Quantum Geometry of Hyperbolic 3-Manifolds

Sergei Gukov

Quantum ~~Geometry~~ Gravity

~~of~~ on Hyperbolic 3-Manifolds

Sergei Gukov

Quantum Theories of "Gravity" in Three Dimensions

- Pure Gravity

  (Chern-Simons gravity, state sum models, ...)

  based on S.G., hep-th/0306165

- String Theory (AdS_3/CFT_2, ...)

  in progress w/ G. Moore & A. Strominger

In this talk I will consider <u>two</u> seemingly unrelated <u>problems</u> about Chern-Simons gauge theory in three dimensions.

$$\int \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)$$

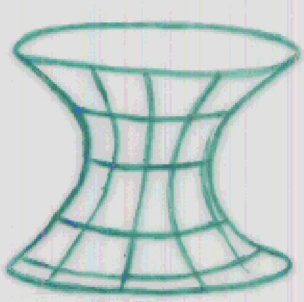
Problem 1 is motivated by connection between Chern-Simons theory with complex gauge group $SL(2,\mathbb{C})$ and three-dimensional quantum gravity:

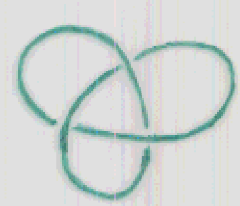
A. Achucarro, P. Townsend
E. Witten

$SL(2,\mathbb{C})$ Chern-Simons

↙ ↓ ↘

de Sitter gravity in 2+1 dimensions	AdS gravity in three dimensions (Euclidean)	knot invariants and geometry of hyperbolic 3-manifolds





$SL(2,\mathbb{C})$ Chern-Simons Theory

M oriented 3-manifold

$\mathcal{A}$ $SL(2,\mathbb{C})$ gauge connection

$$Z(M) = \int \mathcal{D}\mathcal{A} \; e^{iI}$$

where

$$I = \frac{k+\sigma}{8\pi} \int_M \mathrm{Tr}\left(\mathcal{A}\wedge d\mathcal{A} + \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right) +$$

$$+ \frac{k-\sigma}{8\pi} \int_M \mathrm{Tr}\left(\bar{\mathcal{A}}\wedge d\bar{\mathcal{A}} + \frac{2}{3}\bar{\mathcal{A}}\wedge\bar{\mathcal{A}}\wedge\bar{\mathcal{A}}\right) =$$

$$= k \cdot I_{CS} + i\sigma \cdot I_{grav}$$

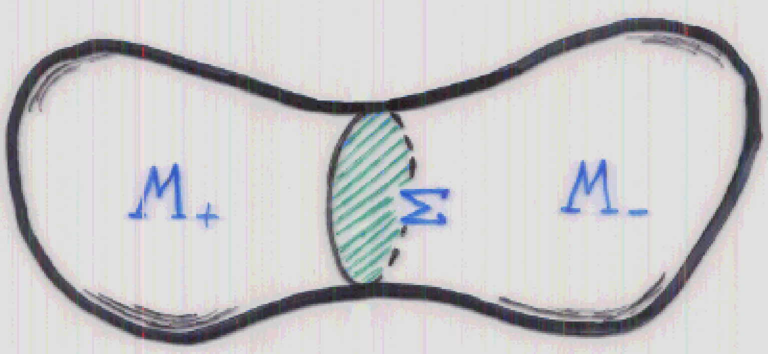
$$\boxed{k \in \mathbb{Z} \qquad \sigma \in \mathbb{R}}$$

- In gravity, the Newton constant $G_N \sim 1/\sigma$

- We shall study $SL(2,\mathbb{C})$ Chern-Simons theory on a 3-manifold M with a single torus boundary $\Sigma = T^2$

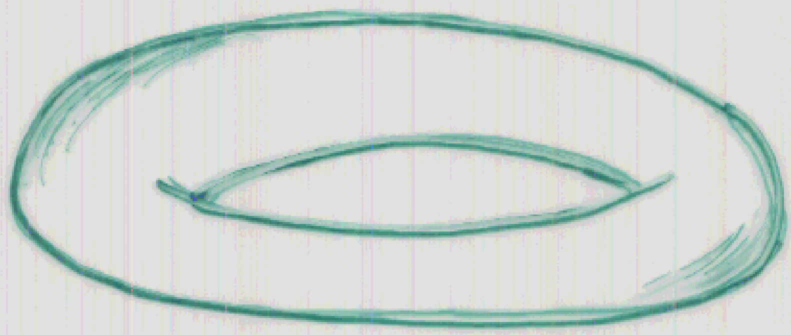
$$Z(M,\Sigma) = ?$$

- According to Thurston, such manifolds can be used as basic building blocks for constructing more general (hyperbolic) 3-manifolds without boundary.



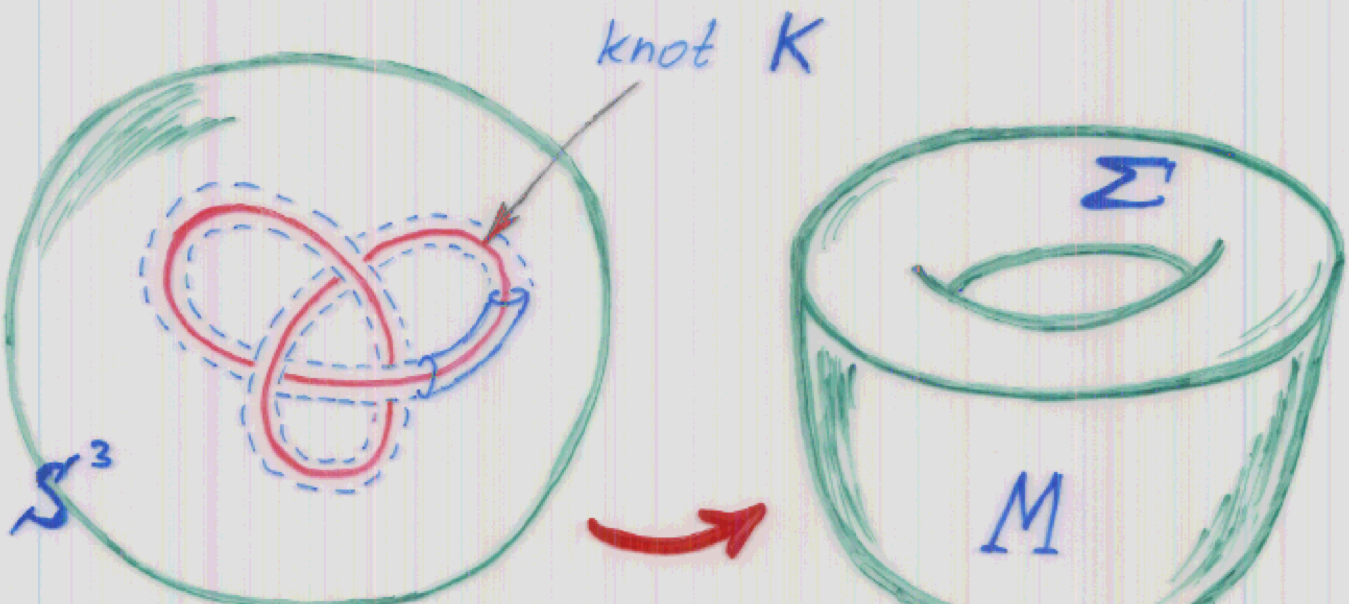
3-Manifolds with Torus Boundary

- BTZ Black Hole:



$M =$ solid torus

- Knot Complements:



$M = S^3 \setminus K$

Problem 2 is motivated by $SU(2)$ Chern-Simons theory and its realization in topological string theory.

$$Z_{SU(2)}(M) = \int \mathcal{D}A \, e^{i I_{SU(2)}}$$

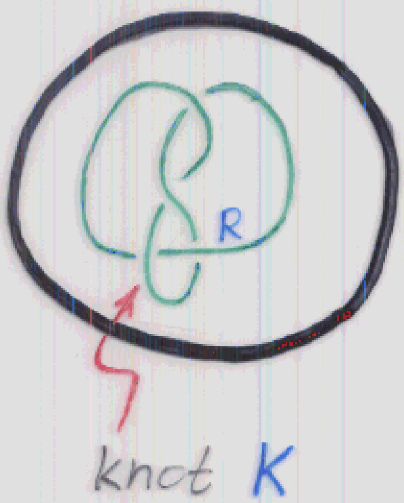
where

$$I_{SU(2)} = \frac{k}{4\pi} \int_M \mathrm{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$$

- Wilson loop observables:

$$W_R(K) = \mathrm{Tr}_R \left(P\exp \oint_K A \right)$$

R: $SU(2)$ representation of dimension $N = 2j+1$



$$\langle W_R(K) \rangle = \frac{1}{Z_{SU(2)}(S^3)} \int \mathcal{D}A \; W_R(K) \, e^{i I_{SU(2)}}$$

- As shown in the famous work by Witten, the vev's of Wilson loop observables lead to polynomial knot invariants:

$$\langle W_R(K) \rangle = \text{Jones polynomial of } K$$

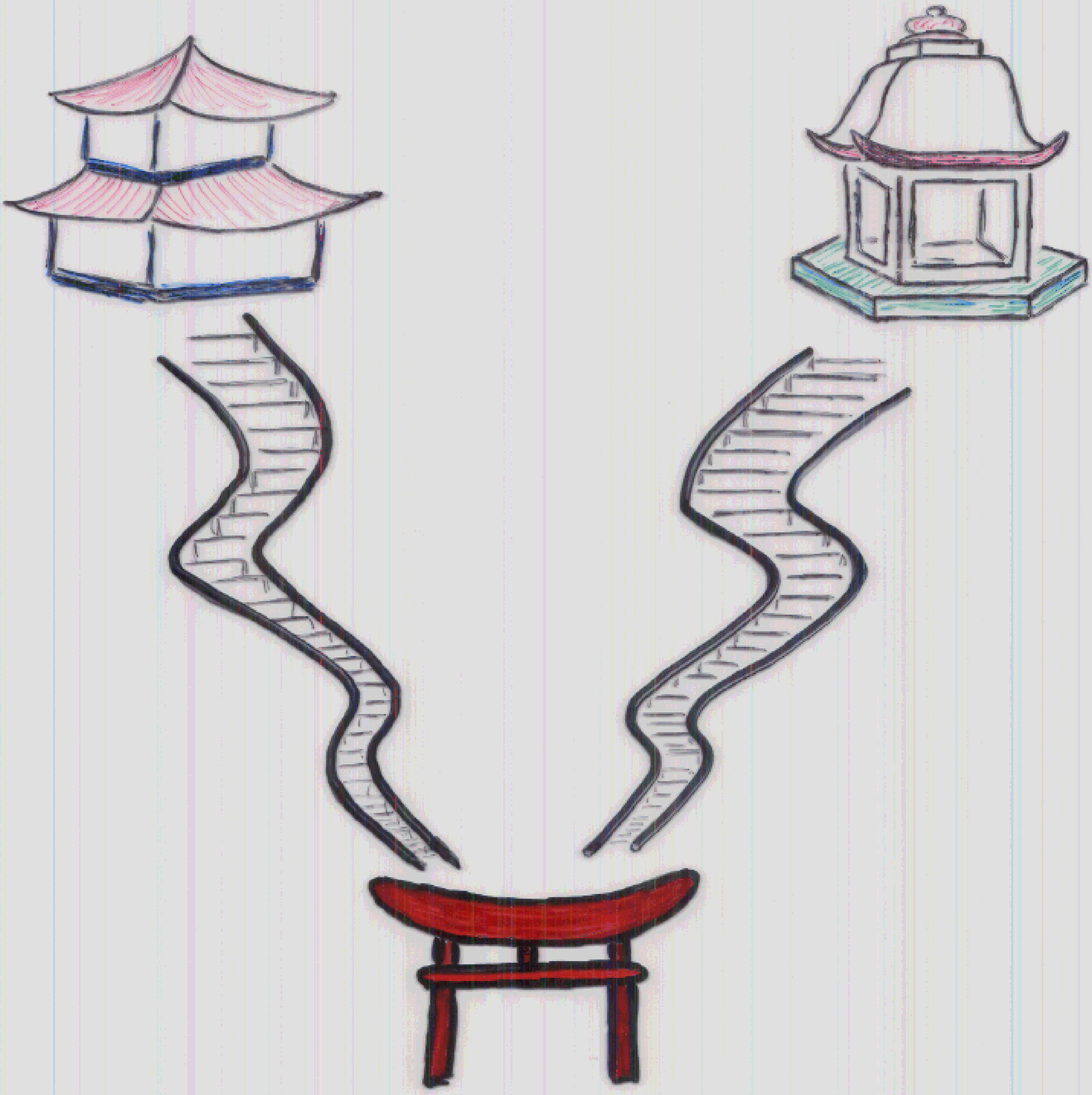
More precisely,

$$\boxed{\langle W_R(K) \rangle = J_N(q)}$$

where $q = e^{\frac{2\pi i}{k+2}}$ is a root of unity.

What is the physical/geometrical interpretation of the Jones polynomial when q is <u>not</u> a root of unity?

It turns out that these two problems



are closely related and, moreover, their solution can be expressed geometrically in terms of an algebraic curve.

- The algebraic curve relevant to these problems is the moduli space of classical solutions in $SL(2,\mathbb{C})$ Chern-Simons theory

$$\mathcal{F}=0 \quad , \quad \overline{\mathcal{F}}=0$$

where $\mathcal{F}=d\mathcal{A}+\mathcal{A}\wedge\mathcal{A}$ and $\overline{\mathcal{F}}=d\overline{\mathcal{A}}+\overline{\mathcal{A}}\wedge\overline{\mathcal{A}}$.

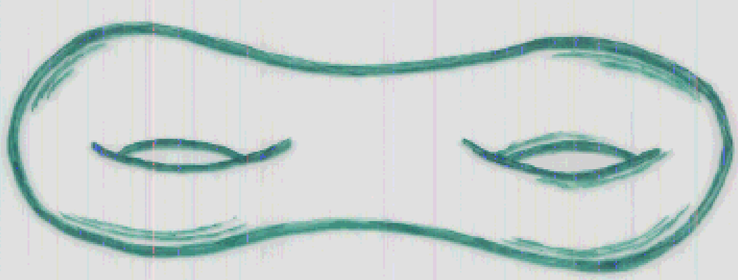
$\Rightarrow$ moduli space of flat connections on M:

$$L = Rep\left(\pi_1(M)\to SL(2,\mathbb{C})\right)/_{conj}$$

- If M is a knot complement, the variety L can be described explicitly as the zero locus of the A-polynomial:

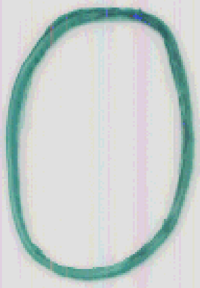
D. Cooper et. al. '94
D. Cooper, D. Long '95-98

$$L: \quad A(\ell,m)=0$$



- $A(\ell, m)$ is a polynomial invariant of knots with integer coefficients.

Examples:

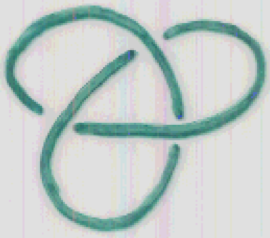
- 

  trivial knot (unknot)

  $$A(\ell, m) = \ell - 1$$

  "mass shell" condition for Euclidean BTZ black hole

  S. Carlip, C. Teitelboim '95

- 

  trefoil knot

  $$A(\ell, m) = (\ell - 1)(\ell m^6 + 1)$$

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  figure-8 knot

  $$A(\ell, m) = (\ell - 1)(-2 + m^4 + m^{-4} - m^2 - m^{-2} - \ell - \ell^{-1})$$

Classical Theory

- The complex variables ℓ and m parametrize the classical phase space,

$$\mathcal{P} = \mathbb{C}^* \times \mathbb{C}^*$$

- The curve

$$L: \quad A(\ell, m) = 0$$

is a Lagrangian submanifold in $\mathcal{P}$.

Quantum Theory

$\mathcal{P}$ Phase Space

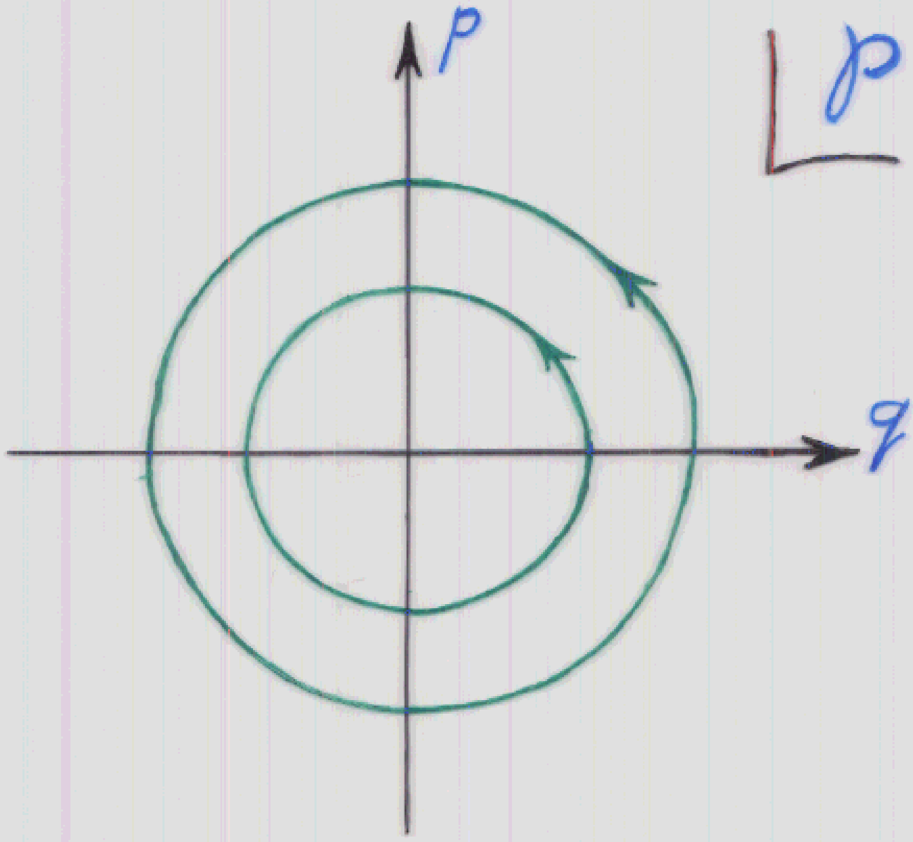
$L \hookrightarrow \mathcal{P}$ Semiclassical State in the effective quantum mechanics

$\Rightarrow$ Quantization of this system leads to the wave function, $Z(M)$, which obeys

$$\hat{A}\, Z_{SL(2,\mathbb{C})} = 0$$

Toy Model

Harmonic Oscillator:



$$L : \quad \frac{1}{2}\left(p^2 + q^2\right) = E$$

- Bohr-Sommerfeld condition:

$$\frac{1}{2\pi\hbar} \oint p\, dq \in \mathbb{Z}$$

$$\Rightarrow \quad E = \hbar\left(n + \frac{1}{2}\right)$$

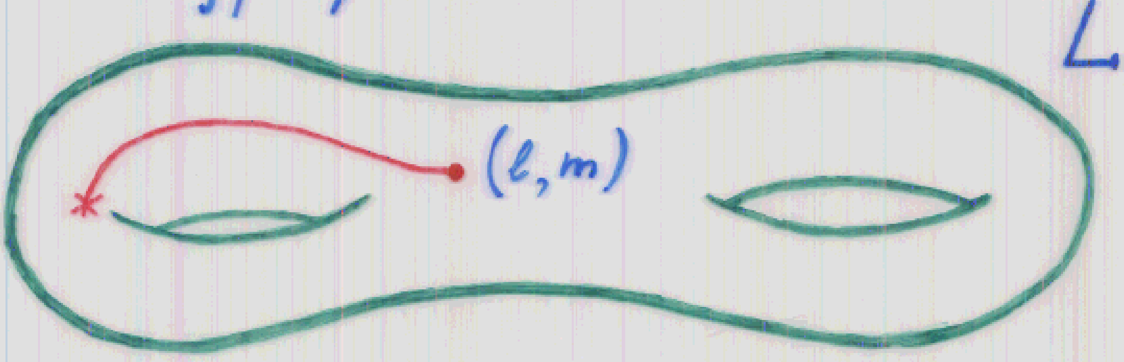
"Maslov correction"

Semi-classical Wave Function:

$$Z = \sum_{\alpha} \varphi \cdot e^{iS/\hbar} + O(\hbar)$$

where the action integral in the effective quantum mechanics is given by

$$S = \int p\, dq$$



- evaluating the action integral, S, and using the result of [Hodgson '86], we obtain the expected expression for the $SL(2,\mathbb{C})$ partition function

$$Z(M) = \sqrt{T}\, \exp\left(\delta \cdot Vol(M) + ik \cdot CS(M)\right) + \ldots$$

• Remarkably, the colored Jones polynomial shows a similar asymptotic behavior in the "semi-classical" limit,

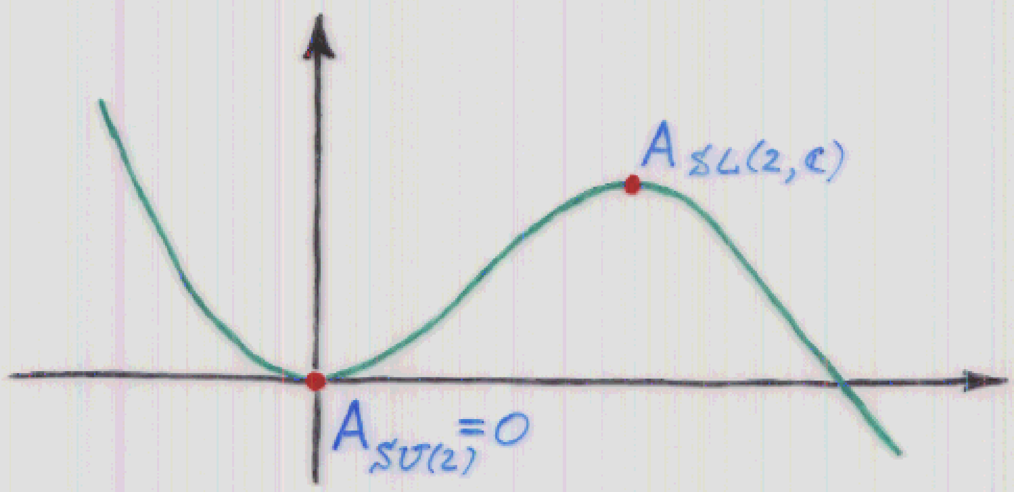
$$k \to \infty, \quad N \to \infty, \quad a := \frac{N}{k} = \text{fixed}$$

Specifically,*

$$J_N(e^{2\pi i/k})$$

Rational a ↙ ↘ Non-rational a

$$\sqrt{T_{SU(2)}}\, e^{ik \cdot I_{CS}(A_{SU(2)}=0)} + \ldots \qquad \sqrt{T_{SL(2,\mathbb{C})}}\, e^{k(Vol + i \cdot CS)} + \ldots$$



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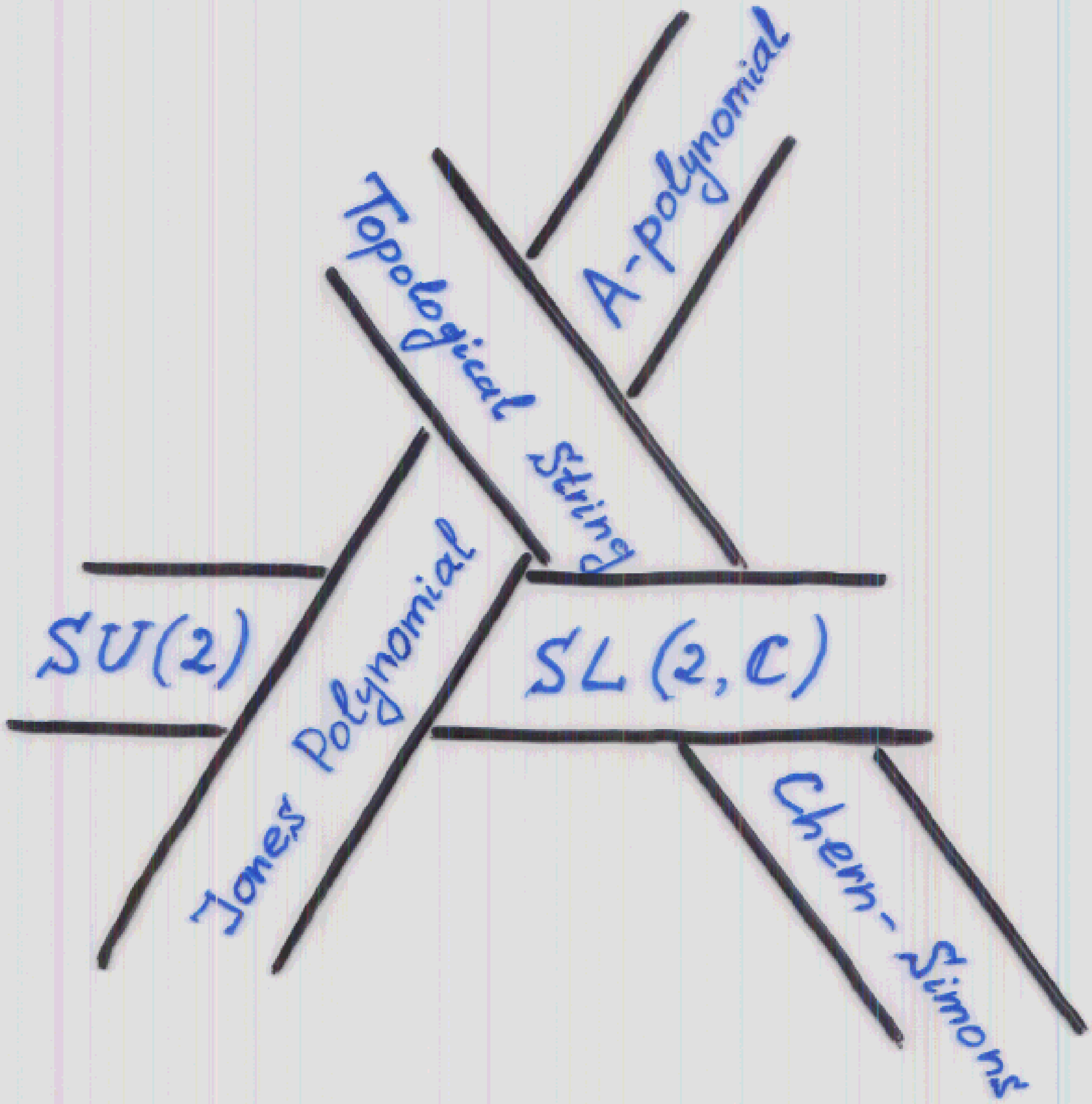
* The case $a=1$ is an exception. It was known before as the "volume conjecture". Kashaev '94, ...

$a = \frac{N}{k}$	asymptotic behavior of $J_N(e^{2\pi i/k})$	Chern-Simons theory	CFT
Rational	Polynomial	$SU(2)$	Rational
Non-Rational	Exponential	$SL(2,\mathbb{C})$	Non-Rational

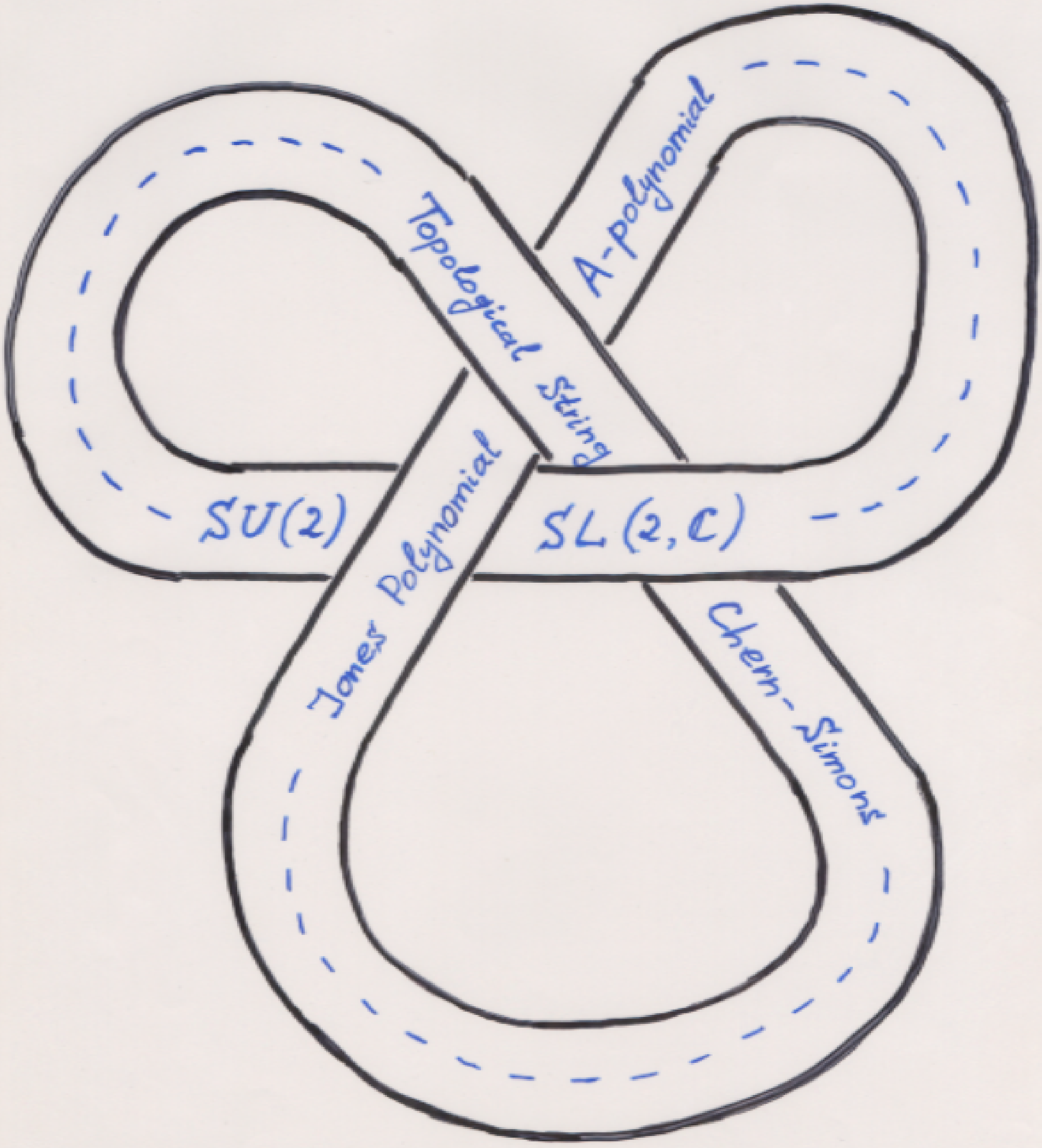
- This suggests a new connection between the $SL(2,\mathbb{C})$ partition function and the colored Jones polynomial,

$$Z(M) \overset{?}{\longleftrightarrow} J_N(q)$$

when q is not a root of unity.



A-polynomial
Topological String
SU(2)
SL(2,C)
Jones Polynomial
Chern-Simons



Topological String
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