Solving the Gluing Identity

- Gluing Identity $\Longrightarrow$

- Theorem: We have the identity of cohomology classes on X:

$$b(V)\cdot J_{D_1,D_2}\pi^*b(V_D) = J_{D_1,O}\pi^*b(V_{D_1})\cdot J_{O,D_2}\pi^*b(V_{D_2}$$

- For general X, complete classification of solutions not available.

- Definition: A T-manifold X is called a balloon manifold if
  i. X^T is finite
  ii. (GKM) T-weights on T_pX at fixed point p are pairwise linearly independent.
  iii. The moment map is injective on X^T.

- Examples: projective toric manifolds, flag manifolds.

Problem

• Let $LT_{g,k}(d,X)$ be the virtual fundamental cycle of $\bar{M}_{g,k}(d,X)$. This is a homology cycle in $A_*(\bar{M}_{g,k}(d,X))$ of dimension

$$\langle c_1(X), d\rangle + (1-g)dim(X) + k - 3.$$

• This cycle plays the role of the fundamental cycle of a compact manifold.

• General Problem: Fix a vector bundle E on $M_{g;k}(d,X)$, and a characteristic class $b(E) \in A^*(M_{g,k}(d,X))$. Fix cohomology classes $\omega_1, .., \omega_k$ on X. Study the integrals

$$K_D := \int_{LT_{g,k}(d,X)} e_1^*\omega_1 \cdots e_k^*\omega_k \, b(E).$$

• For simplicity, will restrict to $\omega_1 = \cdots = \omega_k = 1$. All results here have been generalized to the case when ω_i are arbitrary. The class b will be Euler class, Chern polynomial, or more generally any multiplicative class.

Recent Progresses

in Mirror Principle

Mirror Principle:

Compute characteristic numbers on moduli spaces of stable maps $\Leftarrow$ Hypergeometric type series.

Counting curves: Euler numbers.

Hirzebruch multiplicative classes, total Chern classes.

Marked points and general GW invariants.

Plan of the Lecture:

1. Review of Mirror Principle.(Lian, K. Liu, Yau)

2. Examples. Candelas formula, toric mirror formula.

3. Recent works: Discussions on:

(a). Proof of the Hori-Vafa formula. (Lian, C.H. Liu, K. Liu, Yau)

(b). Proof of the Marino-Vafa formula. (C.-C. Liu, K. Liu, J. Zhou)

We hope to develop a "black-box" method which makes easy the computations of the characteristic numbers and the GW-invariants on the moduli space of stable maps:

Starting data $\Longrightarrow$ Mirror Principle $\Longrightarrow$ Closed Formulas for the invariants.

General set-up: X: Projective manifold.

$\mathcal{M}_{g,k}(d,X)$: moduli space of stable maps of genus g and degree d with k marked points into X, modulo the obvious equivalence.

Points in $\mathcal{M}_{g,k}(d,X)$ are triples: $(f; C; x_1, \cdots, x_k)$:

$f : C \to X$: degree d holomorphic map;

$x_1, \cdots, x_k$: k distinct smooth points on the genus g curve C.

$f_*([C]) = d \in H_2(X, \mathbf{Z})$: identified as integral index $(d_1, \cdots, d_n)$ by choosing a basis of $H_2(X, , \mathbf{Z})$ (dual to the Kahler classes).

Virtual fundamental cycle of Li-Tian, (Behrend-Fantechi): $\mathcal{M}_{g,k}(d,X)^v$, a homology class of the expected dimension on $\mathcal{M}_{g,k}(d,X)$.

Consider the case $k = 0$ first.

V: concavex bundle on X, direct sum of a positive and a negative bundle on X.

V induces sequence of vector bundles V_d^g on $\mathcal{M}_{g,0}(d, X)$: $H^0(C, f^*V) \oplus H^1(C, f^*V)$.

b: a multiplicative characteristic class.

Problem: Compute $K_d^g = \int_{\mathcal{M}_{g,0}(d,X)^v} b(V_d^g)$.

Mirror Principle: Compute

$F(T, \lambda) = \sum_{d,\,g} K_d^g \lambda^g \, e^{d \cdot T}$

in terms of hypergeometric type series.

Here λ, $T = (T_1, \cdots, , T_n)$ formal variables.

Key ingredients for the proof of the Mirror Principle:

(1). Linear and non-linear moduli spaces;

(2). Euler data and Hypergeometric Euler data (HG Euler data).

Non-linear moduli: $M_d^g(X) =$ stable map moduli of degree $(1,d)$ and genus g into $\mathbf{P}^1 \times X$.

$= \{(f,C) : f : C \to \mathbf{P}^1 \times X\}$ with C a genus g (nodal) curve.

Linearized moduli: W_d for toric X. (Witten, Aspinwall-Morrison):

Example: $\mathbf{P}^n$, $[z_0, \cdots, z_n]$

W_d: $[f_0(w_0,w_1), \cdots, f_n(w_0,w_1)]$

Question (Summarize)

Given X (a toric manifold or Balloon manifold)

and a vector bundle
$$\begin{array}{c} V \\ \downarrow \\ X \end{array}$$

It induces vector bundles

on $M_{g,k}(d, X)$ by looking at

$$V_d^g = H^0(C, f^*V) \ominus H^1(C, f^*V)$$

Given multiplicative characteristic class

$$K_d^g = \int_{[M_{g,0}(d,X)]^v} b(V_d^g)$$

So we have a map

$$V \longrightarrow F(T, \lambda) = \sum K_d^g \lambda^g e^{d \cdot T}$$

$f_j(w_0, w_1)$: homogeneous polynomials of degree d.

Simplest compactification.

Lemma:(LLY+Li) There exists an explicit equivariant collapsing map

$$\varphi : \ M_d^g(\mathbf{P}^n) \longrightarrow W_d.$$

$M_d^g(X)$ is very "singular" and complicated. But W_d smooth and simple, we push-forward everything to W_d!

Starting data: $V \longrightarrow X$ a concavex(!) bundle. It induces vector bundles V_d^g on $M_{g,k}(d, X)$ by taking $H^0(C, f^*V)$ or $H^1(C, f^*V)$, or by their sum. This gives

$$\begin{array}{c} V_d^g \\ \downarrow \\ M_{g,k}(d,X) \end{array} \xleftarrow{\ \pi\ } M_d^g(X) \xrightarrow{\ \varphi\ } W_d$$

Or in terms of polynomials in equivariant cohomology rings of the Linear moduli space

$$V \longrightarrow \{ Q_d = \varphi_! (\pi^* b (V_d^g)) \}$$

We computed it when V can be splitted as direct sum of line bundles.

What property does this map have when we change V. The maps give instanton sum for II_A theory. The equality with II_B theory is similar to Rieman-Roch formula in sigma model.

Remark: Physicists compute by approximating on linear moduli. Mathematicians compute on nonlinear moduli.

Balloon manifold X: Projective manifold with torus action and isolated fixed points.

$$H = (H_1, \cdots, H_k)$$

a basis of equivariant Kahler classes.

(1). $H(p) \neq H(q)$ for any two fixed points.

(2). T_pX has linearly independent weights for any fixed point p.

Theorem: Mirror principle holds for balloon manifolds for any concavex bundles.

Remarks: All homogeneous and toric manifolds are balloon manifolds. For $g = 0$ we can identify HG series explicitly. Higher genus needs more work.

1. For toric manifolds and $g = 0 \Longrightarrow$ all mirror conjectural formulas from physics.

2. Grassmannian: Hori-Vafa formula.

2. Direct sum of positive line bundles on $\mathbf{P}^n$ (including the Candelas formula): Two independent approaches: Givental, Lian-Liu-Yau.

One of the simple but key techniques:

Functorial localization formula:

$f : X \to Y$ equivariant map. $F \subset Y$ a fixed component, $E \subset f^{-1}(F)$ fixed components in X. Let $f_0 = f|_E$, then

For $\omega \in H_T^*(X)$, we have identity on F:

$$f_{0*}[\frac{i_E^*\omega}{e_T(E/X)}] = \frac{i_F^*(f_*\omega)}{e_T(F/Y)}.$$

There are many applications of this formula. Several later works follow this line.

Apply such formula to φ, the collapsing map and the pull-back class $\omega = \pi^* b(V_d^g)$:

This formula connects the computations of mathematicians and physicists.

Push all computations on the nonlinear moduli to the linear moduli.

Mirror symmetry formula = Comparison of of computations nonlinear and linearized moduli!?

Introduce the notion of **Euler Data**, which naturally appears on the right hand side of the functorial localization formula:

$$Q_d = \varphi_!(\pi^* b(V_d^g))$$

which is a sequence of polynomials in equivariant cohomology rings of the linear moduli spaces (or restricted to X) with simple quadratic relations.

From functorial localization formula we prove that:

Knowing Euler data $Q_d \equiv$ knowing the K_d^g.

There is another much simpler Euler data, the **HG Euler data** P_d which coincides with Q_d on the "generic" part of the nonlinear moduli.

The Gluing Identity

• Enlarge $M_{g,k}(d,X)$ to

$$M_D := M_{g,k}((1,d), \mathbf{P}^1 \times X).$$

The projection $\mathbf{P}^1 \times X \to X$ induces a map

$$M_D \xrightarrow{\pi} M_{g,k}(d,X).$$

Pulling back $b(V_D)$ via π, we get a cohomology class $\pi^* b(V_D)$ on M_D.

• $\mathbf{C}^\times$ acts on $\mathbf{P}^1$ by the standard rotation. This induces an $\mathbf{C}^\times$ action on M_D. Will do localization on M_D relative to this action.

• Each fixed point in M_D comes from gluing pairs in $M_{g_1,k_1+1}(d_1,X) \times M_{g_2,k_2+1}(d_2,X)$ at a marked point x. Here $D = D_1 + D_2$ where $D_i = (g_i, k_i; d_i)$.

Comparison theorem

• There is a version for stable map moduli:

$$i : F_{D_1,D_2} \to M_D$$

plays the role of $i_F : F \to A$. Evaluation map

$$e : F_{D_1,D_2} \to X$$

evaluating at gluing point plays the role of $g : F \to E$.

• Fix a projective embedding $X \subset \mathbf{P}^n$. Each map stable $(f, C, x_1, .., x_k)$ is a degree $(d, 1)$ map into $X \times \mathbf{P}^1 \subset \mathbf{P}^n \times \mathbf{P}^1$:

• Corresponding to this are $n+1$ polynomials $f_i(w_0, w_1)$ each vanishing of order d_i at $[a_i, b_i] \in \mathbf{P}^1$.

• Call this component F_{D_1,D_2}, and $i : F_{D_1,D_2} \to M_D$ inclusion. There are two natural projection maps

$$p_0 : F_{D_1,D_2} \to M_{g_1,k_1+1}(d_1, X)$$
$$p_\infty : F_{D_1,D_2} \to M_{g_1,k_1+1}(d_1, X)$$

Pulling back $b(V'_{D_1})$ via p_0, and $b(V'_{D_2})$ via p_∞, we get cohomology classes $p_0^* b(V'_{D_1})$ and $p_\infty^* b(V'_{D_2})$ on F_{D_1,D_2}.

• Theorem(Gluing Identity): On F_{D_1,D_2} we have identity of cohomology classes:

$$i^* \pi^* b(V_D) = p_0^* b(V'_{D_1})\, p_\infty^* b(V'_{D_2}).$$

• Elementary idea: $f : A \to B$, a G-equiv. map of G manifolds;

$$\begin{array}{cccc} f^{-1}(E) \supset & F & \xrightarrow{i_F} & A \\ & g \downarrow & & \downarrow f \\ & E & \xrightarrow{j_E} & B. \end{array}$$

For $\omega \in H_G^*(A)$, we have identity on E:

$$\frac{j_E^* f_*(\omega)}{e_G(E/B)} = g_* \frac{i_F^*(\omega)}{e_G(F/A)}.$$

• Theorem(LLY+J. Li): The corrrespondence

$$(f, C, x_1, .., x_k) \mapsto [f_0, .., f_n]$$

defines an equivariant morphism $\varphi : M_D \to N_d$ where N_d is the projective space of $(n+1)$-tuple of polynomials of degree d.

• The fixed points in N_d are copies of $\mathbf{P}^n$. There is a similar theorem if we have an embedding $X \subset \mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_m}$. Then N_d is replaced by a product W_d of N_d's. Label the fixed points by Y_{d_1,d_2}, and inclusion

$$j : X \subset Y_{d_1,d_2} \to W_d.$$

• Putting together a commutative square:

$$\begin{array}{ccc} F_{D_1,D_2} & \xrightarrow{i} & M_D \\ e \downarrow & \circ & \downarrow \varphi \\ X & \xrightarrow{j} & W_d. \end{array}$$

• <u>Theorem</u>: (Comparison Theorem) For any equivariant class ω on M_D, we have an identity on X:

$$\frac{j^*\varphi_*(\omega \cap LT_D)}{e(X/W_d)} = e_* \frac{i^*\omega \cap [F_{D_1,D_2}]}{e(F_{D_1,D_2}/M_D)}.$$

Denote the RHS by $J_{D_1,D_2}\omega$.

• <u>Theorem</u>: Consider the integral

$$K_D = \int_{LT_{g,k}(d,X)} b(V_D).$$

Suppose the integrand has the right degree. Then

$$\int_X e^{-H\cdot t} J_{O,D}\pi^* b(V_D) = (-1)^g (2 - 2g - d \cdot t) K_D.$$

• Thus the goal is to compute the numbers K_D by first computing the classes $J_{D_1,D_2}\pi^* b(V_D)$ on X. Let's restrict to $g = 0$ and $k = 0$ for simplicity.

The quadratic relations + coincidence determine the Euler data uniquely up to certain degree.

Q_d always have the right degree for $g = 0$.

Mirror transformation used to reduce the degrees of the HG Euler data P_d.

Example: CY quintic in $\mathbf{P}^4$,

$P_d = \prod_{m=0}^{5d}(5\kappa - m\alpha)$

with α weight from S^1 action on $\mathbf{P}^1$, and κ generator of equivariant cohomology ring of W_d.

Starting data: $V = \mathcal{O}(5)$ on $X = \mathbf{P}^4$ and the hypergeometric series ($\alpha = -1$) is:

$HG[B](t) = e^{Ht} \sum_{d=0}^{\infty} \frac{\prod_{m=0}^{5d}(5H+m)}{\prod_{m=1}^{d}(H+m)^5} e^{dt},$

H: hyperplane class on $\mathbf{P}^4$; t: parameter.

Introduce

$\mathcal{F}(T) = \frac{5}{6}T^3 + \sum_{d>0} K_d^0\, e^{dT}$.

Algorithm: Take expansion in H:

$HG[B](t) = H\{f_0(t)+f_1(t)H+f_2(t)H^2+f_3(t)H^3\}$.

Candelas Formula: With $T = f_1/f_0$,

$$\mathcal{F}(T) = \frac{5}{2}\left(\frac{f_1}{f_0}\frac{f_2}{f_0} - \frac{f_3}{f_0}\right).$$

Remark: Both the denominator and the numerator in the HG series are equivariant Euler classes. Especially the denominator is exactly from the localization formula. Easily seen from the functorial localization formula.

Example: X, toric manifold; $g = 0$.

$D_1, .., D_N$: T-invariant divisors

$V = \oplus_i L_i$, $c_1(L_i) \geq 0$ and $c_1(X) = c_1(V)$.

$b(V) = e(V)$

$\Phi(T) = \sum K_d^0 e^{d \cdot T}$.

HG Euler series: generating series of the HG Euler data.

$$B(t) = e^{-H \cdot t} \sum_d \prod_i \prod_{k=0}^{\langle c_1(L_i), d\rangle} (c_1(L_i) - k)$$

$$\times \frac{\prod_{\langle D_a, d\rangle < 0} \prod_{k=0}^{-\langle D_a, d\rangle - 1} (D_a + k)}{\prod_{\langle D_a, d\rangle \geq 0} \prod_{k=1}^{\langle D_a, d\rangle} (D_a - k)} e^{d \cdot t}.$$

Mirror Principle $\Longrightarrow$ There are explicitly computable functions $f(t), g(t)$, such that

$$\int_X \left(e^f B(t) - e^{-H\cdot T} e(V)\right) = 2\Phi - \sum T_i \frac{\partial \Phi}{\partial T_i}$$

where $T = t + g(t)$.

Easily solved for Φ.

Note the (virtual) equivariant Euler classes in the HG series.

Similar to index formula

topological index = Analytic index

In general we want to compute:

$$K_{d,k}^{g} = \int_{LT_{g,k}(d,X)} \prod_{j=1}^{k} ev_j^* \omega_j \cdot b(V_d^g)$$

where $\omega_j \in H^*(X)$.

Form a generating series

$F(t, \lambda, \nu) = \sum_{d,g,k} K_{d,k}^{g} e^{dt} \lambda^{2g} \nu^{k}$.

Ultimate Mirror Principle: Compute this series in terms of explicit HG series!

The classes induce Euler data: Euler data encode the geometric structure of the stable map moduli.

Example Consider open toric CY, say $\mathcal{O}(-3) \to \mathbf{P}^2$. $V = \mathcal{O}(-3)$. Let

$Q_d = \sum_{g\geq 0} \varphi_!(\pi^* e_T(V_d^g)\, \lambda^{2g}.$

The corresponding HG Euler data is given explicitly by

$P_d\, J(\kappa, \alpha, \lambda) J(\kappa - d\alpha, -\alpha, \lambda).$

Where P_d is exactly the genus 0 HG Euler data and J is generating series of Hodge integrals (sum over all genus): degree 0 Euler data.

Euler data includes computations of all Gromov-Witten invariants and more general. Some closed formulas can be obtained....

Mirror principle holds!:

Need to handle the **degrees** of the Euler data.

Recent Results:

(1) **Refined linearized moduli** for higher genus constructed: A-twisted moduli space A_d^g, which is induced from the linear sigma model of Witten. New and simpler compactification of the moduli space of maps. (C.-H. Liu-Liu-Yau).

(2) **Open mirror principle**: Open string theory: counting holomorphic discs with boundary inside a Lagrangian submanifold; more generally counting number of open Riemann surfaces with boundary in Lagrangian submanifold. Linearized moduli space being constructed which gives a new compactification of the moduli. (C.-H. Liu-Liu-Yau).

(3) **Mirror principle for Calabi-Yau in Grassmannian.** No known linear moduli. We use the quot scheme to play the role of the linear moduli. The method gives a proof of the

Hori-Vafa formula. Details to be explained. (Lian-C.-H. Liu-Liu-Yau, 2001, Bertram et al 2003.)

(4) **Marino-Vafa formula**: Hodge integrals needed to compute the HG Euler data.

Mariño-Vafa formula: Generating series of all triple Hodge integrals on moduli spaces of curves given by close formulas of finite expression in terms of representations of symmetric groups:

Conjectured from large N duality between Chern-Simons and string theory. Proof by combining combinatorics and functorial localization formula, and differential equation by C.-C. Liu, K. Liu and J. Zhou. Detail to be explained .

W 8

Mirror Principle for Grassmannian: Hori-Vafa formula. and flag manifolds

The existence of linear moduli made easy the computations for toric manifolds. Let

$ev: \mathcal{M}_{0,1}(d, X) \to X$

be evaluation map, and c the first Chern class of the tangent line at the marked point. One of the key ingredients for mirror formula is to compute:

$$ev_*[\frac{1}{\alpha(\alpha - c)}],$$

or more precisely the generating series

$$HG[1]^X(t) = e^{-tH/\alpha} \sum_{d=0}^{\infty} ev_*[\tfrac{1}{\alpha(\alpha-c)}]\, e^{dt}.$$

Remark: Assume the linearized moduli exists. Then *functorial localization formula* applied to the collpasing map: $\varphi: \quad M_d \to N_d$, immediately gives the expression as hypergeometric denominator.

Example: $X = \mathbf{P}^n$, then we have $\varphi_*(1) = 1$, functorial localization:

$$ev_*[\frac{1}{\alpha(\alpha-c)}] = \frac{1}{\prod_{m=1}^{d}(x-m\alpha)^{n+1}}$$

where the denominators of both sides are equivariant Euler classes of normal bundles of fixed points. Here x denotes the hyperplane class.

For $X = Gr(k,n)$, no explicit linearized moduli known. Hori-Vafa conjectured a formula for $HG[1]^X(t)$ by which we can compute this series in terms of those of projective spaces:

Hori-Vafa Formula:

$$HG[1]^{Gr(k,n)}(t) = e^{(k-1)\pi\sqrt{-1}\sigma/\alpha}\frac{1}{\prod_{i<j}(x_i-x_j)}.$$

$$\prod_{i<j}(\alpha\frac{\partial}{\partial x_i}-\alpha\frac{\partial}{\partial x_j})|_{t_i=t+(k-1)\pi\sqrt{-1}}HG[1]^{\mathbf{P}}(t_1,\cdots,t_k)$$

where $\mathbf{P} = \mathbf{P}^{n-1}\times\cdots\times\mathbf{P}^{n-1}$ is product of k copies of the projective spaces. Here σ is the generator of the divisor classes on $Gr(k,n)$ and x_i the hyperplane class of the i-th copy $\mathbf{P}^{n-1}$:

$$HG[1]^{\mathbf{P}}(t_1,\cdots,t_k) = \prod_{i=1}^{k} HG[1]^{\mathbf{P}^{n-1}}(t_i).$$

Main **idea** of proof: We use another smooth moduli, the Grothendieck quot-scheme Q_d to play the role of the linearized moduli, and apply the functorial localization formula, and a general set-up.

Step 1: Plücker embedding: $\tau: \ Gr(k,n)\to \mathbf{P}^N$ induces embedding of the nonlinear moduli

M_d of $Gr(k,n)$ into that of $\mathbf{P}^N$. Composite with the collapsing map gives us a map

$$\varphi: \quad M_d \to W_d$$

into the linearized moduli space W_d of $\mathbf{P}^N$.

On the other hand the Plücker embedding also induces a map:

$$\psi: \quad Q_d \to W_d.$$

Lemma: The above two maps have the same image in W_d: $\mathrm{Im}\,\psi = \mathrm{Im}\,\varphi$.

And all the maps are equivariant with respect to the induced circle action from $\mathbf{P}^1$.

Step 2: Analyze the fixed points of the circle action induced from $\mathbf{P}^1$. In particular we need the distinguished fixed point set to get the equivariant Euler class of its normal bundle.

The distinguished fixed point set in M_d is:

$\mathcal{M}_{0,1}(d, Gr(k,n))$

with equivariant Euler class of its normal bundle: $\alpha(\alpha - c)$, and φ restricted to ev.

Lemma: The distinguished fixed point set in Q_d is a union: $\cup_s E_{0s}$, where each E_{0s} is a fiber bundle over $Gr(k,n)$ with fiber given by flag manifold.

It is a complicated work to determine the fixed point sets E_{0s} and the weights of the circle action on their normal bundles.

Step 3: Let p denote the projection from E_{0s} onto $Gr(k,n)$. Functorial localization formula gives us

Lemma: We have equality:

$$ev_*[\frac{1}{\alpha(\alpha-c)}] = \sum_s p_*[\frac{1}{e_T(E_{0s}/Q_d)}]$$

where $e_T(E_{0s}/Q_d)$ is the equivariant Euler class of the normal bundle of E_{0s} in Q_d.

Step 4: Compute $p_*[\frac{1}{e_T(E_{0s}/Q_d)}]$.

Method 1: Homogeneous bundle and representations: LLLY 2001.

Method 2: Tautological Euler sequences and filtrations: Bertram et al 2003.

Standard push-forward formula on flag manifolds for p_* gives the final formula.

Proof of the Marino-Vafa formula (C.-C. Liu, K. Liu and J. Zhou). The formula was conjectured based on several physical theories:

Witten 92: The open topological string theory on the N D-branes on S^3 of T^*S^3 is equivalent to $U(N)$ Chern-Simons gauge thoery on S^3.

Gopakumar-Vafa 98, Ooguri-Vafa 00: The open topological string theory on the N D-branes on S^3 of the deformed conifold is equivalent to the closed topological string theory on the resolved conifold.

Mariño-Vafa formula 01: From large N duality and framing dependence, they predicted:

$$\langle Z(U,V)\rangle = \exp(-F(\lambda,t,V))$$

U: holonomy of the $U(N)$ Chern-Simons gauge field around the unknot K; V: $U(M)$ matrix

$\langle Z(U,V)\rangle$: knot invariants of K.

$F(\lambda, t, V)$: Generating series of the open Gromov-Witten invariants of $(\tilde{X}, L)$, where $L \cong S^1 \times \mathbb{R}^2$ is a Lagrangian submanifold of the resolved conifold $\tilde{X}$ "canonically associated to" to the unknot K.

(Canonical identifications of parameters similar to mirror formula).

Compare with computations of Katz-Liu, they derived a striking formula about triple Hodge integrals on moduli space of curves.

Let $n_1, \cdots n_h$ be a partition of d and p an arbitrary integer. Write, for $g = 0$,

$$\begin{aligned} & C_{g;n_1,\ldots,n_h}(p) \\ = \; & (-1)^{pl}(p(p+1))^{h-1}\left(\prod_{i=1}^{h} \frac{\prod_{j=1}^{n_i-1}(j+n_i p)}{(n_i-1)!}\right) l^{h-3} \end{aligned}$$

and

$$
\begin{aligned}
& C_{g;n_1,\ldots,n_h}(p) \\
= \; & (-1)^{pl}(p(p+1))^{h-1}\left(\prod_{i=1}^{h}\frac{\prod_{j=1}^{n_i-1}(j+n_i p)}{(n_i-1)!}\right) \\
& \cdot \int_{\overline{M}_{g,h}} \frac{\Lambda_g^\vee(1)\Lambda_g^\vee(p)\Lambda_g^\vee(-p-1)}{\prod_{i=1}^{h}(1-n_i\psi_i)}
\end{aligned}
$$

for $g > 0$. Here

$$
\Lambda_g^\vee(u) = u^g + c_1(\mathbb{E}^\vee)u^{g-1} + \cdots + c_g(\mathbb{E}^\vee)
$$

is the total Chern class of the dual Hodge bundle induced by $H^1(C, \mathcal{O})$.

Mariño-Vafa conjecture:

$$
\begin{aligned}
&\sum_{g=0}^{\infty} C_{g;n_1,\ldots,n_h}(p)\lambda^{2g-2+h} \\
= \; & \frac{(-1)^{pl}}{\sqrt{-1}^{h+l}\prod_{i=1}^{h} n_i}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sum_{\sum_{\sigma=1}^{n} k_\sigma = k} \\
& \sum_{\sum_{\sigma=1}^{n}|R_\sigma| = l}\prod_{\sigma=1}^{n}\frac{\chi_{R_\sigma}(C(k_\sigma))}{z_{k_\sigma}} e^{\sqrt{-1}p\kappa_{R_\sigma}\lambda/2} \\
& \cdot \prod_{1\le i<j\le c_{R_\sigma}} \frac{\sin((l_i^\sigma - l_j^\sigma + j - i)\lambda/2)}{\sin((j-i)\lambda/2)} \\
& \cdot \prod_{i=1}^{c_{R_\sigma}} \frac{\prod_{v=-i+1}^{l_i^\sigma - i} e^{\sqrt{-1}v\lambda/2}}{\prod_{v=1}^{l_i^\sigma} 2\sin((v-i+c_{R_\sigma})\lambda/2)}
\end{aligned}
$$

Remark: Just note that the right hand side is a closed formula, a finite expression in terms of representations of symmetric groups.

Cut-Join: The combinatorics and geometry:

Combinatorics: Denote by $[s_1, \cdots, s_k]$ a k-cycle in the permutation group:

Cut: a k-cycle is cut into an i-cycle and a j-cycle:

$$[s, t] \cdot [s, s_2, \cdots, s_i, t, t_2, \cdots t_j]$$

$$= [s, s_2, \cdots, s_i][t, t_2, \cdots t_j].$$

Join: an i-cycle and a j-cycle are joined to an $(i + j)$-cycle:

$$[s, t] \cdot [s, s_2, \cdots, s_i][t, t_2, \cdots t_j]$$

$$= [s, s_2, \cdots, s_i, t, t_2, \cdots t_j].$$

Geometry:

Cut: One curve split into two lower degree or lower genus curves.

Join: Two curves joined together to give a higher genus or higher degree curve.

The combinatorics and geometry of cut-join are reflected in the following two differential equations, like heat equation:

proved either by direct computations in combinatorics or by localizations on moduli spaces of relative stable maps:

Proof of Mariño-Vafa formula:(Chiu-Chu Melissa Liu, Kefeng Liu, Jian Zhou)

Let $\tilde{C}_{g;n_1,\dots,n_h}(p)$ be defined by the combinatorial side of the Mariño-Vafa formula. Consider generating functions

$$\tilde{H}(\lambda, p, z) = \sum_{g=0}^{\infty} \sum_{h=1}^{\infty} \lambda^{2g-2+h} \cdot \sum_{n_1,\dots,n_h=1}^{\infty} \sqrt{-1}^{n_1+\cdots+n_h-h} \tilde{C}_{g;n_1,\dots,n_h}(p) z_{n_1} \cdots z_{n_h}$$

and let us write

$$H(\lambda, p, z) = \sum_{g=0}^{\infty} \sum_{h=1}^{\infty} \lambda^{2g-2+h} \cdot \sum_{n_1,\dots,n_h=1}^{\infty} \sqrt{-1}^{n_1+\cdots+n_h-h} C_{g;n_1,\dots,n_h}(p) z_{n_1} \cdots z_{n_h}$$

The proof is based on the **Cut-Join** equation: a beautiful match of Combinatorics and Geometry.

Aganagic- Marino-Vafa et al interpret cut-join equation in terms of Chiral Boson

Combinatorics: Computation:

Theorem 1:

$$\frac{\partial \tilde{H}}{\partial p} = \frac{1}{2}\sqrt{-1}\lambda \sum_{i,j=1}^{\infty} \left((i+j) z_i z_j \frac{\partial \tilde{H}}{\partial z_{i+j}} + ij z_{i+j} \left(\frac{\partial \tilde{H}}{\partial z_i} \frac{\partial \tilde{H}}{\partial z_j} + \frac{\partial^2 \tilde{H}}{\partial z_i \partial z_j} \right) \right)$$

Geometry: Localization:

Theorem 2:

$$\frac{\partial H}{\partial p} = \frac{1}{2}\sqrt{-1}\lambda \sum_{i,j=1}^{\infty} \left((i+j) z_i z_j \frac{\partial H}{\partial z_{i+j}} + ij z_{i+j} \left(\frac{\partial H}{\partial z_i} \frac{\partial H}{\partial z_j} + \frac{\partial^2 H}{\partial z_i \partial z_j} \right) \right)$$

Initial Value: Ooguri-Vafa formula

$$H(\lambda, 0, z) = \sum_{d=1}^{\infty} \frac{z_d}{2d \sin\left(\frac{\lambda d}{2}\right)} = \tilde{H}(\lambda, 0, z).$$

The solution is unique!

Remark: Cut-join equation is more fundamental: encodes both geometry and combinatorics: Vafa: Virasoro operators come out of the cut-join.

Applications: Computing GW invariants on Toric Calabi-Yau:

Physical approaches: Aganagic-Mariño-Vafa (2002): BPS numbers for toric Calabi-Yau by using large N dulaity and Chern-Simons invariants.

Vafa et al (2003): Topological vertex. Complete formula for computations of GW invariants and BPS numbers: Chern-Simons.

Iqbal's instanton counting in terms of Chern-Simons.

Mathematical approach by Zhou: Mariño-Vafa formula can be used to compute BPS numbers and GW invariants for toric Calabi-Yau:

Re-organize contributions of fixed points as combinations of Mariño-Vafa formula.

Recovered the formula of Iqbal (based on a slight generalization of the MV formula).

Working in progress to understand the topological vertex of Vafa et al in terms of the MV formula.

The physical and mathematical approaches should be equivalent:

Bridge: The Mariño-Vafa formula.

The approach to compute instantons for II_A model should be applicable to compute coupling in (2,0) model if

$$\begin{array}{c} V \\ \downarrow \\ X \end{array}$$

is a bundle

that come from some toric manifold which is invariant under Torus action.

Jun Li - Yau

Let $C \subset \mathbb{C}P^3$ be

the cubic $z_0^3 + z_1^3 + z_2^3 + z_3^3 = 0$

Define

$$\tilde{X} \subset C \times C$$

$\mathbb{C}P^3 \times \mathbb{C}P^3$

to be

$$\sum_{i=0}^{3} x_i y_i = 0$$

It admits a free Z_3 action

$$([x_i], [y_i]) \longrightarrow ([x_0, x_1, \zeta x_2, \zeta^2 x_3], [y_0, y_1, \zeta^2 y_2, \zeta y_3])$$

$$X = \tilde{X} / Z_3 \, .$$

There is a natural bundle

$$0 \to \mathcal{O} \oplus \mathcal{O} \to E \to T_X \to 0$$

which is stable (after deformation) Hence a natural SU(5) theory.

A. Strominger :

$$\begin{array}{c} V \\ \downarrow \\ X \end{array}$$

Let F be the curvature of the bundle.

ω be a Hermitian metric on X.

① $\mathrm{Tr}_\omega F = 0$

② $F^{2,0} = F^{0,2} = 0$

③ $\partial\bar\partial\omega = \frac{\sqrt{-1}}{30}\mathrm{Tr}_{ad}\, F\wedge F - \sqrt{-1}\,\mathrm{Tr}\, R_\omega\wedge R_\omega$

④ $d^*\omega = \sqrt{-1}\,(\partial - \bar\partial)\ln\|\Omega\|$

where Ω is the holo $(3,0)$ form.