neq 0$

  - KKS cosmological constant does not vanish pointwise

- **Chiral Measure for twisted chiral blocks**

$$d\mu[\delta,\varepsilon,p] = \frac{Z_C}{Z_M}\left\{\frac{\Xi_6[\delta]\vartheta[\delta]^4}{16\pi^2\Psi_{10}} + \Gamma[\delta,\varepsilon]\left(i\pi p^2 - 4\partial_\tau \ln\vartheta[\mu_i](0,\tau)\right)\right\}$$

- **Chiral Splitting and Z_2 twistings**
  - yields the non-trivial correction proportional to p^2
    (due to precise relation between $\hat{\tau}$ and $\hat{\Omega}$)
  - Explicit calculation yields

$$\frac{Z_C}{Z_M}\Gamma[\delta,\varepsilon] = \pm i\langle\mu_i|\nu_0\rangle\frac{\vartheta[\mu_i](0,\tau)^4}{(2\pi)^7\eta(\tau)^{12}}$$

- **Application to KKS model**
  - in progress