A Perturbative Window into Non-perturbative Physics

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$N=1$ Gauge theories $d=4$

Theory with four supercharges

+ 4 super directions $(\theta^{\alpha}, \theta^{\dot{\alpha}})$

$(\alpha, \dot{\alpha})$ spinors

$$\int d^4x\, d^4\theta \ldots + \left[\int d^4x\, d^2\theta \ldots + \text{c.c.} \right]$$

F_terms

$\downarrow$

dictates IR physics

A "Long" time ago:

(Veneziano - Yankielowicz + Taylor)

Considered glueball superfield

$$S = \mathrm{Tr}\, W_\alpha W^\alpha$$

W_α: Gluino chiral superfield

Assumption: IR dynamics is described in terms of S.

$$\int d^4x\, d^2\theta\, \underbrace{W(S)}_{\text{superpotential}} = \int d^4x\, d^2\theta \left(N S \left[\ln \tfrac{S}{\Lambda^3} - 1 \right] + \tau S \right)$$

for pure $U(N)$ YM

follows from ABJ anomaly

IR: $\frac{dW}{dS} = 0 \rightarrow \ln \frac{S}{\Lambda^3} = -\tau/N \Rightarrow S = \Lambda^3 e^{-\tau/N}$

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With massive matter → integrate out the matter and obtain glueball superpotential:

$$W(S) = W_{vr} + \sum_{n\geq 1} \alpha_n S^n$$

Proposal with Dijkgraaf:

α_n can be computed for gauge theories admitting large N description. In particular for U(N) with adjoints:

$$W(S) = N \frac{\partial F_0}{\partial S}$$

where F_0 planar amplitudes of associated matrix model

$$Z = e^{\sum \lambda^{2g-2} F_g(S)} = \int D\phi_i \, e^{-\frac{W(\Phi_i)}{\lambda}} \qquad (4$$

$S = \lambda M$ = 't Hooft parameter of MM

This was motivated by string dualities.

Moreover the string dualities implied that the computation of glueball superpotentia[l] should be visible perturbatively in the original gauge theory:

diagrams with n-loops of charged matter $\longrightarrow$ $\alpha_n S^n$

$$W(S) = N\,S\left(\ln \frac{S}{\Lambda^3} - 1\right) + \sum_{n \geq 1} \alpha_n S^n - \tau S$$

$$\frac{dW}{dS} = 0 \implies W\Big|_{min} = \sum_m b_m e^{-\frac{m\tau}{N}}$$

$$S^m \to e^{-\frac{m\tau}{N}}$$

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perturbative n-loop $\longleftrightarrow$ non-perturbative n fractional instanton

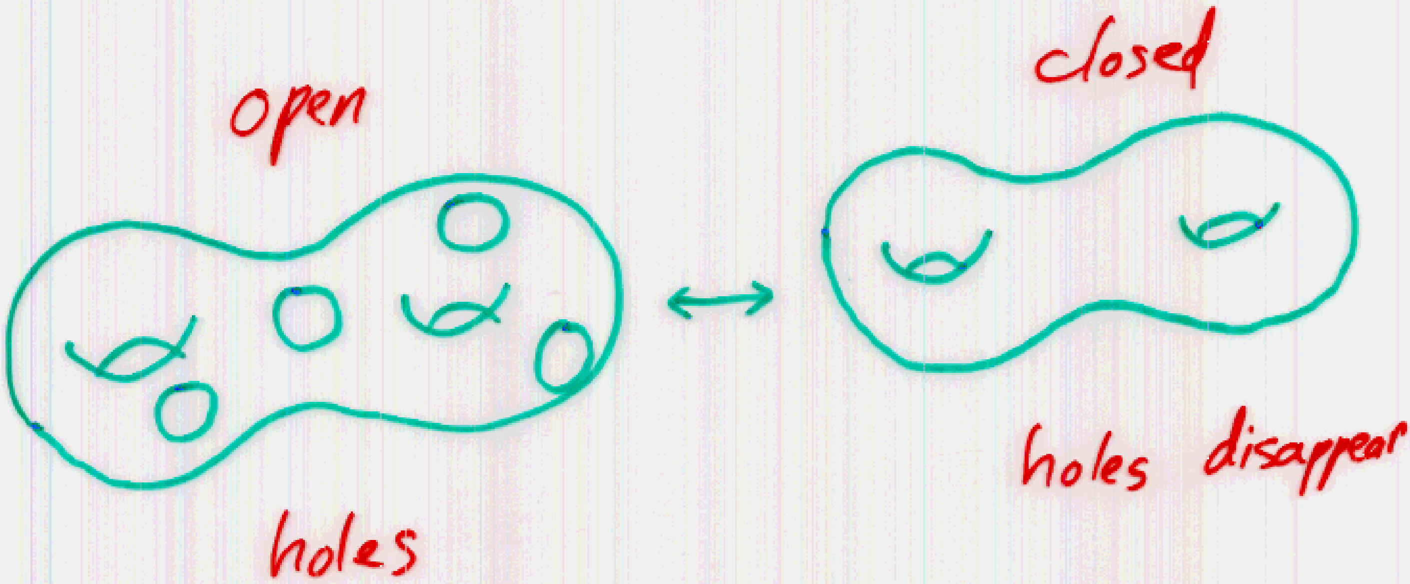
This has led to a new perturbative understanding of non-perturbative effects of $\mathcal{N}=1$, $d=4$ susy gauge theories:

$\mathcal{N}=1$, 2, 4

- $\mathcal{N}=1$ $\dashrightarrow$ Seiberg-like dualities
- $\mathcal{N}=2$ $\dashrightarrow$ Seiberg-Witten geometry
- $\mathcal{N}=4$ $\dashrightarrow$ Montonen-Olive duality

Also: New results have been obtained for various $\mathcal{N}=1$ theories for which no previous results were known. Also new dualities have been discovered using these results.

The stringy motivation for this conjecture came from open/closed string dualities:



Step 1 : Open/closed duality in topological string — Gopakumar, V.

Step 2 : Embed this in superstring — V.
Cachazo, Intriligator, V.

Step 1 : Improved understanding via a worldsheet derivation. Ooguri, V.

<u>idea</u>: Consider closed string side near $t = N g_s \to 0$ and obtain "holes":

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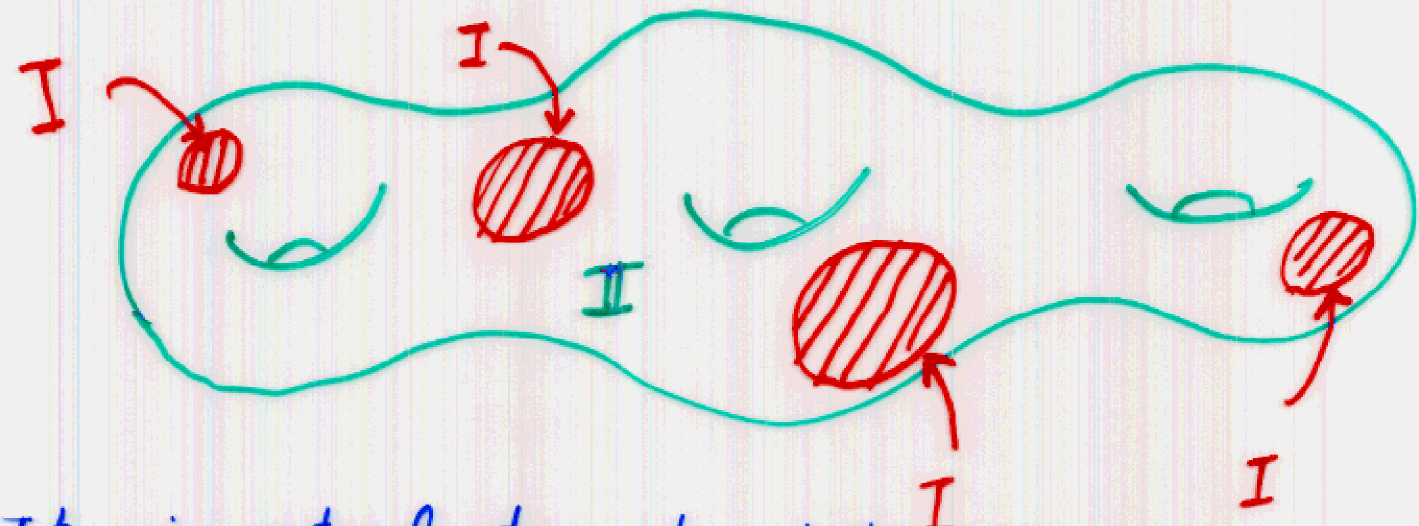
closed string worldsheet theory develops

2 phases $t \to 0$:

phase I $\longrightarrow$ integrate out $\longrightarrow$ holes factor t

phase II $\longrightarrow$ Bulk of Riemann surface



It is natural to ask whether the step 2 can also be improved by a direct worldsheet derivation of this duality also for superstrings.

There is a progress in this direction recently: Berkovits, Ooguri, V. (work in progress)

Consider Type II superstrings on

$CY^3 \times R^4$

Berkovits formalism:

$$\Big(\underbrace{X^\mu , \underset{s=0}{\theta^\alpha_{L,R}} , \underset{1}{p^\alpha_{L,R}}}_{\text{topological } \sigma\text{-model on } \mathbb{C}^2} ; \underbrace{\underset{0}{\theta^{\dot\alpha}_{L,R}} , \underset{1}{p^{\dot\alpha}_{L,R}} , \underset{\text{ghost}}{\rho}}_{\text{decouple from F-term computations}} \Big) + \underbrace{\text{topological } \sigma\text{-model on } CY^3}$$

topological σ-model on $\mathbb{C}^2$ → topological string on $\mathbb{C}^2 \times CY^3$ ← topological σ-model on CY^3

The main new ingredient is to incorporate

RR-flux on the closed string side:

Befor turning on flux

$$\int t \cdot \Sigma \xrightarrow{\text{flux}} \int \left(t + N(\theta_L - \theta_R)^2\right) \cdot \Sigma$$

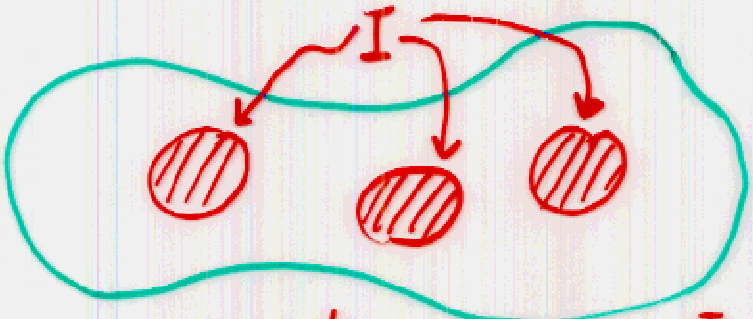
super-worldsheet

size of $\mathbb{P}^1$

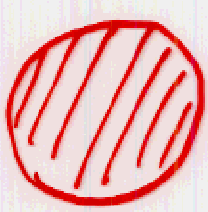
('t Hooft parameter)

Kähler class field

Again phase I, phase II:



integrate out phase I:

 $\longrightarrow \left(t + N(\theta_L - \theta_R)^2\right)|0\rangle$

$$= \mathrm{Tr}\, e^{\xi W^\alpha p_\alpha} (\theta_L - \theta_R)^2$$

where we introduce N-dimensional D-brane Hilbert space.

$$\left(p_\alpha \to \frac{\partial}{\partial \theta_\alpha} \right) \Rightarrow \boxed{\mathrm{Tr}\, W_\alpha^2 = t} \text{ as expected.}$$

$(\theta_L - \theta_R)^2 \longleftrightarrow$ Neumann b.c. for D-brane.

The stringy motivation for the conjecture also predicts that diagram by diagram gauge theory Feynman diagrams should reduce to the corresponding matrix model diagram. This prediction motivated a direct field theory derivation of the superpotential for the glueball superfield $W(S)$.

Dijkgraaf, Grisaru, Lam, Zanon, V.

(See Daniela's talk)

Another field theoretic derivation has also been found, based on a generalization of Konishi anomaly. Cachazo, Douglas, Seiberg, Witten

Extensions

$d > 4$

What about susy theories $d \neq 4$?

What is MQM good for?

Dijkgraaf, V.

See Robbert's talk

Other Groups + Reps

$W(S) = ?$

Aganagic, Intriligator, Warner, V.

$g > 0$

What do non-planar diagrams correspond to in susy gauge theory?

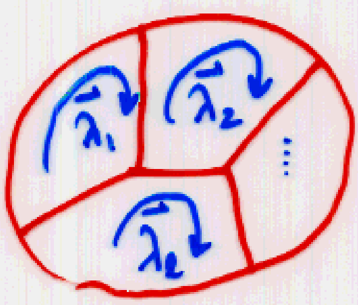
Ooguri, V.

Generalization to other Groups / Reps

Aganagic, Intriligator, Warner, V.

Using Feynman Diagram techniques (see Daniela's talk) one obtains the glueball superpotential:

$$W(S)=\sum_{\substack{\ell\text{-loops}\\ \vec{\lambda}}} S^{\ell}\, F(\vec{\lambda})\cdot\left(\det_{\ell\times\ell}(\vec{\lambda}_a\cdot\vec{\lambda}_b)\,\frac{(r-\ell)!}{r!}\right)$$



$\vec{\lambda}_a$: Cartan weights in each loop

$F(\vec{\lambda})$: corresponding matrix model amp. with weight $\vec{\lambda}$

r: rank of G.

for $U(N)$ + adjoints

$\det(\vec{\lambda}_a\cdot\vec{\lambda}_b)\propto \mathbb{P}_{roj.}$ Planar diagrams

Note: this expression for $W(S)$ is unambiguous for $\ell\leq r$: $\left\{\begin{array}{l}\det\to 0\\ (r-\ell)!\to\infty\end{array}\right.$

related: $S^k_{\text{classic}}=0$, k large enough.

For classical groups (A, B, C, D) the ambiguity for S^k $k>l$ can be resolved: Compute $\sum \alpha_n(N) S^n$

for $N > n$: $\alpha_n(N)$ unambiguous

analytically continue $\alpha_n(N)$ to all N. (Equivalent to insertion of gluino prescr.: two gluino insertions per 't Hooft index loop)

Interpretation:

consider $U(N)$ case as an example.

$$U(N) \overset{\text{embed}}{\subset} U(N+k) \times U(k)$$

(related to embeding in $U(N+k|k)$ supergroup; see Robbert's talk)

k: brane/anti-brane pairs

$k \to \infty$ leads to the above prescription.

$$U(N+k)\times U(k) \xrightarrow[\text{Higgses}]{} U(N+k-1)\times U(k-1)$$

$$\to \cdots \to U(N)$$

One can show that F-terms generically do not depend on k → justifying the prescription. For some very special reps. there are residual instanton effects which change the F-terms for small k.

(Explains puzzles raised by Kraus + Shigemori)

For more general groups (E, F, G) no general extension for S^m known for large m. 15

In some special cases one can use "analytic continuation in group" to get answer.

(e.g. $\mathcal{N}=1^*$ $\quad W(S)=\sum c_\ell S^\ell \quad \dfrac{c_\ell(G)}{c_\ell(U(N))}=\dfrac{\sum_i p_i^\ell}{N}$

p_i = Dynkin #'s

C-Deformation
+
non-planar diagrams

Ooguri + V.

Standard $\mathcal{N}=1$ gauge theory

Glueball superpotential $\Leftrightarrow$ Planar diagrams of associated matrix model

$$W(S) = N \frac{\partial F_0}{\partial S}$$

? $\leftarrow$ non-planar

Hint: Closed topological string: turning on self-dual 4d graviphoton field strength $F_{\alpha\beta}$ measures non-planarity ($\lambda_s^2 \Leftrightarrow F_{\alpha\beta} F^{\alpha\beta}$)

BCOV
AGNT '93

We thus ask what is the effect of turning on graviphoton on D-branes.

$$S = \int p^L_\alpha \partial\theta^\alpha_L + p^R_\alpha \bar\partial\theta^\alpha_R + \alpha'^2 \underline{F^{\alpha\beta} p^L_\alpha p^R_\beta} + \cdots$$

Very similar reasoning to bosonic non-commutativity:

$$\{\theta^\alpha, \theta^\beta\} = \alpha'^2 F^{\alpha\beta}$$

Ooguri, V.

In other words superspace is deformed.

We can consider a field theory limit for which this effect survive: $\begin{cases} \alpha' \to 0 \\ F^{\alpha\beta} \to \infty \end{cases}$ $\alpha'^2 F^{\alpha\beta} = f^{\alpha\beta}$ finite

This has been studied further

de Boer, Grassi, Van Nieuwenhuizen

Seiberg

Berkovits + Seiberg

⋮

Even though this is an interesting limit to consider in its own right, it is not the limit which yields connection between non-planar diagrams of matrix model + F-terms of $\mathcal{N}=1$ gauge theories.

Some susy's are broken in this limit $\longrightarrow$ F-terms don't make sense.

to restore susy $\rightarrow$ add a boundary tepm on Riemann surface

$$\int P^L_\alpha \partial \theta^{\alpha L} + P^R_\alpha \bar{\partial} \theta^{\alpha R} + \alpha'^2 F_{\alpha\beta} P^\alpha_L P^\beta_R \underbrace{+ \alpha'^2 \oint P^\alpha_{(z)} \int P^\beta \cdot F_{\alpha\beta}}_{\text{boundary term}}$$

$\rightarrow$ undeforms superspace

$\rightarrow$ restores susy ($\simeq$ topological invariance)

Boundary $e^{\alpha' \oint W^\alpha P_\alpha}$; $P_\alpha = P^L_\alpha + P^R_\alpha$ (18

$\Longrightarrow$ boundary term can be intepreted non-standard W_α:

$$\{W_\alpha, W_\beta\} = F_{\alpha\beta} \quad (+\bar{D}(\cdots))$$

: C-deformation

we consider limit $\begin{cases} \alpha' \to 0 \\ F_{\alpha\beta} : \text{finite} \end{cases}$

One can also consider gravitational backgrounds $E_{\alpha\beta\gamma}$ (gravitino super field)

$$\{W_\alpha, W_\beta\} = F_{\alpha\beta} + \underbrace{E_{\alpha\beta\gamma} W^\gamma}_{\text{easy to show}} + \bar{D}(\ldots)$$

easy to show using standard s-tensor calculus

The corresponding F-terms:

$$\Gamma = \int d^4x\, d^2\theta \left[\int d^2\hat{\theta} \left(F_{\alpha\beta} + \hat{\theta}^\gamma E_{\alpha\beta\gamma} \right)^{2g} F_g (S + \hat{\theta}^\alpha w_\alpha + N \hat{\theta}^2 \right.$$

includes $\boxed{W(S) = \lambda^2 N \frac{\partial F}{\partial S}(S, \lambda^2) + \tau S}$

$F(S, \lambda^2) = \sum \lambda^{2g-2} F_g(S)$

$\lambda^2 = F_{\alpha\beta} F^{\alpha\beta}$

This also includes

$$\Gamma = \sum_{g=1}^{\infty} \int d^4x \, d^2\theta \, \left(E_{\alpha\beta\gamma} E^{\alpha\beta\gamma} \right) F_g(S) \lambda^{2g-2}$$

These can be derived again using diagrammatic technique. Ooguri, V.

(some subtlety for $(E_{\alpha\beta\gamma})^2$ contribution at genus 0 → can be removed by field redefinition)

Dijkgraaf, Grisaru, Ooguri, Zanon

Γ has been checked for certain cases against gauge theory prediction for $\lambda = 0, g = 1$

Dijkgraaf + V.

Dijkgraaf, Sinkovics, Temurhan

Klemm, Marino, Theisen

These results have also been derived using Konishi anomaly.

{ Adlay, Cirafici, David, Gava, Narain
{ David, Gava, Narain

C-deformed Pure $\mathcal{N}=1$ YM

$$\{W_\alpha, W_\beta\} = F_{\alpha\beta}$$

$$W(S) = \lambda^2 N \frac{\partial F(S,\lambda^2)}{\partial S} + \tau S$$

$$F(S,\lambda^2) = F_{\text{measure}}(S,\lambda^2) + F_{\text{pert.}}(S,\lambda^2)$$

for pure $U(N)$ $\quad F_{\text{pert.}} = 0$

$$F_{\text{measure}} = \log \mathrm{Vol}(U(N)) \leftarrow \text{"R.S. = hole"}$$

$$= \frac{1}{2}\mu^2 \ln\mu - \frac{1}{12}\ln\mu + \sum_{g>1} \frac{B_{2g}}{2g(2g-2)} \mu^{-2g+2}$$

$$\mu = \frac{S}{\lambda}$$

$$\lambda^2 = F_{\alpha\beta}F^{\alpha\beta}$$

$$\frac{1}{\lambda} W = \underbrace{N\left[\mu\ln\mu - \sum_{g>0}\frac{B_{2g}}{2g}\mu^{1-2g}\right]}_{\text{generalization of VY superpot. to include c-deformation}} + \tau\mu$$

$$F_{\text{measure}}(\mu) = \text{free energy of non-critical bosonic string } c=1 \text{ at self dual radius} = \log \mathrm{Vol}[U(N)]$$

$\mu \longrightarrow$ cosmological constant

This is not an accident:

$\mathcal{N}=1$ pure YM is related to $\leftrightarrow$ B-model on conifold $\longleftrightarrow$ $c=1$ at self dual radius

Ghoshal, V.

(See Robbert's talk)

Integrability

Special class of $\mathcal{N}=1$ susy gauge theories $\longleftrightarrow$ Solvable matrix models

captured by: $F(y, x) = 0$

Riemann Surface encodes large $\underline{N}$ eigenvalue distribution of matrix model.

Loop eqns

Virasoro costraints

⋮

capture integrability of matrix model

$\updownarrow$

Can be also understood directly from the "closed string" side (See Mina's talk)

Kodaira-Spencer theory $F(x,y) = 0$

The fact that Riemann surfaces are made of pants

suggests that once we solve the pant diagram all integrable matrix models (captured by a surface). Thus all F-terms for all integrable models can be obtained from this basic object.

(Mina's talk)

Thus: $\left\{\begin{array}{l} \mathcal{N}=1 \text{ susy gauge theories} \\ \text{integrable structures} \\ \text{non-critical bosonic strings} \end{array}\right\}$ connected

Questions for Future Work

- Connection with Nekrasov's Instanton calculus
  (similar features: integrable structure)

- How to deal with non-integrable matrix models? (No geometry?
  No duality?)

- Deeper understanding of $W(S)$ for higher powers of S especially for non-classical groups

- Gauge theoretic interpretation of $N=4$ topological string?

- Completing closed $\to$ open derivation for superstrings

– Can one recover all non-pert. BPS type string amplitudes from string perturbation theory ?