- **Based on:**

  hep-th/0206161, N.N.

  hep-th/0302191, Andrei Losev, Andrei Marshakov, N.N.

  hep-th/0306238, Andrei Okounkov, N.N.

- **Foundations were set down in:**

  N. Nekrasov, hep-th/9609219 ;

  A. Lawrence, N. Nekrasov, hep-th/9706025

  A. Losev, N. Nekrasov, S. Shatashvili, hep-th/9711108, hep-th/9801061

  N. Nekrasov, A. Schwarz, hep-th/9802068

  G. Moore, N. Nekrasov, S. Shatashvili, hep-th/9712241, hep-th/9803265

- *Related work on instanton integrals:*

  W. Krauth, H. Nicolai, M. Staudacher, hep-th/9803117

  N. Dorey,T. J. Hollowood, V. V. Khoze, M. P. Mattis, hep-th/0206063

  N. Dorey, V.V. Khoze, M.P. Mattis, hep-th/9706007, hep-th/9708036

  N. Dorey,V.V. Khoze, M.P. Mattis, hep-th/9607066

  T. Hollowood, hep-th/0201075, hep-th/0202197

  U. Bruzzo, F. Fucito, J.F. Morales, A. Tanzini, hep-th/0211108;

  D.Bellisai, F.Fucito, A.Tanzini, G.Travaglini, hep-th/0002110, hep-th/0003272, hep-th/0008225

  R. Flume, R. Poghossian, hep-th/0208176;

  R. Flume, R. Poghossian, H. Storch, hep-th/0110240, hep-th/0112211

  V. Kazakov, I. Kostov, N. Nekrasov, "D-particles, Matrix Integrals and KP hierarchy", hep-th/9810035

- *Related work on SW curves, fivebrane constructions, fractional branes, AdS/CFT duals, topological strings, graviphoton backgrounds ....*

http://www.arXiv.org/hep/

SUMMARY

- $\mathcal{N} = 2$ supersymmetric four dimensional gauge theories, in a certain $\mathcal{N} = 2$ supergravity background, called Ω-background.

- Calculate the partition function of the gauge theory in this background.

- Relation to the prepotential of the low-energy effective action.

- From this: the exact expression for the prepotential.

- The calculation reveals the existence of the saddle point in the gauge theory path integral, where the rôle of the Planck constant is played by the parameter $\hbar$ of the Ω-background, which is taken to zero, when the background approaches the flat spacetime $\mathbf{R}^4$.

- Thus we find a **novel kind of the MASTER FIELD in the gauge theory**

- We relate field theory partition function to a string partition function: via geometric engineering, and via D-engineering.

- $\hbar$-expansion becomes a genus expansion, in all cases. **Novel embedding of the topological string into superstrings**. Not necessarily critical topological string, $\widehat{c} \neq 3$.

- Gauge/string duality for the $\mathcal{N} = 2$ chiral rings. Gravitational descendants. More general Ω-backgrounds: λ_k-classes.

-

$\mathcal{N} = 2$ GAUGE THEORY,

DEFORMATIONS AND BACKGROUNDS

Lagrangian, fields, couplings

- START WITH:

$$S_0 = \frac{1}{4g_0^2} \int \sqrt{G} d^4x \, \mathrm{Tr}\Big\{ -F_{IJ}F^{IJ} - 2D_I\phi D^I\overline{\phi} - [\phi, \overline{\phi}]^2 - i\overline{\lambda}^{\dot\alpha}_{\mathbf{i}} \sigma^I_{\alpha\dot\alpha} D_I \lambda^{\alpha\mathbf{i}} + \frac{i}{2}\left(\phi\epsilon^{\mathbf{ij}}[\overline{\lambda}_{\dot\alpha\mathbf{i}}, \overline{\lambda}^{\dot\alpha}_{\mathbf{j}}] - \overline{\phi}\epsilon_{\mathbf{ij}}[\lambda^{\alpha\mathbf{i}}, \lambda^{\mathbf{j}}_{\alpha}]\right)\Big\} + \frac{\vartheta_0}{2\pi}\int \mathrm{Tr} F\wedge F$$

- UV CUT-OFF μ. LOW-ENERGY EFFECTIVE (COMPLEXIFIED) SCALE

$$\Lambda^{2N} = \mu^{2N} e^{-\frac{8\pi^2}{g_0^2} + 2\pi i \vartheta_0}$$

- THE ACTION (3.1) IS THE LIMIT OF THE SIX DIMENSIONAL ACTION OF THE THEORY PUT ON A SIX MANIFOLD $\mathbf{T}^2 \times \mathbf{R}^4$ WITH THE STANDARD FLAT PRODUCT METRIC.

Ω-background

- Replace $\mathbf{T}^2 \times \mathbf{R}^4$ by the space $\mathcal{N}_6$ with the metric:

$$ds^2 = R^2 dz d\bar{z} + g_{IJ}\left(dx^I + V^I dz + \overline{V}^I d\bar{z}\right)\left(dx^J + V^J dz + \overline{V}^J d\bar{z}\right)$$

- $V^I = R\Omega^I_J x^J,\ \overline{V}^I = R\overline{\Omega}^I_J x^J.$

- For $[\Omega, \overline{\Omega}] = 0$ this metric is flat.

- In addition we turn on a flat connection (Wilson loops) in the R-symmetry group, which should compensate some part of the metric induced holonomy on the spinors.

- Two choices: Ω^+ or Ω^-. Depends on what you like more: instantons, or anti-instantons, chiral or antichiral, flying to Kyoto from Paris or from Boston....

- **Take the limit $R \to 0$ with $\Omega, \overline{\Omega}$ fixed**. The SYM action gets extra terms. The bosonic part:

$$\boxed{\begin{aligned} S^{bos}_{\Omega} = -\frac{1}{4g_0^2}\int \mathrm{Tr} F_{MN}F^{MN} + \frac{\vartheta_0}{2\pi}\int \mathrm{Tr} F \wedge F \\ + \frac{1}{4g_0^2}\int \mathrm{Tr}(D_M\phi - V^N F_{MN})(D_M\overline{\phi} - \overline{V}^N F_{MN}) + \\ + \frac{1}{4g_0^2}\int \mathrm{Tr}\left([\phi, \overline{\phi}] - V^M D_M\overline{\phi} + \overline{V}^M D_M\phi\right)^2 \end{aligned}}$$

- The unbroken susy is generated by two supercharges Q_1, Q_2 which anticommute to a Lorentz + R-symmetry rotation.

- For future use:

$$\Omega = \begin{pmatrix} 0 & \epsilon_1 & & \\ -\epsilon_1 & 0 & & \\ & & 0 & \epsilon_2 \\ & & -\epsilon_2 & 0 \end{pmatrix}$$

- If Ω is purely self-dual, $\epsilon_1 = \epsilon_2$, or purely antiself-dual, $\epsilon_1 = -\epsilon_2$, then the supersymmetry is enchanced to the algebra with four supercharges (all of the same chirality).

- It also enhances when Ω fixes a two-plane in $\mathbf{R}^4$, $\epsilon_1 = 0$, or $\epsilon_2 = 0$ (two of each chirality).

- Similar algebra, for $\epsilon_1 + \epsilon_2 = 0$, was considered by Vafa, Ooguri, Seiberg and Berkovits, however the sugra background they look at is seemingly different.

GAUGE THEORY PARTITION FUNCTION

- Euclidean path integral over the field configurations, such that

$$\phi(x) \to \mathbf{a} = \mathrm{diag}(a_1, \ldots, a_N), \qquad x \to \infty$$

- on the Ω-background

$$Z(\mathbf{a}, \epsilon_1, \epsilon_2; \Lambda) = \int DAD\psi D\eta D\chi D\Phi D\overline{\Phi}\ e^{-S_\Omega}$$

- The partition function is holomorphic in $\mathbf{a}, \epsilon_1, \epsilon_2, \Lambda$ (susy). Nonrenormalization of ϵ's.

- Take the limit $\overline{\epsilon}_{1,2} \to \infty, \overline{\mathbf{a}} \to \infty$ while keeping $\mathbf{a}$, etc. fixed. Partition function is not sensitive to that.

- Field theory is interested in the limit $\epsilon_1, \epsilon_2 \to 0$ of the partition function, where, as we argued last year:

$$Z(\mathbf{a}; \epsilon_1, \epsilon_2, \Lambda) = \exp\left(-\frac{1}{\epsilon_1\epsilon_2}\mathcal{F}(\mathbf{a}; \epsilon_1, \epsilon_2, \Lambda)\right)$$

where $\mathcal{F}$ is analytic in ϵ_1, ϵ_2 for $\epsilon_{1,2} \to 0$. We prove the conjecture, which identifies $\mathcal{F}_0(\mathbf{a}, \Lambda) \equiv \mathcal{F}(\mathbf{a}; 0, 0, \Lambda)$ with the Seiberg-Witten prepotential of the low-energy effective theory, and we get, *par consequence*, the instanton expression for the prepotential.

Perturbative part

- Expand around $A = 0$. Piece of cake. Ratio of determinants.

- Spectrum of fluctuations becomes discrete.

$$Z^{pert}(\mathbf{a}, \epsilon_1, \epsilon_2, \Lambda) = \prod_{i,j\geq 1} \frac{a_l - a_n + \epsilon_1(i-1) + \epsilon_2(j-1)}{\Lambda} =$$

- Proper time regularization

$$\exp \sum_{l,n} \gamma_{\epsilon_1,\epsilon_2}(a_l - a_n; \Lambda)$$

Where

$$\gamma_{\epsilon_1,\epsilon_2}(x;\Lambda) = \frac{d}{ds}\bigg|_{s=0} \frac{\Lambda^s}{\Gamma(s)} \int_0^\infty \frac{dt}{t} t^s \frac{e^{-tx}}{(e^{t\epsilon_1}-1)(e^{t\epsilon_2}-1)}$$

- For $\epsilon_2 = -\epsilon_1 = \hbar$:

$$\begin{aligned}\gamma_\hbar(x;\Lambda) =& \frac{x^2}{2\hbar^2}\left(\log\frac{x}{\Lambda} - \frac{3}{2}\right) \\ & - \frac{1}{12}\log\frac{x}{\Lambda} \\ & + \sum_{g=2}^{\infty} \frac{B_{2g}}{2g(2g-2)} \frac{\hbar^{2g-2}}{x^{2g-2}}\end{aligned}$$

- Something we have seen in $c = 1$ [talks by V.Kazakov, I.Klebanov, ...], Liouville [talk by J.Teschner], $c = 1$ string/topological string on conifold [talk by C.Vafa], superpotential of pure $\mathcal{N} = 1$ SYM, etc.

Instanton part

$$Z = Z^{pert} \sum_{k=0}^{\infty} \Lambda^{2kN} Z_k(\mathbf{a}, \epsilon_1, \epsilon_2)$$

- In my lecture at Strings'02 I gave an expression for $Z_k(\mathbf{a}, \epsilon_1, \epsilon_2)$ in terms of some contour integral over k eigenvalues ϕ_I, $I = 1, \ldots, k$ – positions of D(-1) instantons, transverse to the D3 branes.

- Follows straightforwardly from the ADHM

- Exists for any A,B,C,D type group

- for $SU(N)$:

$$Z_k(\mathbf{a}, \epsilon_1, \epsilon_2) = \oint \prod_{I=1}^{k} \left[\frac{\epsilon_1 + \epsilon_2}{2\pi\epsilon_1\epsilon_2} \frac{d\phi_I}{P(\phi_I)P(\phi_I + \epsilon_1 + \epsilon_2)} \right] \prod_{I \neq J} \frac{\phi_{IJ}(\phi_{IJ} + \epsilon_1 + \epsilon_2)}{(\phi_{IJ} + \epsilon_1)(\phi_{IJ} + \epsilon_2)}$$

where

$$P(x) = \prod_{l=1}^{N} (x - a_l)$$

$$\phi_{IJ} = \phi_I - \phi_J$$

- Grand canonical ensemble of instantons. Λ^{2N} — Fugacity

- The integral can be explicitly evaluated via residues.

- The Ω-background lifts instanton moduli, leaving only a finite number of isolated points on the appropriately compactified instanton moduli space as a full set of supersymmetric minima of the action. The evaluation of the partition function is then reduced to the calculation of the ratios of the bosonic and fermionic determinants near each critical point. These points are labeled by the N-tuples of partitions.

- Explicitly:

$$Z(\mathbf{a};\epsilon_1,\epsilon_2,\Lambda) = Z^{pert}(\mathbf{a};\Lambda,\epsilon_1,\epsilon_2)\sum_{\vec{\mathbf{k}}}\Lambda^{2N|\vec{\mathbf{k}}|}Z_{\vec{\mathbf{k}}}(\mathbf{a};\epsilon_1,\epsilon_2)$$

where:

$$\begin{aligned}
Z_{\vec{\mathbf{k}}}(\mathbf{a};\epsilon_1,\epsilon_2) &= \prod_{l,n;i,j}\frac{a_l-a_n+\epsilon_1(i-1)+\epsilon_2(-j)}{a_l-a_n+\epsilon_1(i-\widetilde{k}_{nj}-1)+\epsilon_2(k_{li}-j)} \\
&= \prod_{l,n;i,j}\frac{a_l-a_n+\epsilon_1(-i)+\epsilon_2(j-1)}{a_l-a_n+\epsilon_1(\widetilde{k}_{lj}-i)+\epsilon_2(j-k_{ni}-1)} \\
&= \frac{1}{\epsilon_2^{2N|\vec{\mathbf{k}}|}}\prod_{(l,i)\neq(n,j)}\frac{\Gamma\left(k_{li}-k_{nj}+\nu(j-i+1)+b_{ln}\right)\Gamma\left(\nu(j-i)+b_{ln}\right)}{\Gamma\left(k_{li}-k_{nj}+\nu(j-i)+b_{ln}\right)\Gamma\left(\nu(j-i+1)+b_{ln}\right)}\ ,
\end{aligned}$$

$$b_{ln} = \frac{a_l-a_n}{\epsilon_2}, \qquad \nu = -\frac{\epsilon_1}{\epsilon_2}$$

- The special case $\epsilon_2 = -\epsilon_1 = \hbar$, deserves a special attention.

In this case, the expression for the partition function simplifies to:

$$
\begin{aligned}
Z(\mathbf{a};\hbar,\Lambda) &= \sum_{\vec{\mathbf{k}}} \Lambda^{2N|\vec{\mathbf{k}}|} Z_{\vec{\mathbf{k}}}(\mathbf{a};\hbar) \\
Z_{\vec{\mathbf{k}}}(\mathbf{a};\hbar) &= Z^{pert}(\mathbf{a};\hbar)\mu^2_{\vec{\mathbf{k}}}(\mathbf{a},\hbar) \\
\mu^2_{\vec{\mathbf{k}}}(\mathbf{a},\hbar) &= \prod_{(l,i)\neq(n,j)} \left(\frac{a_l - a_n + \hbar(k_{l,i} - k_{n,j} + j - i)}{a_l - a_n + \hbar(j-i)} \right)
\end{aligned}
$$

- *THIS IS WHERE WE WERE IN CAMBRIDGE LAST YEAR*

A fork in the road

- **Option 1**. Limit $\epsilon_1, \epsilon_2 \to 0$. Derive prepotential. Can do for any classical gauge group, any reasonable matter content. See instanton gas, perhaps liquid, perhaps confinement.

- **Option 2.** Study full partition function, for $SU(N)$, expand in $\hbar$, see strings.

PREPOTENTIAL STORY

- Typical instanton charge is of the order $k \sim \frac{1}{\epsilon_1\epsilon_2}$.

- A useful function:

$$\delta_{\epsilon_1,\epsilon_2}(x) = |x-\epsilon_1| + |x-\epsilon_2| - |x| - |x-\epsilon_1-\epsilon_2|$$

- Density of eigenvalues:

$$\rho(x) = \sum_I \delta_{\epsilon_1,\epsilon_2}(x-\phi_I)$$
$$\longrightarrow -2\epsilon_1\epsilon_2 \sum_{I=1}^{k} \delta(x-\phi_I)$$

- Difference with the ordinary 't Hooft-like limit of the ordinary matrix integrals is the presence of the equal number of the ϕ_{IJ} terms in the numerator and the denominator of the measure.

- Changes qualitatively the density dependence of the effective potential on the eigenvalues $\Longrightarrow$ the resulting equilibrium distribution of the eigenvalues.

- The superymmetric matrix integral scales as $\exp\, k\, F$
  [*Seems to be what BFSS ordered*]

- as opposed to the 't Hooft's $\exp k^2 F$. Here $k = k_{saddle}$.

- Nevertheless we have a sharp peak in the measure, which justifies the application of the saddle point method. Indeed, by expanding in ϵ_1, ϵ_2 we map the instanton contribution onto:

$$Z^{pert}\Lambda^{2kN}Z_k(\mathbf{a},\epsilon_1,\epsilon_2) \sim \exp\left(\frac{1}{\epsilon_1\epsilon_2}\mathbf{E}_\Lambda[\rho]\right)$$

where

$$\begin{aligned}\mathbf{E}_\Lambda[\rho] =& \frac{1}{2}\sum_{l,n}(a_l-a_n)^2\left(\log\left(\frac{a_l-a_n}{\Lambda}\right)-\frac{3}{2}\right)\\ &-\frac{1}{4}\int_{x\neq y} dxdy\ \frac{\rho(x)\rho(y)}{(x-y)^2}-\int\ dx\ \rho(x)\log\left(\frac{P(x)}{\Lambda^N}\right)\end{aligned}$$

- Now get the saddle point equation on ρ, or, better yet, on

$$f(x)=\sum_{l=1}^{N}|x-a_l|+\rho(x)$$

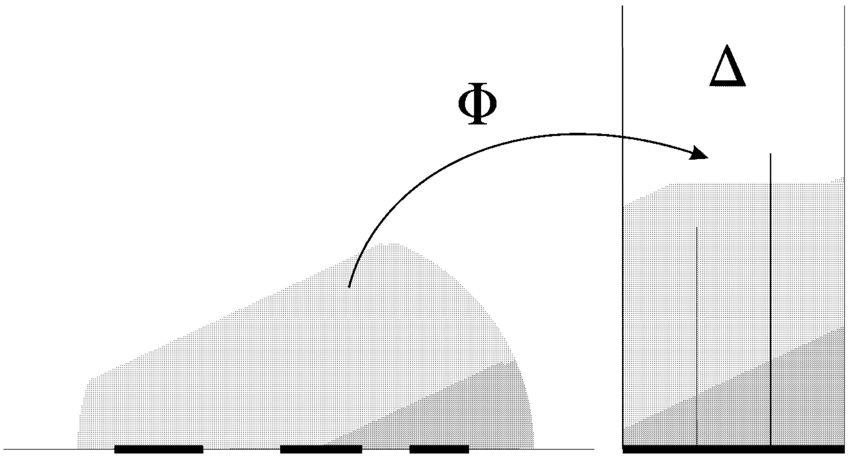
and the resolvent

$$R_f(z)=\tfrac{1}{2}\int\frac{f''(x)}{z-x}dx$$

- Skipping details: the conditions on f can be concisely formulated in terms of analytic maps:

- $$\varphi(x) = f'(x) + \frac{i}{\pi} \int_{y \neq x} dy \, \log\left|\frac{y-x}{\Lambda}\right| f''(y)$$

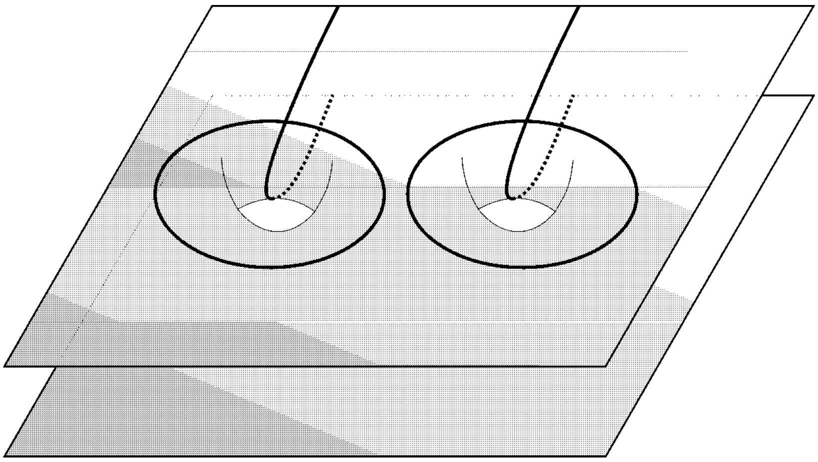
is a boundary value of the analytic map



Conformal map for $N = 3$

- FROM THIS ONE CONCLUDES, AFTER SOME DISCUSSION:

$$
\begin{aligned}
&\mathcal{F}_0(\mathbf{a},\Lambda) = \mathbf{E}_\Lambda(\rho) \\
&R_f(z)dz = d\log(w) \\
\mathcal{C}_u: \quad &\Lambda^N(w+\frac{1}{w}) = x^N + u_1 x^{N-1} + \ldots + u_N \\
&a_l = \oint_{\mathbf{a}_l} dS \\
&\frac{\partial \mathcal{F}_0}{\partial a_l}_l = 2\pi i \oint_{\mathbf{b}_l} dS
\end{aligned}
$$



THE CURVE $\mathcal{C}_u$ AND THE CYCLES ON IT, THE CLOSED ONES ARE $\mathbf{a}_l$'S, THE NONCOMPACT ONES ARE $\mathbf{b}_l$'S

- THE DENSITY ρ HAS N BANDS. *Similar to N-cut solutions.*

- THIS APPROACH EXTENDS TO ANY MATTER AND CLASSICAL GAUGE CONTENT.

- **We have therefore confirmed** [SEIBERG-WITTEN, KLEMM-LERCHE-THEISEN-YANKIELOWICZ ARGYRES-FARAGGI, HANANY-OZ, NACULIC-SCHNITZER] AND HAVE TOOLS TO EXTEND THEM.

- **Instanton density**. In many respects, Ω-background looks like a box of a finite size:

$$V \sim \frac{1}{\epsilon_1 \epsilon_2}$$

- At the saddle point:

$$\frac{k}{V} = u_2 - \sum_l \frac{a_l^2}{2} = \frac{\Lambda^{2N}}{a^{2(N-1)}} + \dots$$

is small in the quasiclassical region.

- Thus we have some sort of dilute instanton gas. We can trace the saddle point to the point where the bands in the density start overlapping. Some kind of transition? Analyticity severely constrains the choices.

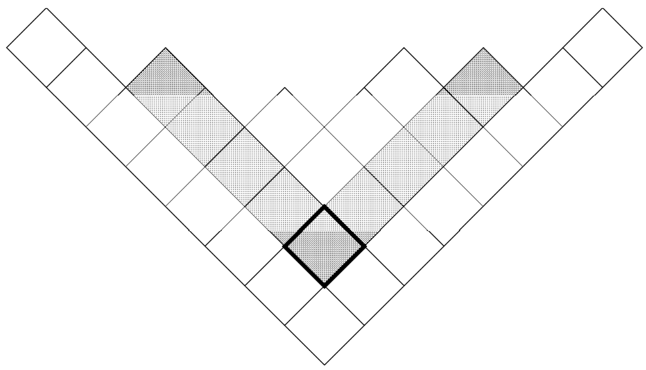
Plancherel measure on partitions

- WE NOW TURN TO THE FULL PARTITION FUNCTION. FIRST FOR $\nu = 1$.

- TAKE $N = 1$. NEED TO SUM OVER A SINGLE PARTITION $\mathbf{k}$ AND THE WEIGHT $\mu_{\vec{\mathbf{k}}}(\mathbf{a}, \hbar)$ REDUCES TO

$$\mu(\mathbf{k}) = \frac{1}{\hbar^{|\mathbf{k}|}} \prod_{i<j} \left(\frac{k_i - k_j + j - i}{j - i} \right) = \prod_{\square \in \mathbf{k}} \frac{1}{\hbar h(\square)} ,$$

THE PRODUCT IS OVER ALL SQUARES $\square$ IN THE YOUNG DIAGRAM OF THE PARTITION $\mathbf{k}$ AND $h(\square)$ DENOTES THE CORRESPONDING HOOK-LENGTH.



BOX, HOOK, ARM, LEG, CONTENT.....

THE WEIGHT $\mu(\mathbf{k})^2$ IS KNOWN AS THE PLANCHEREL MEASURE ON PARTITIONS BECAUSE OF THE RELATION

$$\mu(\mathbf{k}) = \frac{\dim R_{\mathbf{k}}}{\hbar^{|\mathbf{k}|} \; |\mathbf{k}|!} ,$$

WHERE $R_{\mathbf{k}}$ IS THE IRREDUCIBLE REPRESENTATION OF THE SYMMETRIC GROUP CORRESPONDING TO THE PARTITION $\mathbf{k}$. IT FOLLOWS THAT

$$\sum_{|\mathbf{k}|=n} \mu^2(\mathbf{k}) = \frac{\hbar^{-2n}}{n!}$$

AND

$$Z_{N=1} = \exp \frac{1}{\hbar^2}\left(t_1 \frac{a^2}{2} + e^{t_1}\right)$$

$$\Lambda^2 = e^{t_1}$$

- GROMOV-WITTEN THEORISTS WOULD IMMEDIATELY RECOGNISE THE TYPE A TOP.STRING PARTITION FUNCTION ON $\mathbf{CP}^1$.

$$t_1 \sim \omega, a \sim \mathbf{1}$$

WHAT IS THE PHYSICS OF THIS RESULT?

- WE ARE TALKING ABOUT $U(1)$ THEORY WITH INSTANTONS:

- EITHER HIGGSED $U(k) \times U(k+1)$ WITH $(k, \overline{k+1}) \oplus (k+1, \overline{k})$.

- FRACTIONAL D3 BRANE AT $\mathbf{R}^4/\mathbf{Z}_2$ SINGULARITY WITH FRACTIONAL D(-1) INSTANTONS (PREVIOUS PICTURE IS OBTAINED BY ADDING k REGULAR BRANES AND TAKING THEM AWAY), WITH ADJUSTED TWISTED SECTOR SCALARS.

- NONCOMMUTATIVE $U(1)$ THEORY.

- BLOW UP THE SINGULARITY. FRACTIONAL D3 BRANE BLOWS UP TO D5 WRAPPING $\mathbf{P}^1$, D-INSTANTONS INTO D1 BOUND TO D5 = **little string** WORLDSHEET INSTANTONS.

- This extends to more general deformations:

- Space-time: put a holomorphic wave of the twisted sector fields [Klebanov-Nekrasov, 1999]

- Gauge theory: add higher Casimirs to the prepotential

- Worldsheet of the little strings: Gravitational descendants of ω are turned on.

- Dictionary:

$t_k \Phi^{k+1}$	$\leftrightarrow$	$t_k \sigma_{k-1}(\omega)$
$\hbar$	$\leftrightarrow$	g_s

- We have got a version of the gauge theory/closed string duality. Works for chiral sector.

- Topological type A string on $\mathbf{P}^1$. Effective field theory (analogue of the supergravity in the AdS/CFT): what is it?

- ANSWER: FREE CHIRAL FERMION/BOSON ON THE SPHERE, IN A GENERIC $\mathcal{W}_\infty$ BACKGROUND:

$$Z(a;t_1,t_2,\ldots;\hbar) = e^{\sum_{g=0}^{\infty}\hbar^{2g-2}\mathcal{F}_g(a,t)} =$$
$$\left\langle \frac{a}{\hbar}\right| e^{-\frac{J_1}{\hbar}}\ \exp\sum_{k=1}^{\infty} t_k \hbar^{k-1}\oint :\ \widetilde{\psi}\left((D+\tfrac{1}{2})^{k+1}-(D-\tfrac{1}{2})^{k+1}\right)\psi :\ e^{-\frac{J_{-1}}{\hbar}}\left|\frac{a}{\hbar}\right\rangle$$

- ACTION IS

$$\int \widetilde{\psi}\bar{\partial}\psi$$

THE INGREDIENTS:

$$J =: \widetilde{\psi}\psi :$$

$$\frac{a}{\hbar} = J_0 \qquad \text{eigenvalue}$$

AND $D = z\partial_z$.

General Ω-background

- What if $\epsilon_1 + \epsilon_2 \neq 0$?

- We still get the string on $\mathbf{P}^1$, but this time with extra worldsheet gravitational stuff:

$$\mathcal{F}_g = \langle\langle \exp\left[\int \ a\ \mathbf{1} + \sum_k t_k \sigma_{k-1}(\omega)\right] \quad \mathbf{c}(\epsilon_1)\mathbf{c}(\epsilon_2)\rangle\rangle_g$$

- where $\mathbf{c}(x) = x^{g-1} + \lambda_1 x^{g-2} + \ldots + \lambda_g x^{-1}$ is the Chern polynomial (up to x^{-1} factor) of the Hodge bundle over $\overline{\mathcal{M}}_g$.

- THIS IS A NEW PREDICTION OF STRING THEORY (more general then the CS-inspired recent predictions [Aganagic-Mariño-Vafa]).

- Now turn to $N > 1$.

- Change picture. Engineer [Katz-Klemm-Vafa] the gauge theory using A_{N-1} fibration over $\mathbf{P}^1$.

- From 1997 [Strominger-Becker, Katz-Klemm-Vafa, Nekrasov] we have learned that this leads to the five dimensional gauge theory compactified on a circle, of radius β.

- Cycles in A_{N-1}: kahler moduli: βa_l. (replace by ALF if want $\sum_l a_l \neq 0$).

- Base's kahler class: $\log(\beta\Lambda)$.

- A simple extension of our instanton counting technique gives for the partition function:

$$
\begin{aligned}
Z(\mathbf{a};\hbar,\beta,\Lambda) &= \sum_{\vec{\mathbf{k}}} (\beta\Lambda)^{2N|\vec{\mathbf{k}}|} Z_{\vec{\mathbf{k}}}(\mathbf{a};\hbar,\beta) \\
Z_{\vec{\mathbf{k}}}(\mathbf{a};\hbar,\beta) &= Z^{pert}(\mathbf{a};\hbar,\beta)\ \mu^2_{\vec{\mathbf{k}}}(\mathbf{a},\beta,\hbar) \\
\mu^2_{\vec{\mathbf{k}}}(\mathbf{a},\beta,\hbar) &= \prod_{(l,i)\neq(n,j)} \frac{\sinh\frac{\beta}{2}(a_l - a_n + \hbar(k_{l,i} - k_{n,j} + j - i))}{\sinh\frac{\beta}{2}(a_l - a_n + \hbar(j-i))} \\
Z^{pert}(\mathbf{a};\hbar,\beta) &= \exp \sum_{l,n} \gamma_\hbar\,(a_l - a_n|\beta;\Lambda)
\end{aligned}
$$

- TOPOLOGICAL TYPE A STRING ON THE CONIFOLD SHOWS UP:

$$\begin{aligned}\gamma_{\hbar}\left(x;\beta;\Lambda\right) =& \frac{\beta x^3}{12\hbar^2} - \frac{x^2}{2\hbar^2}\log\left(\beta\Lambda\right) - \frac{\beta x}{24} + \sum_{n=1}^{\infty}\frac{1}{n}\frac{e^{-\beta n x}}{(e^{-\beta n\hbar}-1)(e^{\beta n\hbar}-1)} \\ &= \sum_{g=0}^{\infty}\gamma_g(x;\beta)\hbar^{2g-2} \\ &\gamma_0(x;\beta) = -\frac{x^2}{2}\log\left(\beta\Lambda\right) + \beta\frac{x^3}{12} - \frac{1}{\beta^2}\ \mathrm{Li}_3\left(e^{-\beta x}\right) \\ &\gamma_1(x;\beta) = -\frac{1}{12}\log\left(2\mathrm{sinh}\frac{\beta x}{2}\right) \\ &\gamma_g(x;\beta) = \frac{B_{2g}\beta^{2g-2}}{2g(2g-2)}\ \mathrm{Li}_{3-2g}\left(e^{-\beta x}\right)\end{aligned}$$

- LAST YEAR I CONJECTURED THAT THIS FIVE DIMENSIONAL PARTITION FUNCTION COINCIDES WITH THE FULL TOPOLOGICAL STRING PARTITION FUNCTION ON THE A_{N-1} FIBERED OVER $\mathbf{P}^1$ GEOMETRY. RECENTLY THIS WAS CONFIRMED (INDIRECTLY, VIA GOPAKUMAR-VAFA DUALITY) IN [IQBAL,KASHANI-POOR].

- This implies a **NOVEL EMBEDDING** of the topological string in the superstring.

- After the talk of Eric d'Hoker I would not dare to sketch the argument for this.

- what I planned to say was to lift Ω-background to M-theory and then reduce back to the IIA picture. We would get a varying dilaton, nontrivial 4d metric, and varying graviphoton field strength. Then I would try to mimic the worldsheet argument of [BCOV].

- What I still can say, is that unlike the background of BCOV+ [Antoniadis-Gava-Narain-Taylor], which was:

flat $\mathbf{R}^4$ with constant graviphoton field strength times CY, on which the superstring amplitudes for zero momentum vertex operators calculated the prepotential terms in the effective action, to be inserted in the action as

$$\int d^4x d^d\theta \sum_g \mathcal{F}_g(a,\ldots)\mathcal{W}^{2g}$$

- On Ω-background even the center of mass mode of the string is normalizable, and we get the prepotential plus gravicorrections as the full free energy of the string:

$$\sum_g \mathcal{F}_g(a,\ldots)\hbar^{2g-2}$$

thereby establishing more detailed equivalence with the topological string amplitudes.

DUAL PARTITION FUNCTION AND CHIRAL FERMIONS

- The full partition function has a natural fermionic representation. The physical origin of these fermions is still not completely clear. One way to understand them is to invoke the Chern-Simons/closed string duality of Gopakumar-Vafa, and then the fermionic representation of the large N topological gauge theories [Douglas]. Another possible origin is through the M-theory fivebrane realization of the gauge theory. Mathematically this story is very much related to the Heisenberg algebra representation in the cohomology of the moduli space of the torsion free sheaves on $\mathbf{CP}^2$, studied by H. Nakajima.

QUANTUM ELECTRO-MAGNETIC DUALITY

- LET $\xi_1, \ldots, \xi_N$ BE THE COMPLEX PARAMETERS,

$$\sum_l \xi_l = 0$$

CONSIDER

$$Z^D(\xi; p; \hbar, \Lambda) =$$

$$\sum_{\substack{p_1,\ldots,p_N \in \mathbf{Z}, \\ \sum_l p_l = p}} Z\left(\hbar\left(p_l + \rho_l\right); \hbar, \Lambda\right) \exp\left(\frac{i}{\hbar} \sum_l p_l \xi_l\right)$$

CLEARLY,

$$Z^D(\xi; p+N; \hbar, \Lambda) = Z^D(\xi; p; \hbar, \Lambda)$$

(SHIFT ALL p_l'S BY 1). THUS, THERE ARE ESSENTIALLY N PARTITION FUNCTIONS ONE COULD CONSIDER. THEY ARE LABELED BY THE LEVEL 1 INTEGRABLE HIGHEST WEIGHTS OF $\widehat{\mathbf{sl}}_N$. MOREOVER:

$$Z^D(\xi; p+1; \hbar, \Lambda) = Z^D(\xi^+; p; \hbar, \Lambda)$$

WHERE

$$\xi^+ = (\xi_2, \xi_3, \ldots, \xi_N, \xi_1)$$

TO EXTRACT FROM Z^D THE PARTITION FUNCTION OF INTEREST WE PERFORM A CONTOUR INTEGRAL. IN THE SEARCH FOR PREPOTENTIAL WE ARE ACTUALLY INTERESTED IN THE EXTREMELY HIGH FREQUENCY FOURIER MODES OF Z^D, AS WE WANT $\mathcal{F}_0(\mathbf{a}; \Lambda)$ AS A FUNCTION OF FINITE $a_l = \hbar(p_l + \rho_l)$, WITH $\hbar \to 0$. THIS MEANS THAT THE INVERSE FOURIER TRANSFORM CAN BE EVALUATED USING THE SADDLE POINT, WHICH WE ALREADY ANALYZED.

$$Z^D \sim e^{\sum_{g=0}^{\infty} \hbar^{2g-2} \mathcal{F}_g^D(\xi, \Lambda)}$$

WHERE $\mathcal{F}_0^D$ IS THE ELECTROMAGNETIC DUAL TO $\mathcal{F}_0$.

Free fermions

Introduce N free chiral fermions $\psi^{(l)}$:

$$\psi^{(l)}(z) = \sum_{r\in\mathbf{Z}+\frac{1}{2}} \psi^{(l)}_r \ z^{-r} \left(\frac{dz}{z}\right)^{\frac{1}{2}}$$

$$\widetilde{\psi}^{(l)}(z) = \sum_{r\in\mathbf{Z}+\frac{1}{2}} \widetilde{\psi}^{(l)}_r \ z^{-r} \left(\frac{dz}{z}\right)^{\frac{1}{2}}$$

$$\{\psi^{(l)}_r, \widetilde{\psi}^{(m)}_s\} = \delta_{lm}\delta_{r+s}$$

which can also be packed into a single chiral fermion Ψ

$$\Psi_r, \widetilde{\Psi}_r, \qquad r \in \mathbf{Z} + \tfrac{1}{2}$$

$$\{\Psi_r, \widetilde{\Psi}_s\} = \delta_{r+s}$$

$$\Psi(z) = \sum_{r\in\mathbf{Z}+\frac{1}{2}} \Psi_r z^{-r} \left(\frac{dz}{z}\right)^{\frac{1}{2}}, \ \widetilde{\Psi}(z) = \sum_{r\in\mathbf{Z}+\frac{1}{2}} \widetilde{\Psi}_r z^{-r} \left(\frac{dz}{z}\right)^{\frac{1}{2}}$$

in the standard Kyoto fashion:

$$\Psi_{N(r+\rho_l)} = \psi^{(l)}_r, \quad \widetilde{\Psi}_{N(r-\rho_l)} = \widetilde{\psi}^{(l)}_r$$

The operators Ψ_r, $\widetilde{\Psi}_s$ act in the standard fermionic Fock space $\mathcal{H}$ (sometimes called an infinite wedge representation). It splits as a sum of Fock subspaces with fixed $U(1)$ charge:

$$\mathcal{H} = \oplus_{p\in\mathbf{Z}}\mathcal{H}[p]$$

Introduce affine $\widehat{U(N)}_1$ currents, which act within $\mathcal{H}[p]$ for any p:

$$J^{ln}(z) =: \psi^{(l)}\widetilde{\psi}^{(n)} := \sum_{k\in\mathbf{Z}} \frac{dz}{z^{k+1}} \sum_{r\in\mathbf{Z}+\frac{1}{2}} : \psi^{(l)}_r \widetilde{\psi}^{(n)}_{k-r} :$$

Here we normal order with respect to the vacuum $|0\rangle$, which is annihilated by

$$\Psi_r|0\rangle = 0, \quad r > 0$$

$$\widetilde{\Psi}_s|0\rangle = 0, \quad s > 0$$

which is equivalent to

$$\psi_r^{(l)}|0\rangle = 0, \quad r > 0,$$
$$\widetilde{\psi}_s^{(l)}|0\rangle = 0, \quad s > 0$$

The normal ordered product is simply:

$$: \Psi_r \widetilde{\Psi}_s := \begin{cases} \quad \Psi_r \widetilde{\Psi}_s, & s > 0 \\ - \quad \widetilde{\Psi}_s \Psi_r, & r > 0 \end{cases}$$

$$: \psi_r^{(l)} \widetilde{\psi}_s^{(m)} := \begin{cases} \quad \psi_r^{(l)} \widetilde{\psi}_s^{(m)}, & s > 0 \\ - \quad \widetilde{\psi}_s^{(m)} \psi_r^{(l)}, & r > 0 \end{cases}$$

One can also introduce vacua with different overall $U(1)$ charges:

$$\Psi_r|p\rangle = 0, \quad r > p,$$
$$\widetilde{\Psi}_s|p\rangle = 0, \quad s > -p$$
$$|p\rangle \in \mathcal{H}[p]$$

It is also useful to work with Ψ and the corresponding $\widehat{U(1)}_1$ currents:

$$J(z) =: \Psi\widetilde{\Psi} : (z) = -\frac{1}{N}\sum_{l,m} J^{lm}(z^{-N})z^{l-m}$$

and Virasoro generators:

$$L_0 = \sum_{r\in\mathbf{Z}+\frac{1}{2}} r \quad : \Psi_r \widetilde{\Psi}_{-r} :$$

Note:

$$L_0|p\rangle = \frac{p^2}{2}|p\rangle$$

Bosonization

It is sometimes convenient to work with the chiral boson:

$$\phi(z) = q - i\mathcal{J}_0 \, \log{(z)} + i\sum_{n\neq 0}\frac{1}{n}J_n \; z^{-n}$$
$$\partial\phi \equiv z\partial_z\phi = -iJ$$
$$[q, J_0] = i$$
$$\Psi(z) =: e^{i\phi(z)} : , \qquad \widetilde{\Psi}(z) =: e^{-i\phi(z)} :$$

We will also use a truncated boson, with the zero mode q removed:

$$\varphi(z) = -i\mathcal{J}_0 \log(z) + i\sum_{n\neq 0}\frac{1}{n}J_n\ z^{-n}$$

The space $\mathcal{H}[p]$ is actually an irreducible representation of the Heisenberg algebra generated by $\mathcal{J}$:

$$\mathcal{H}[p] = \mathrm{Span}_{n_1,\dots,n_k>0}\ J_{-n_1}\dots J_{-n_k}|p\rangle, \qquad J_0\Big|_{\mathcal{H}[p]} = p$$

We also recall:

$$\Psi : \mathcal{H}[p] \to \mathcal{H}[p+1], \qquad \widetilde{\Psi} : \mathcal{H}[p] \to \mathcal{H}[p-1]$$

Dual partition function as a current correlator

- WE CLAIM:

$$Z^D(\xi; p; \hbar, \Lambda) = \langle p | e^{\frac{1}{\hbar} \oint \mathrm{Tr} E_+(z) J(z)} e^{\oint \frac{1}{\hbar} \mathrm{Tr} H(z) J(z)} \Lambda^{2L_0} e^{-\frac{1}{\hbar} \oint \mathrm{Tr} E_-(z) J(z)} | p \rangle$$

- THE MATRICES $E_\pm, H$ ARE GIVEN BY:

$$E_+(z) = z E^{N,1} + \sum_{l=2}^{N} E^{l-1,l},$$

$$H(z) = \sum_l \xi_l \; E^{l,l}$$

$$E_-(z) = z^{-1} E^{1,N} + \sum_{l=2}^{N} E^{l,l-1}$$

Affine algebras and arbitrary gauge groups

- Note that Z^D can be expressed in terms of the Chevalley generators e_i, f_i, h_i of the affine Lie algebra $\widehat{\mathbf{sl}}_N$:

$$Z^D(\xi; p; \hbar, \Lambda) = \left(u_\hbar, \Lambda^{2h_0} e^{\mathbf{H}_\xi} u_\hbar\right)_{V_{\omega_p}},$$

$$u_\hbar = \exp\left(\frac{N}{\hbar}\sum_{l=1}^{N} f_{l-1}\right) v_0$$

$$\mathbf{H}_\xi = \frac{1}{\hbar}\sum_{l=1}^{N-1} (\xi_l - \xi_{l+1})\, h_l$$

where V_{ω_p} is the integrable highest weight module with the highest weight vector v_0 (annihilated by the simple roots e_l) and the highest weight ω_p, $p = 1, \ldots, N$.

- We get an obvious generalization to any simple Lie algebra $\widehat{\mathbf{g}}$.

- Conjecturally, it gives the partition function in the Ω-background in the $\mathcal{N} = 2$ gauge theory with the gauge group, S-dual to G (Langlands dual).

- **THIS MAY WORK EVEN FOR** E, F, G seria as well.

Dual partition function and $\mathfrak{gl}(\infty)$

- BACK TO $SU(N)$.

- IN THE LANGUAGE OF THE SINGLE FERMION Ψ:

$$Z^D(\xi;p;\hbar,\Lambda) = \langle p| e^{\frac{\mathcal{J}_1}{\hbar}} e^{\mathbf{H}_\xi} \Lambda^{2L_0} e^{\frac{\mathcal{J}_{-1}}{\hbar}} |p\rangle$$

WHERE $\mathbf{H}_\xi$ IS A DIAGONAL MATRIX:

$$\mathbf{H}_\xi = \frac{1}{\hbar} \sum_r \xi_{(r+\frac{1}{2})\mathrm{mod}N} : \Psi_r \widetilde{\Psi}_{-r} :$$

CLEARLY, $[\mathbf{H}_\xi, L_0] = 0$. THIS FORMULA EXPRESSES THE DUAL PARTITION FUNCTION AS AN AVERAGE WITH THE PLANCHEREL MEASURE OF THE N-PERIODIC WEIGHT $e^{\mathbf{H}_\xi}$.

- Z^D IS THE TAU-FUNCTION OF THE TODA LATTICE HIERARCHY, THANKS TO THE RESULTS OF [ORLOV-TAKEBE-OKOUNKOV-PANDHARIPANDE], WITH THE SPECIFIC PARAMETERIZATION OF THE TIMES, LEADING TO [UENO-TAKASAKI] FORMULATION OF TODA HIERARCHY.

- THE FUNCTION Z^D OBEYS TODA EQUATION:

$$4\partial^2_{\log(\Lambda)} \log\left(Z^D(p)\right) = \frac{Z^D(p+1)Z^D(p-1)}{(Z^D(p))^2}$$

IT IS EASY TO CALCULATE YUKAWA COUPLINGS IN KYOTO, NEAR RIMS

$\mathcal{N}=2$ THEORY WITH ADJOINT HYPERMULTIPLET

- THIS THEORY IS ULTRAVIOLET FINITE, AND IS THUS CHARACTERIZED BY THE MICROSCOPIC COUPLING $\tau_0 = \frac{\vartheta_0}{2\pi} + \frac{4\pi i}{g_0^2}$, AND BY THE MASS $\mathbf{m}$ OF THE HYPERMULTIPLET.

$$q = e^{2\pi i \tau_0}$$

COUNTS INSTANTONS.

- THE PREPOTENTIAL OF THE LOW-ENERGY EFFECTIVE THEORY IS EXPECTED TO HAVE THE FOLLOWING EXPANSION:

$$\begin{aligned}\mathcal{F}_0(\mathbf{a},\mathbf{m},q) =& \pi i \tau_0 \sum_l a_l^2 - \\ & -\frac{1}{2}\sum_{l\neq n}\left[(a_l-a_n)^2 \log(a_l-a_n) - (a_l-a_n+\mathbf{m})^2\log(a_l-a_n+\mathbf{m})\right] + \\ & + \sum_{k=1}^{\infty} q^k f_k(\mathbf{a},\mathbf{m})\end{aligned}$$

THE PREVIOUS CALCULATIONS OF THE COEFFICIENTS f_k FOR LOW VALUES OF k [HOLLOWOOD, NEKRASOV].

- ONE OF THE REMARKABLE FEATURES OF THE PREPOTENTIAL OF THE LOW-ENERGY EFFECTIVE THEORY OF THE THEORY WITH ADJOINT HYPERMULTIPLET IS ITS RELATION TO THE ELLIPTIC CALOGERO-MOSER SYSTEM [DONAGI-WITTEN + GORSKY-NEKRASOV].

- THE COEFFICIENTS f_k OF THE PREPOTENTIAL OF THE CALOGERO-MOSER SYSTEM WERE CALCULATED FOR $k = 1, 2$ IN, E.G. [D'HOKER-PHONG-KRICHEVER].

- OUR METHOD ALLOWS TO **DERIVE** THIS.

- PARTITION FUNCTION AND ITS DUAL ARE THE CONFORMAL BLOCKS ON A TORUS WITH THE MODULAR PARAMETER τ_0, E.G.:

$$Z^D(\xi, p; \hbar, \mathbf{m}, q) = \mathrm{Tr}_{\mathcal{H}_p}\left(q^{L_0}\mathcal{V}_\mu(1)\ e^{\mathbf{H}_\xi}\right)$$

WHERE

$$\mathcal{V}_\mu(z) = :e^{i\mu\varphi(z)}: = \exp\left(-\mu\sum_{n>0}\mathcal{J}_{-n}\frac{z^n}{n}\right) z^{\mu\mathcal{J}_0} \exp\left(\mu\sum_{n>0}\mathcal{J}_n\frac{z^{-n}}{n}\right)$$

- MODULAR PROPERTIES OF THIS PARTITION FUNCTION REFLECT MONTONEN-OLIVE S-DUALITY OF THE $\mathcal{N}=4$ THEORY [VAFA-WITTEN].

CONCLUSIONS AND FUTURE DIRECTIONS

- We have got the theory of SW curves and differentials under control.
- *Extend to non-classical groups?*
- *String loop corrections in terms of the chiral theory on the curve*
- *Superpotential is allowed in the case* $\epsilon_1 + \epsilon_2 = 0$. *Break susy and derive Dijkgraaf-Vafa.*

- Some hints [work in progress with S.Shatashvili]: In the case of pure theory and the theory with adjoint our partition function is:

$$\sum_{\widehat{\Phi}} \Delta^2(\widehat{\Phi})$$

$$\sum_{\widehat{\Phi}} \Delta^2(\widehat{\Phi}) \frac{1}{\mathrm{Det}(\mathbf{m} + \mathrm{Ad}\widehat{\Phi})}$$

respectively, modulo regularization etc. The matrix

$$\widehat{\Phi}$$

is infinite, but the spectrum is discrete:

$$\widehat{\Phi}_{l,i} = a_l + \hbar(k_{li} - i)$$

And we take $\hbar \to 0$ limit. Compare with DV, who take finite matrix, integrate over eigenvalues, and take the limit where it is infinite.

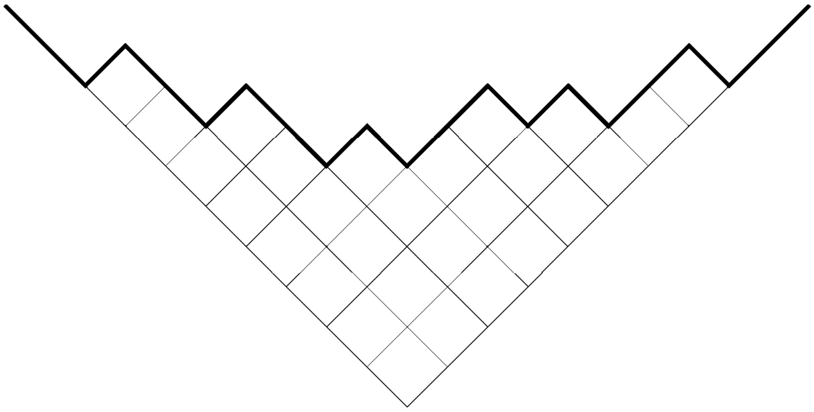
- **Word of caution**: The analogy is suggestive, but breaks down for the theory with fundamental matter.

- We have evaluated the partition function by the saddle point. This, in fact, was the *saddle point of saddle points.* It only exists for real a_l's, Λ, and in the quasiclassical region. Analyticity is enough to prove the relation to SW curves everywhere on the moduli space. It also predicts an interesting analytic structure of the full partition function Z, *which would be nice to characterise more precisely. Holomorphic anomaly equation?.*

- We can easily perturb the theory by the single trace operators. This leads to the simple Whitham-like deformations [Krichever] of the curves+differentials. *What happens for multi-trace operators? What happens to dual string?Make contact to [Aharony-Silverstein-...]*

- *Gromov-Witten dual of Calogero-Moser?* In $N = 1$ case with $\mathbf{m} = 0$ this is simply the mirror of the GW theory of elliptic curve [Okounkov-Pandharipande-Dijkgraaf-BCOV-Douglas-Gross-Taylor].

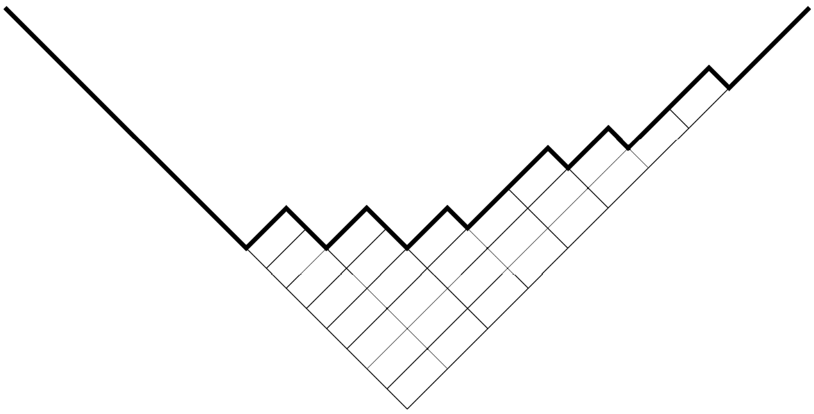
The standard geometric object associated to a partition is its Young diagram. For example, the following figure



Young diagram in the Russian form

shows the diagram of the partition $(8, 6, 5, 3, 2, 2, 1, 1)$. The upper boundary of the diagram of $\mathbf{k}$ is a graph of a piecewise-linear function $f_{\mathbf{k}}(x)$ which we call the *profile* of the partition $\mathbf{k}$.

For general $\epsilon_2 > 0 > \epsilon_1$, it is convenient to extend the definition of $f_{\mathbf{k}}(x)$ by scaling the two axes by $-\epsilon_1$ and ϵ_2, respectively. For example, for $(\epsilon_1, \epsilon_2) = (-1, \frac{1}{2})$, the scaled diagram of the same partition $(8,6,5,3,2,2,1,1)$ and the corresponding profile $f_{\mathbf{k}}(x|\epsilon_1, \epsilon_2)$ will look as follows



Squeezed Young diagram

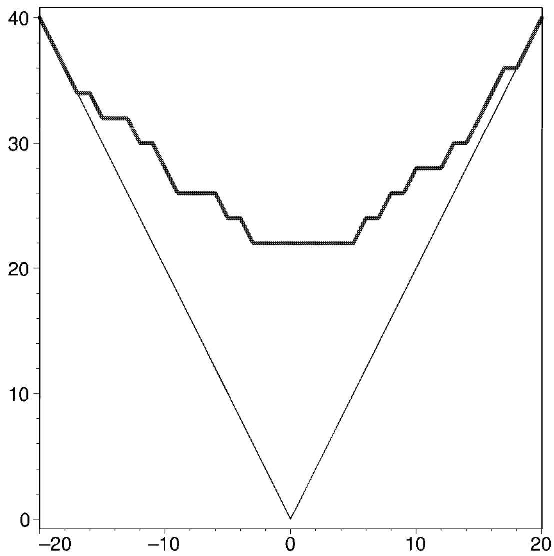
For an N-tuple of partitions $\vec{\mathbf{k}} = (\mathbf{k}_1, \ldots, \mathbf{k}_N)$ and a vector $\mathbf{a}$ we define:

$$f_{\mathbf{a};\vec{\mathbf{k}}}(x|\epsilon_1, \epsilon_2) = \sum_{l=1}^{N} f_{a_l;\mathbf{k}_l}(x|\epsilon_1, \epsilon_2)$$

For example, for $\epsilon_2 = -\epsilon_1 = \hbar$, $a_1 = -a_2 = 11\hbar$ and the partition

$$\{(7,4,3,3,2,1),(8,7,4,4,3,1)\}$$

the corresponding profile looks as follows:



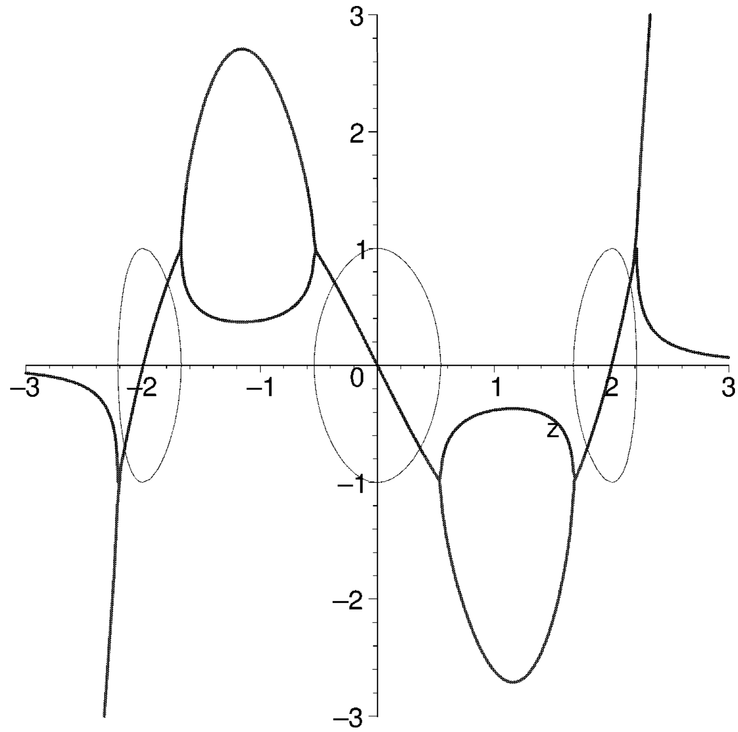
The profile of the colored partition $\{(7,4,3,3,2,1),(8,7,4,4,3,1)\}$

An $SU(3)$ example

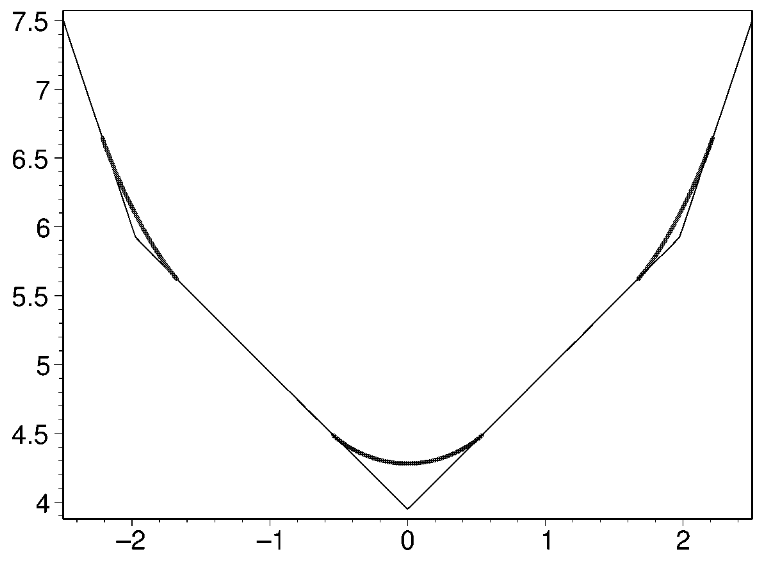
Here is an example of a limit shape $f(x)$. Take

$$P(z) = z^3 - 4z\,.$$

To visualize the curve $w + \frac{1}{w} = P(z)$, let's look at the plots of $\Re(w)$ and $\Im(w)$ for $z \in [-3, 3]$ plotted in bold and normal, respectively



And here is the limit shape



7.5
7
6.5
6
5.5
5
4.5
4
−2
−1
0
1
2

Example of the series evaluation by the saddle point

Suppose you want to evaluate the leading $\hbar \to 0$ asymptotics of

$$\sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{x}{\hbar}\right)^n$$

For $x/\hbar \in \mathbf{R}_+$ this can be done by the saddle point. Note that the saddle point is at:

$$n = x/\hbar$$

which may not be integer. This is analogous to our evaluation of the sum over partitions:

$$n \quad \leftrightarrow \quad \textit{profile of the partition } f_{\vec{a};\mathbf{k}}$$

$$n = x/\hbar \textit{ at the saddle point} \quad \leftrightarrow \textit{ limit profile } f(x)$$

The answer in our toy model is analytic in x and once we evaluated it by the saddle point for large positive $x/\hbar$ can be continued to any complex x. Similarly, our answer can be continued to any complex a_l's.