Homological Mirror symmetry

and

Asymptotic expansion of Special function

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See

Multivalued Morse theory, Asymptotic Analysis and Mirror symmetry

(http://www.math.kyoto-u.ac.jp/~fukaya/)

Homological mirror symmetry conjecture
(Kontsevich around '93)

(M^{2n}, ω) symplectic mfd.	$(M^\vee, J)$ complex mfd
$L^n \subset M^{2n}$ Lagrangian sub. mfd	$\mathcal{E}(L) \to M^\vee$ holomorphic vector bundle (coherent sheaf)
$p \in L_1 \cap L_2$	$S_p : \mathcal{E}(L_1) \to \mathcal{E}(L_2)$ homomorphism (holomorphic)
(HF Floer hom.	Ext extension)

$L \subset M$

$M^\vee$

$\Downarrow$

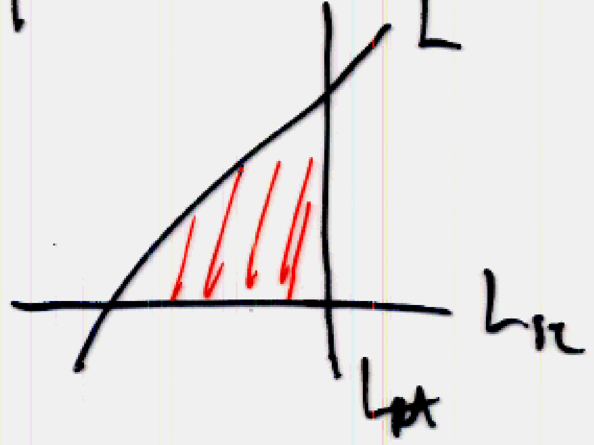
$L_{pt} \longleftrightarrow$ point R

$\phantom{L_{pt} \longleftrightarrow}$ $\in \mathcal{E}(L_{pt})$

$L_{st} \longleftrightarrow$ trivial bundle $\mathbb{C} \times M^\vee \to M^\vee$

$p \in L \cap L_{st} \longleftrightarrow s_p$ section of $\mathcal{E}(L)$

$$\sum_{\varphi} \pm e^{-2\pi \int \varphi^* \omega / g} = s_p(R)$$



$\varphi : D^2 \to M$ holo.

$\partial D \to L \cup L_{st} \cup L_{pt}$

Example ③

1) (Kontsevich '93, Polishchuk-Zaslow '98)

$M = T^2 \qquad L = S^1$

$s_p = \text{theta f.c.n.} = \sum e^{-2\pi k^2/g}$

2) (F '98, Kontsevich-Soibelman (2000))

$M = T^{2n} \qquad L = \text{flat } T^n$

$s_p = $ theta f.c.n. (in general indefinite theta)

$$= \sum_{\gamma \in \mathbb{Z}^n \cap V} e^{-2\pi Q(\gamma,\gamma)/g} \qquad Q > 0 \text{ on } V$$

Function we thus obtained so far are classical.

Are there any new special function we can find?

1) $T^2, T^{2n} \longmapsto$ K3, Calabi Yau

2) $L \subset T^{2n} \longmapsto$ non flat L

flat

Today I want to concern with (2).

($\because$ (2) is easier than (1).)

$$\begin{array}{ccc} L & \subset & T^{2n} \\ & & \downarrow \pi \\ & & T^n \end{array}$$

$x_1 \cdots x_n \, y_1 \cdots y_n$

$\omega = \sum dx_i \wedge dy_i$

$x_1 \cdots x_n$

(Strominger-Yau-Zaslow)

$\pi|_L : L \to T^n$ has a singularity.

~

Ex Σ_2 genus 2

$J(\Sigma_2)$ Jacobian $= T^4$

$$\begin{array}{ccc} \Sigma & \subset & T^4 \\ & & \downarrow \pi \\ & & T^2 \end{array}$$

after hyperkähler twist Σ is a Lagrangian submfd

$\pi|_\Sigma : \Sigma \to T^2$ branched cover

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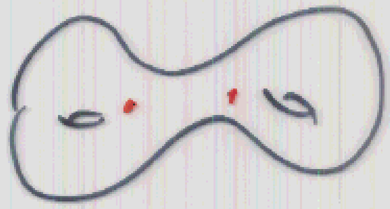
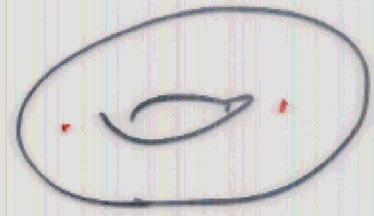




Σ $\qquad\qquad$ T^2

S : two pts $\subset T^2$

branched point

$$L \subset T^4 \qquad\qquad \mathcal{E}(L) \to T^4$$

$$\begin{array}{c} \downarrow \\ T^2 \end{array}$$

$x_1 \; x_2$

$y_1^* \; y_2^*$

$x_1 + \sqrt{-1}\, y_1^* = z_1$

$\pi|_L : L \to T^2$

branched covering

$\mathrm{rk}(\mathcal{E}(L))$

$= \# \, \pi|_L^{-1}(pt)$

$$T^2_{st} \hookrightarrow T^4$$

$y_1 = y_2 = 0$

section of $\mathcal{E}(L)$

$\Downarrow$

$$T^2_{st} \cap L \ni p \rightsquigarrow s_p(x, y^*)$$

Th'm

$$s_p(x,y^*) \sim \sum_{\gamma \in \Lambda} \pm e^{-2\pi\left(\frac{E_\gamma(x)}{\xi} + \sqrt{-1}\frac{\partial E_\gamma}{\partial x_i} y_i^*\right)}$$

Λ: set of Feynman diagram drawn on T^2 (described by ODE)

ξ : parametr of T^4_ξ (mirror tori)

(if $\xi = \varepsilon > 0$ ε = diamer of fiber)

$T^4 \supset L$ holo disk $\varphi : (D^2, \partial D) \to (T^4, L)$

$\mathcal{E}(L) \to T^4_\xi$ $\quad s_\xi \in \Gamma(\mathcal{E}(L))$

fiber → shrink

Asymptotic expansion as $\xi \to 0$

T^2

ODE on T^2

(Morse homotopy)

Witten: CS G-th as a string theory

F-Oh: Asian J '97

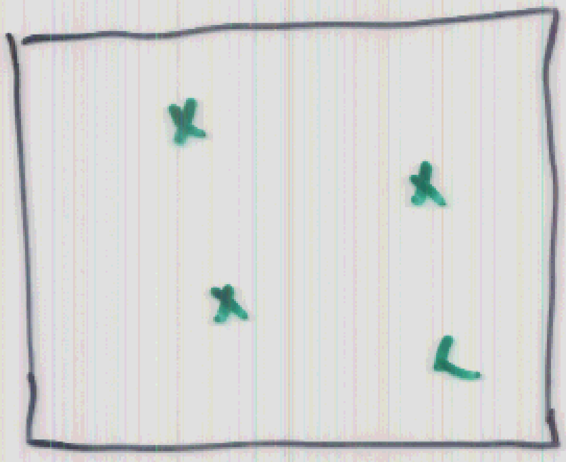
(Floer '88)

Witten S.S. & Morse theory '83

⇑

We are doing multi-valued analogue of it.

What is Λ: set of F.D. drawn on T^2.

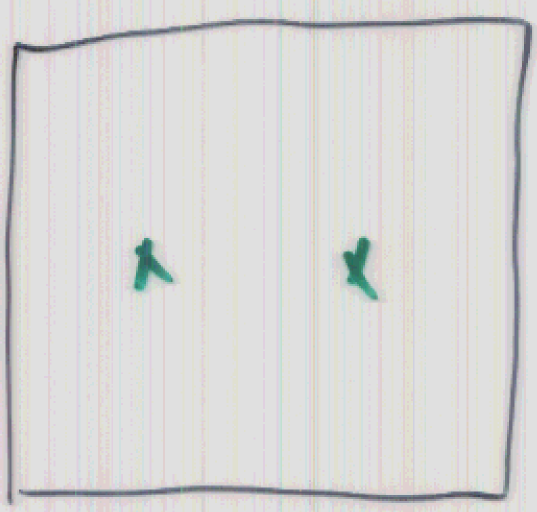


T^2

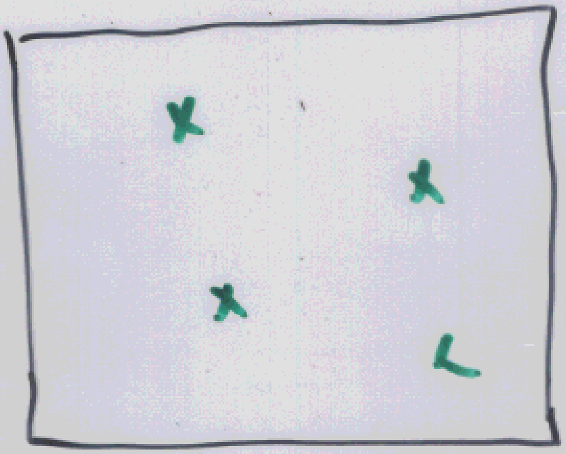
x = S

branched locus

$L \to T^2$

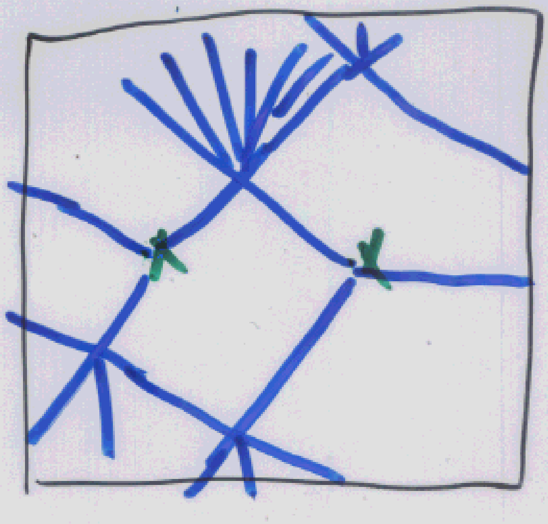


What is Λ: set of F.D. drawn on T^2.



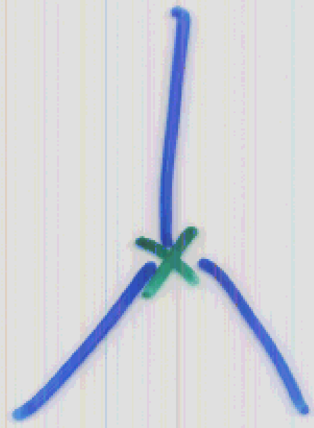
T^2

x = S branched locus $L \to T^2$



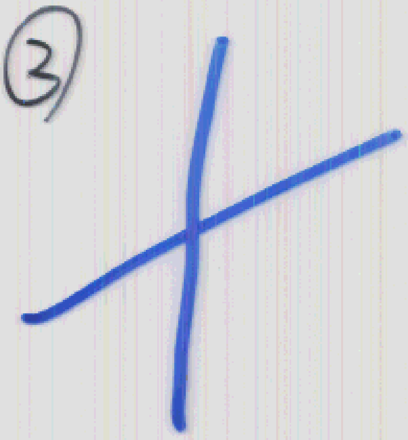
Instanton effect to connection

1 $x \in S$



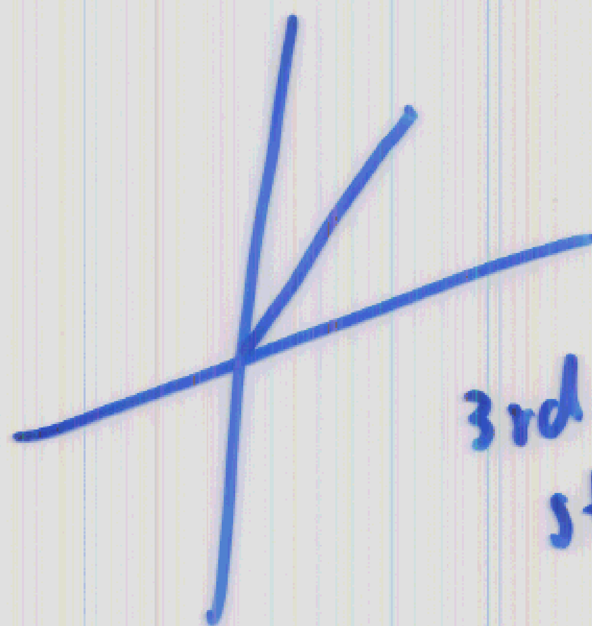
3 lines stars

3



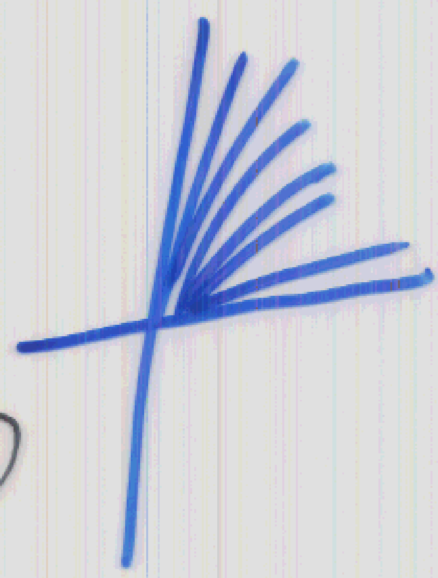
2 lines intersects

$\Rightarrow$



3rd line start

$\Rightarrow$



∞ lines starts

$$\begin{array}{c} L \subset T^4 \quad x_1\, x_2\, y_1\, y_2 \\ \downarrow \\ T^2 \end{array}$$

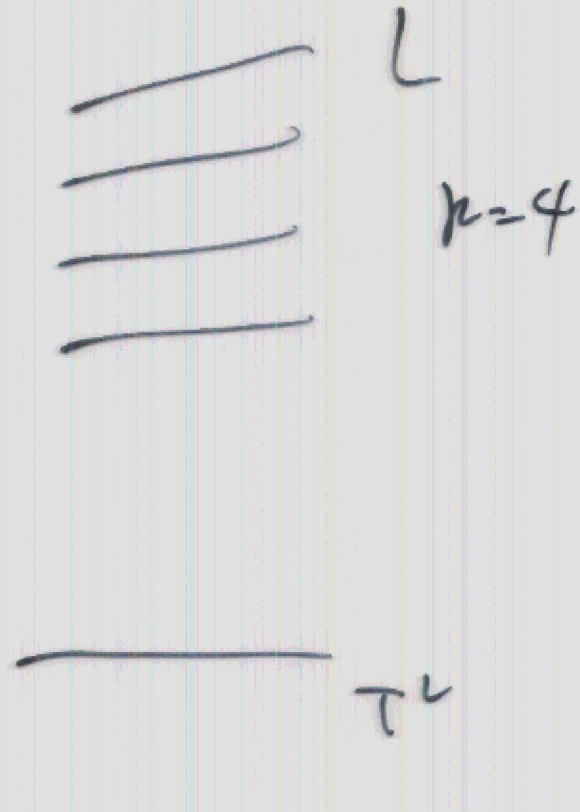
locally L is defined by

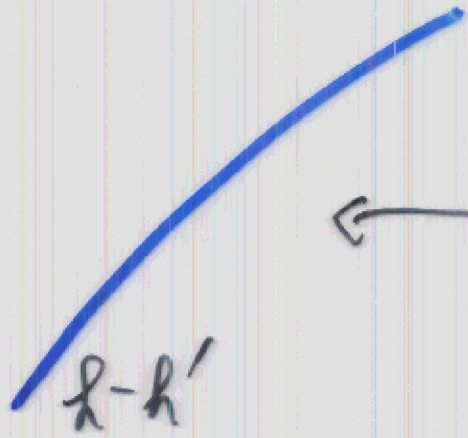
$$y_i = \frac{\partial h}{\partial x_i} \qquad h: \text{generating function}$$

h is multi valued

$\#\,(\pi|_L^{-1})(pt) = k$

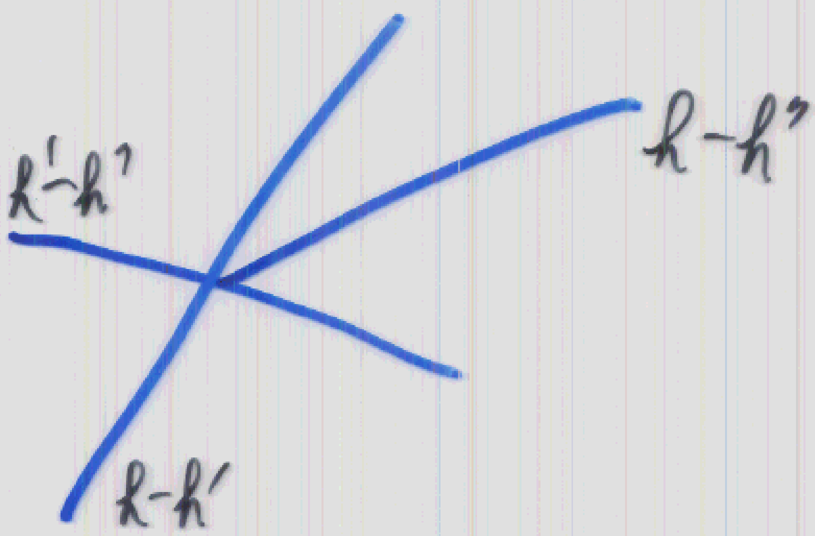
$\Rightarrow$ h is k valued



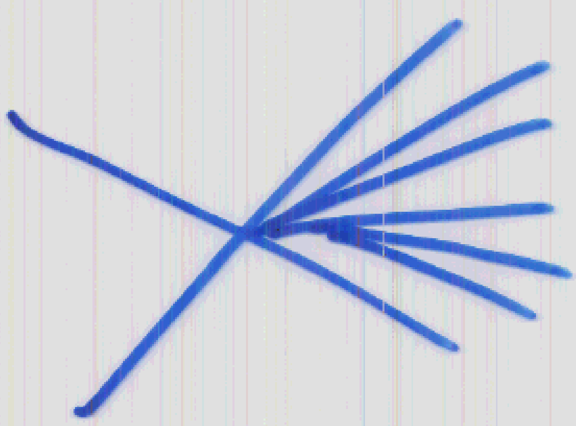


$$\frac{d\ell}{dt} = \operatorname{grad}(h-h')$$

h, h' branches of h



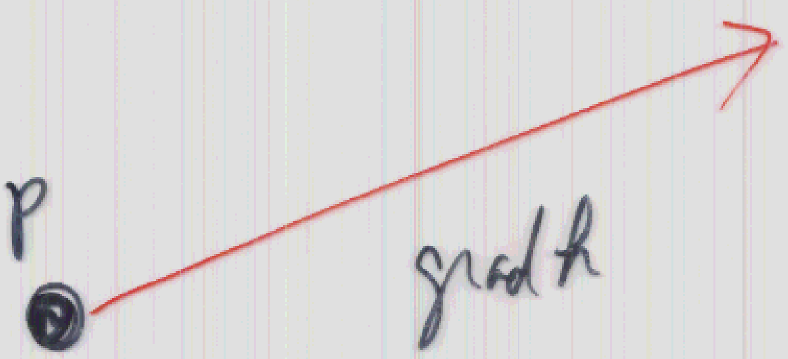
$h \neq h''$



$h = h''$

$$p \in \underset{\substack{\| \\ y = \frac{\partial h}{\partial x}}}{L} \cap \underset{\substack{\| \\ y = 0}}{T^2}$$

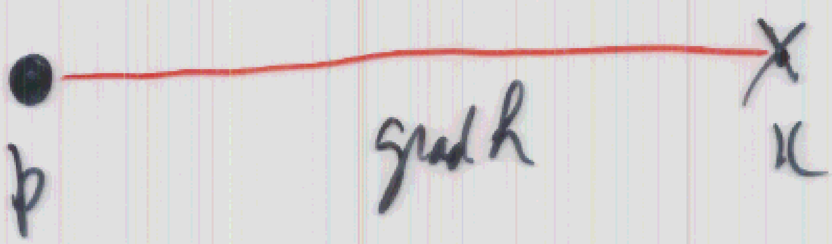
$p \longleftrightarrow$ critical point of h



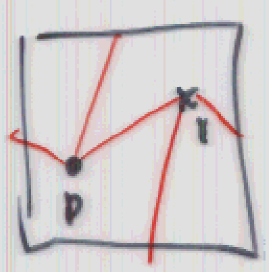
$$\frac{dl}{dt} = \operatorname{grad} h$$

$$S_p(x, y^*) = \sum_{\Lambda} \pm e^{-2\pi\left(\frac{h(x)}{\hbar} + 2\pi\sqrt{-1}\,\frac{\partial h}{\partial x} y^*\right)}$$

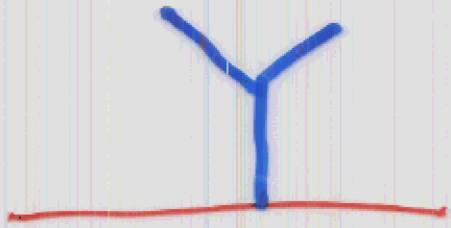
Feynman Diagram

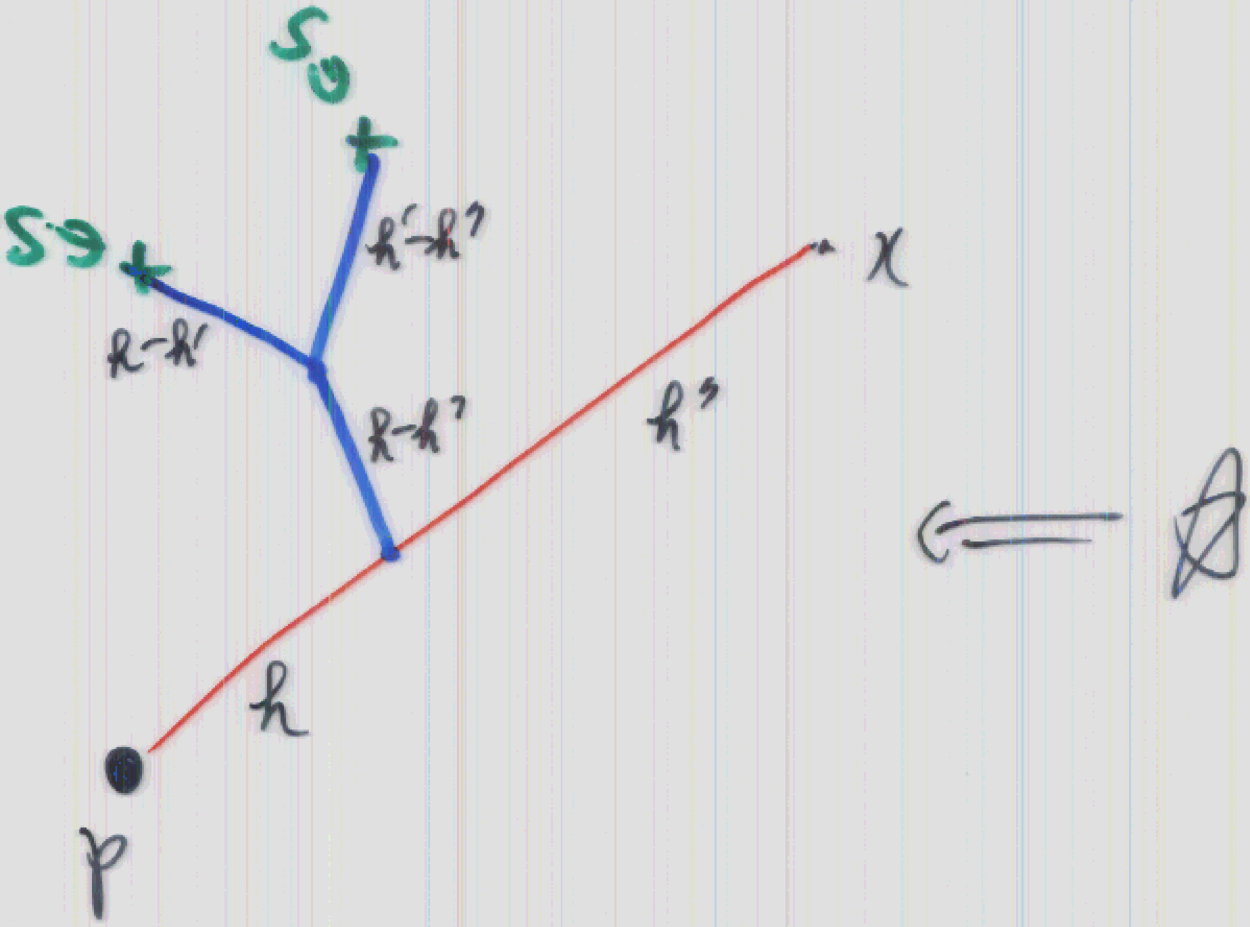


(can be many)



For general F.D.





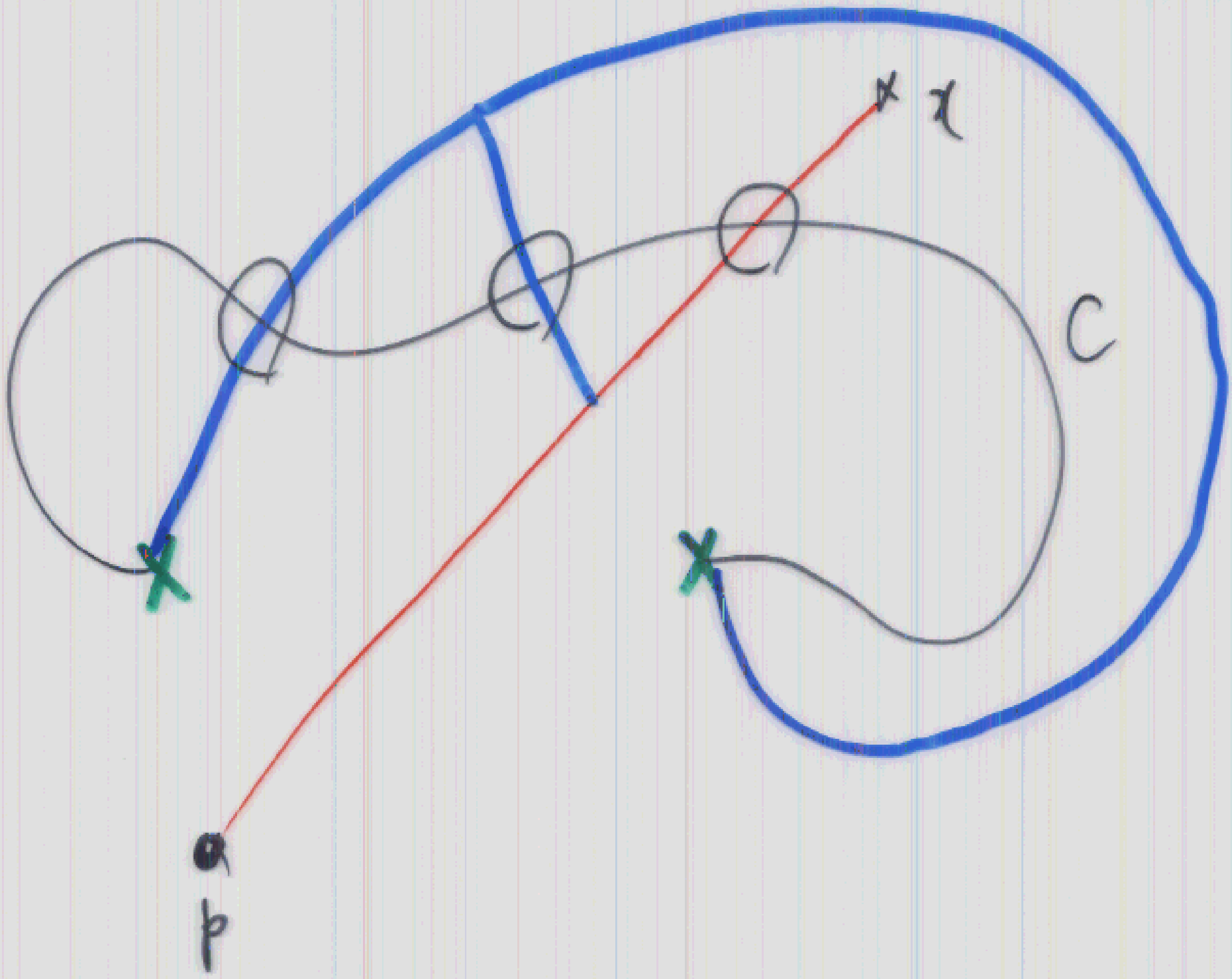
<u>Note</u> S_p is vector valued.

each corresponds to branches of h

$\pm$ sign $\Longleftarrow$ spin str. of L

Spin str. of L $\Longleftrightarrow$ union of Arcs C

$\partial C = S$ ← branched locus



$\cap\ C = 3$ pts

$(-1)^3 = -1$ sign

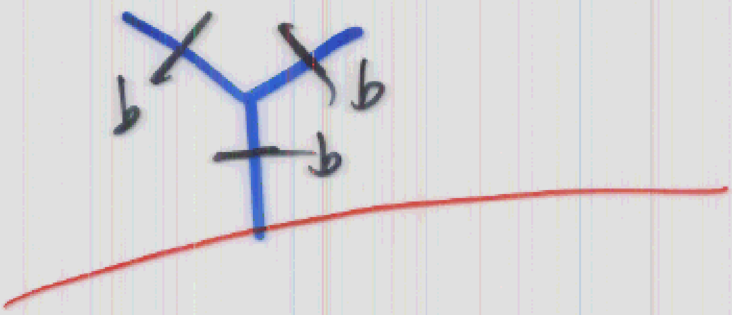
Generalization

① $T^2 \rightsquigarrow T^{2n}$

a) We need to include obstr. theory, F-Oh-Ohta-Ono. If L is special

$\theta \in H^2(L;\mathbb{Q})$ $\theta = \partial b \Rightarrow \Sigma(L,b)$ exists

Extra term in the formula



b: codim. 1 chain

b) We need to include more general Lag. singularity than branched covering.

(clarified by Catastroph theory Arnold, Thom, Mather)

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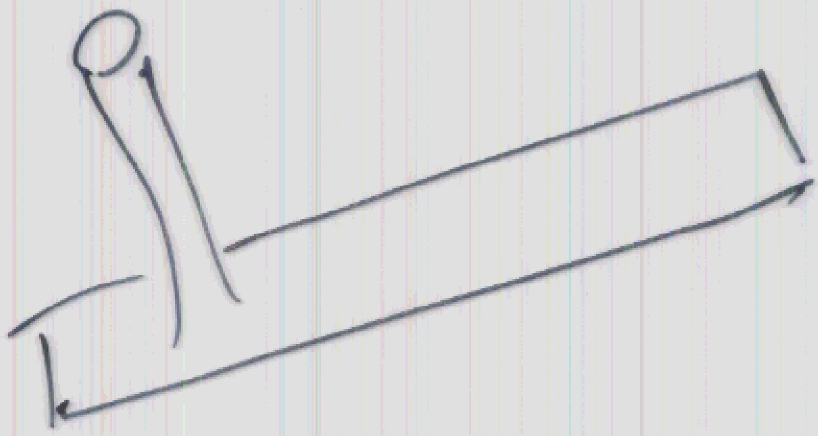
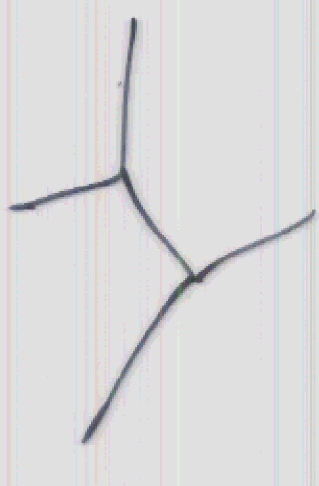
② $T^{2n} \longrightarrow K3$ or CY

We need to include closed string.



③ higher genus

$\Longrightarrow$