Topological String Theory

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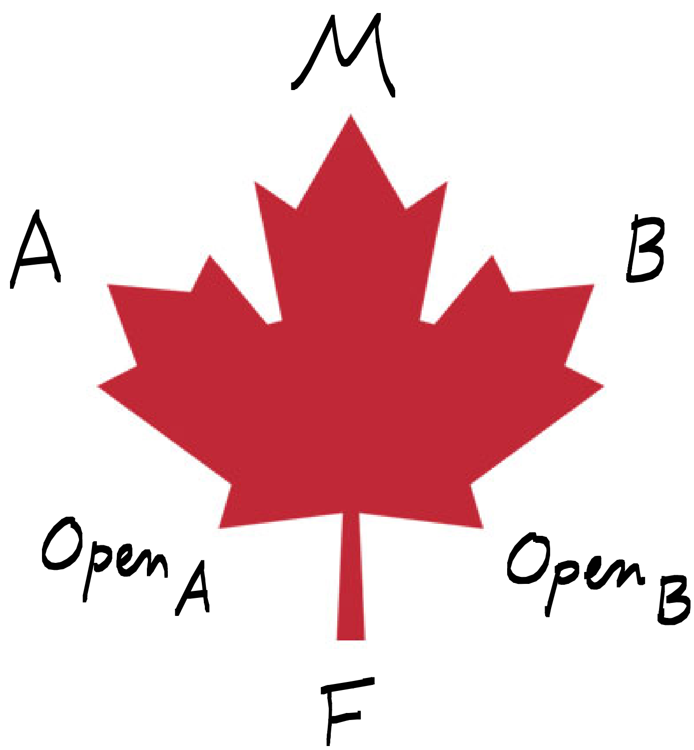
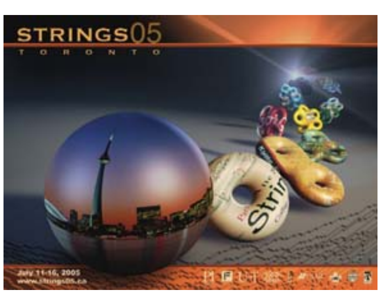




Strings 2005,
Toronto, Canada

If String Theory is an answer,
what is the question?

What is String Theory?

If Topological String Theory is an answer, what is the question?

What is Topological String Theory?

1. Formal Theory

Topological String Theory

Witten; DVV 1990

(1) Start with a Calabi-Yau 3-fold.

$\mathcal{N}=2$ superconformal sigma model

$$\Sigma_g \to CY_3$$

(2) Topologically twist the sigma-model.

(3) Couple it to the topological gravity on Σ_g .

$$F_g = \int_{\mathcal{M}_g} \left\langle \left| \prod_{i=1}^{3g-3} (\eta_i, G^-) \right|^2 \right\rangle$$

Beltrami differential

supercurrent

$$Z^{top}_{pert.} = \exp\left[\sum_{g=0}^{\infty} \lambda^{2g-2} F_g \right]$$

This is an asymptotic expansion.

Methods to compute F_g

Holomorphic Anomaly Equation

BCOV 1993

$$\partial_{\bar{i}} F_g = \frac{1}{2} C_{\bar{i}\bar{j}\bar{k}} e^{2K} G^{j\bar{j}} G^{k\bar{k}}$$

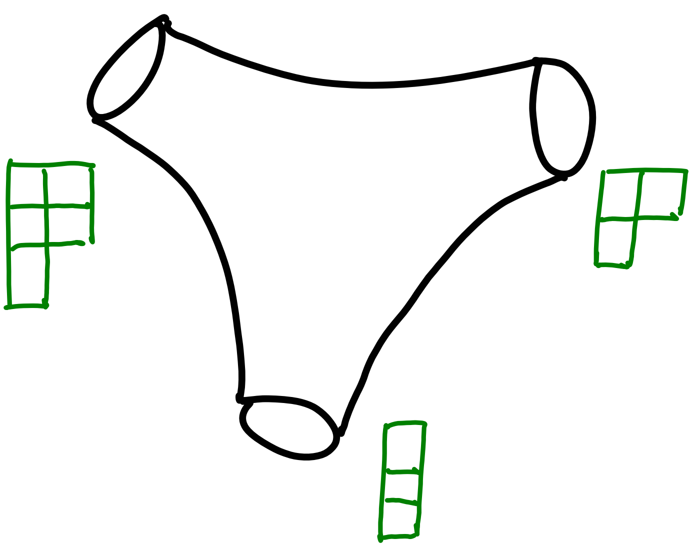
$$\times \left\{ D_j D_k F_{g-1} + \sum_{a=1}^{g-1} D_j F_a \, D_k F_{g-a} \right\}$$

Topological Vertex

Aganagic, Klemm, Marino, Vafa 2003

for noncompact toric CY3,
to all order in perturbation

Iqbal, Kashani-Poor,
Dijkgraaf, ...



Quantum Foams

Okounkov, Reshetikhin, Vafa
2003

Iqbal, Nekrasov, ...
Maurik, Pandharipande, ...

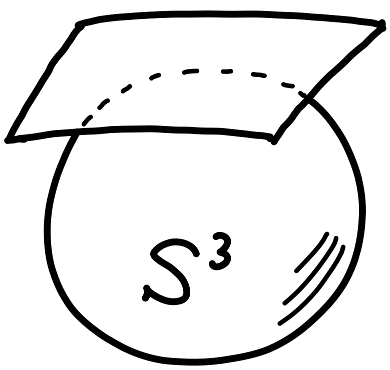
more recent mathematical
developments ...

Open String Field Theory

A-model => Chern-Simons gauge theory

Witten 1992

e.g. T^*S^3



N D branes on S^3

$\Downarrow$

$U(N)$ CS theory on S^3

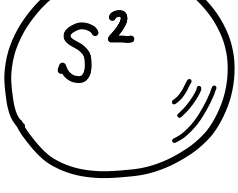
Computation of knot invariants

Vafa, H.O. 1999; Gukov, A.Schwarz, Vafa 2005

B-model => Matrix Model

Dijkgraav, Vafa 2002

e.g. resolved conifold



N D branes on S^2

$\Downarrow$

Gaussian Matrix Model

Laboratory for Large N Dualities

Gopakumar,Vafa 1998

Open string on T^*S^3 (CS theory)

$\parallel$

Closed string on resolved conifold

For the B-model:

Dijkgraav, Vafa 2002

Matrix Model (open string)

eigenvalue distribution

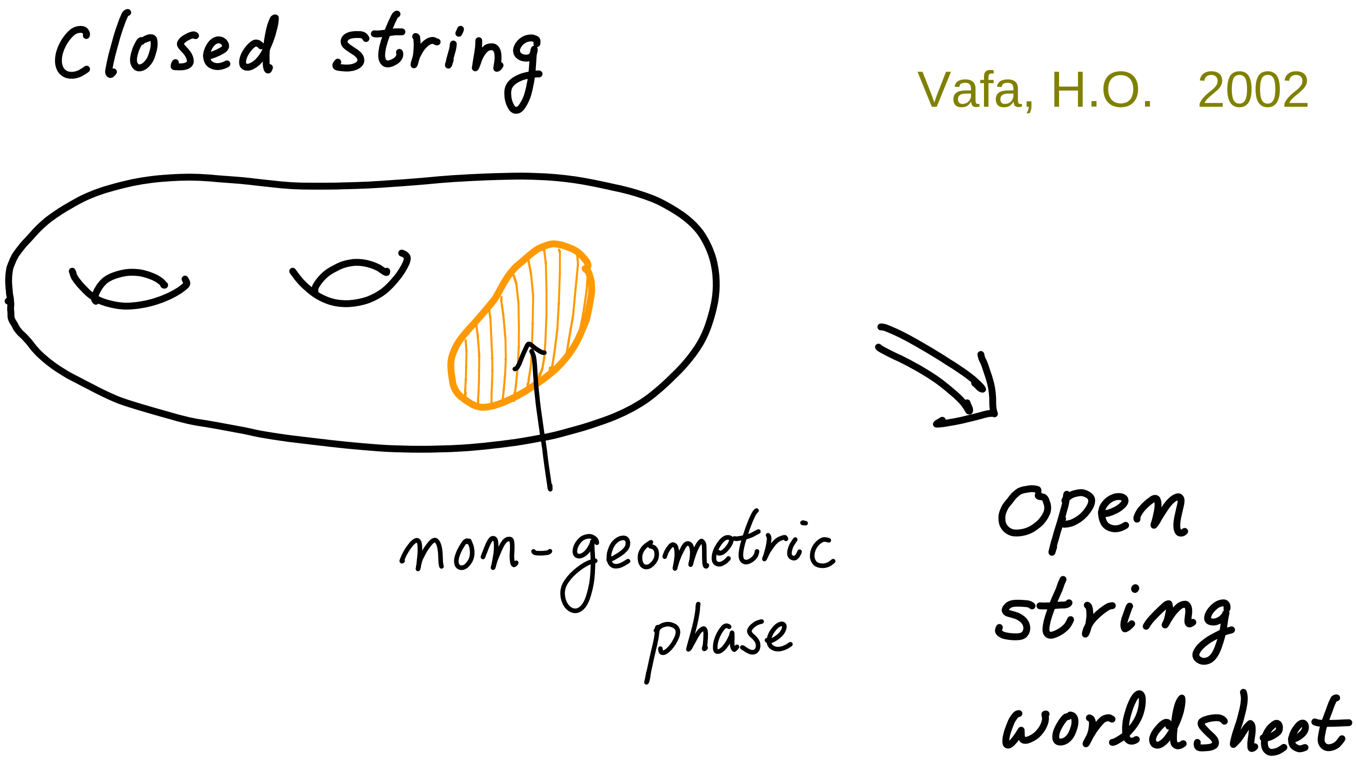
$\parallel$

Kodaira-Spencer Theory (closed string)

Calabi-Yau geometry

The topological string large N duality
is amenable to worldsheet derivation:

Vafa, H.O. 2002



The large N duality is a statement that is perturbative in closed string. That is why the worldsheet derivation is possible.

A similar derivation may be possible for the AdS/CFT correspondence.

Topological String Partition Function = Wave Function

Consider the topological B-model

moduli space $\mathcal{M}_B$: z^i complex structure

λ topological string coupling

$$\text{tangent space to } \mathcal{M}_B = H^3(CY_3, \mathbb{R})$$

$$\dim_{\mathbb{C}} H^3 = \underbrace{h^{2,1}}_{\delta z^i} + \underbrace{1}_{\delta \lambda}$$

For a given background, we can define the B-model.

F_g is a holomorphic function of $H^3(CY_3)$.

$$F_g \sim \int_{\mathcal{M}_g} \left\langle (G_L^-)^{3g-3} (G_R^-)^{3g-3} e^{\chi^I \int_{\Sigma_g} \mathcal{O}_I} \right\rangle$$

$$\chi^I \in H^3, \quad I = 0, 1, \cdots, h^{2,1}$$

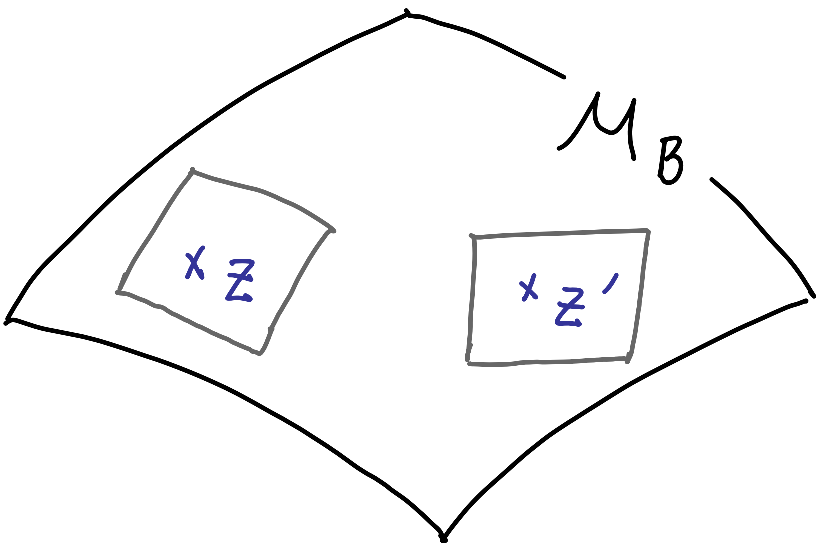
$$\psi_{top}(\chi^I ; \underbrace{z^i, \bar{z}^i}_{\text{background}}) = \exp\left(\sum_g F_g\right)$$

The holomorphic anomaly equations

derived in BCOV;
interpreted by Witten;
refined by D-V-Vonk.

$$\frac{\partial}{\partial z^I}\psi_{top}(x;z,\bar{z}) = \left(\frac{\partial}{\partial x^I}+\cdots\right)\psi_{top}$$

$$\frac{\partial}{\partial \bar{z}^I}\psi_{top}(x;z,\bar{z}) = \left(\bar{C}_I^{\ JK}\frac{\partial^2}{\partial x^J \partial x^K}+\cdots\right)\psi_{top}$$



Interptetation:

(1) On each tangent space, there is a Hilbert space.

(2) The holomorphic anomaly equation is describing the paralell transport between tangent spaces at different points.

More on (1):

- $H^3(CY_3, \mathbb{R})$ has a symplectic structure.
- Topological string uses a holomorphic polarization.

$$\underbrace{H^{3,0}\oplus H^{2,1}}_{x^I}\oplus H^{2,1}\oplus H^{0,3}$$

More on (2):

- The polarization depends on $(z^i, \bar{z}^i)$

$\Rightarrow$ Wave-functions are related by Fourier transformation.

The topological string wave function

$$\psi_{top} = \exp\left(\sum_g F_g \right)$$

may have an interpretation from
7 dimensional pointe of view.

Dijkgraaf, Gukov, Neitzke, Vafa;
Nekrasov;
Gerasimov, Shatashvili 2004

Topological M theory

$\Downarrow$ Canonical Quantization

ψ_{top}

?

2. Superpotentials

It was known that the Yukawa coupling in four dimensions is given by the g=0 topological string amplitude.

Dine, Seiberg, Witten, Wen, Martinec, Distler, Greene, Lerche, Vafa, Warner, ...

This was generalized for g > 0 in type II superstring compactified on CY3

BCOV;
Antoniadis, Gava, Narain, Tayor 1993

$$\int d^4\theta \; F_g(t(\theta)) \left(W(\theta)^2 \right)^g$$

$$W_{\alpha\beta}(\theta) = \underbrace{F_{\alpha\beta}}_{\text{graviphoton}} + \cdots + R_{\alpha\beta\gamma\delta}\, \theta^\gamma_a \theta^\delta_b \epsilon^{ab}$$

$$t(\theta) = \underbrace{t}_{CY_3 \text{ moduli}} + \cdots$$

$$\int d^4\theta \; F_g(t(\theta))\,\left(W(\theta)^2\right)^g$$

$$W_{\alpha\beta}(\theta) = \underbrace{F_{\alpha\beta}}_{\text{graviphoton}} + \cdots + R_{\alpha\beta\gamma\delta}\,\theta^{\gamma}_{a}\,\theta^{\delta}_{b}\,\epsilon^{ab}$$

topological string coupling

$\sim F^2$

Analogously, open topological string theory can be used to compute superpotentials for type II string on CY3 with D branes.

BCOV	1993
Vafa, H.O.	1999
Vafa	2000

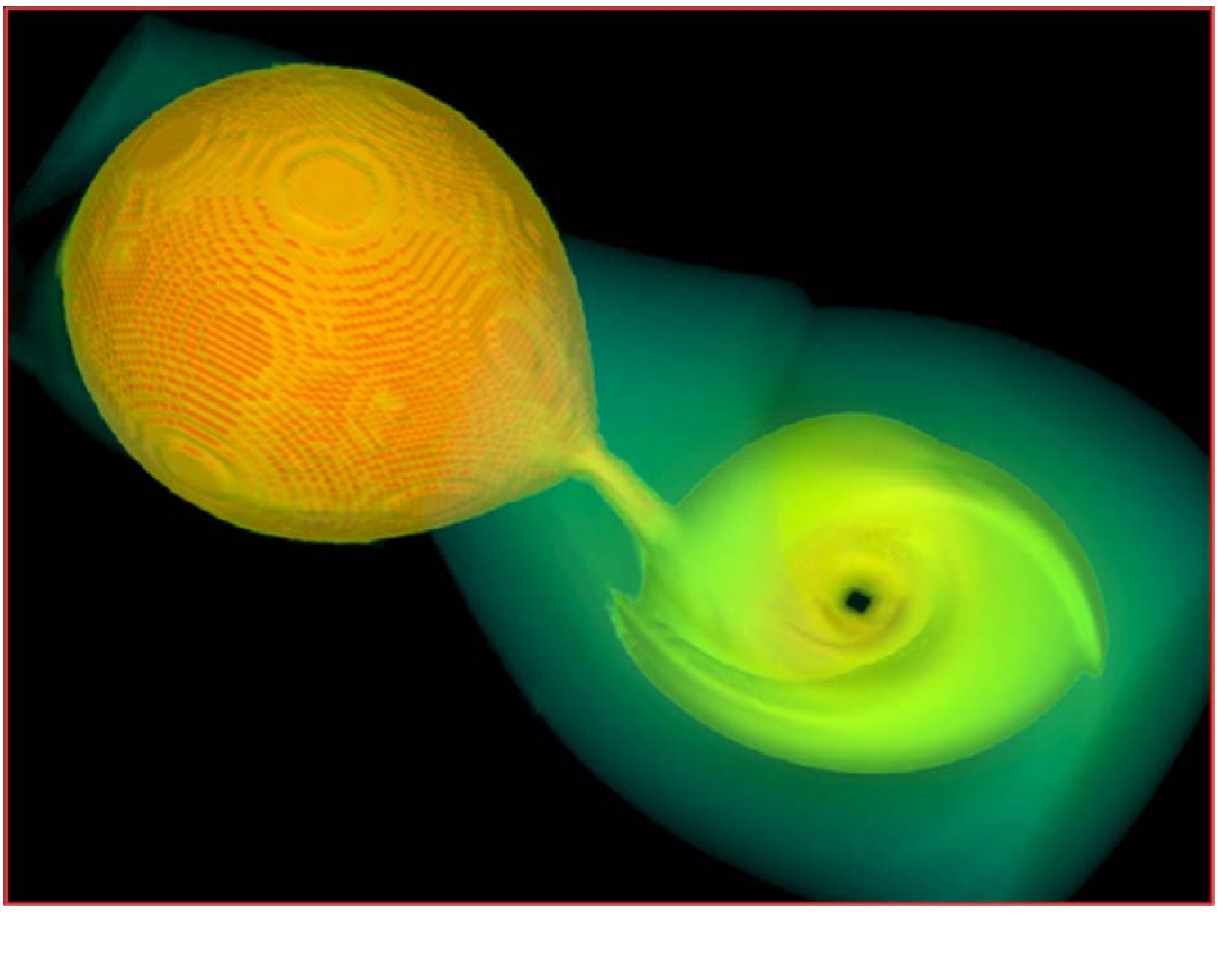
Gauge theories in 4 dimensions
with $\mathcal{N}=1$ supersymmetry

When topological open string field theory is a matrix model, the superpotential of the 4d gauge theory on the branes is given by the partition function of the matrix model.

Dijkgraaf, Vafa 2002

If Topological String Theory is an answer, what is the question?

3. Black Holes



What are questions for which graviphoton field strength is important?

Type II superstring complactified on CY3 has many charged extremal black holes constructed as D branes wrapping cycles in CY3.

The number of ground states of the black hole is computable using the gauge theory on the branes.

$$\Omega(p, q)$$

When the charges p, q are large, the classical gravity descrpition is good, and the number of states is given by the Bekenstein-Hawking formula:

$$\ln \Omega(p, q) \sim \frac{1}{4} A_{\text{HORIZON}}$$

$$p \cdot q \gg 1$$

This has been successfully tested.

Can we make this more precise, by incorporating quantum corrections to the right-hand side?

Extremal Black Hole in String Theory

Consider type IIB superstring on $CY_3 \times \mathbb{R}^{3,1}$

RR 4-form potential $\Rightarrow$ gauge fields A_μ^I on $\mathbb{R}^{3,1}$

$I = 0, 1, \cdots, h^{2,1}$

Choose a symplectic basis of 3-cycles on CY_3.

$\{ A_I , B^I \}$ $\quad I = 0, 1, \cdots, h^{2,1}$

$A_I \cap A_J = 0 = B^I \cap B^J$

$A_I \cap B^J = \delta_I{}^J$

D3 branes wrapping on 3 cycles

q_I times on A_I, p^I times on B^I

= BPS black hole in four dimensions

with electric charges q_I, magnetic charges p^I

$\Omega(p, q)$ = number of BPS states

Conjecture (Strominger, Vafa + H.O.)

$$|\psi_{top}(\chi)|^2 = \sum_q \Omega(p,q)\, e^{-q\phi}$$

where $\chi^I = p^I + \frac{i}{\pi}\phi^I$

(1) Calabi-Yau moduli χ^I are determined by the magnetic charges p^I and the electric potentials ϕ^I.

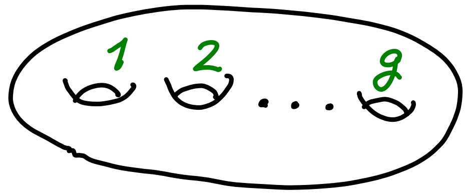
attractor mechanism

(2) This identity is supposed to hold to all order in the string perturbation theory.

There are important non-perturbative corrections to the formula.

Quantum corrections to the formula can be evaluated using the topological string amplitudes.

Cardoso, de Wit, Mohaupt 1998

$$F_g = \int$$ 

The quantum corrections modify the entropy formula.
CdWM computed the modified formula.
OSV noticed that their formula can be re-expressed as:

$$S(p,q) = \mathcal{F}(p,\phi) + q_I \phi^I$$

$$q_I = -\frac{\partial}{\partial \phi^I} \mathcal{F}(p,\phi)$$

$$\mathcal{F}(p,\phi) = \sum_{g=0}^{\infty} F_g\left(X = p + \frac{i}{\pi}\phi\right) + c.c.$$

Quantum corrections can also change the nature of solutions. For example, when a classical solution has a naked singularity, the one-loop correction may generate a horizon to cloak the singularity.

Dabholkar;
Dabholkar, Kallosh, Maloney;
Sen 2004

The OSV conjecture as AdS/CFT correspondence

(1) AdS story:

The near horizon geometry of the black hole is

$$AdS_2 \times S^2 \times CY_3$$

$|\psi_{top}|^2 =$ Partition function of type II string in this geometry

(2) CFT story:

$\Omega(p, q) =$ Number of BPS states on the D brane worldvolume

Gauge theory computation on the D branes.

This may help us understand
the (still mysterious) AdS2/CFT1 correspondence.

$$|\psi_{top}(X)|^2 = \sum_q \Omega(p,q)\, e^{-q\phi}$$

This formula has been examined in various examples:

Vafa; Dabholkar; Sen;
Aganagic, Saulina, Vafa + H.O.;
Dabholkar, Denef, Moore, Pioline; .....

When the effective string coupling is parametrically small and the CY geometry is smooth, all the results are consistent with the OSV conjecture.

When there is a heterotic dual, the OSV formula has been used to clarify the correspondence between elementary strings states and black holes.

The formula is supposed to hold to all order in the perturbative expansion.

I will discuss non-perturbative corrections in the following example.

An example:

Type IIA theory on the following noncompact CY.

$$\mathcal{L}_{-m} \oplus \mathcal{L}_{2g-2+m} \to \Sigma_g$$

The total space is a Calabi-Yau manifold.

g=0; Vafa
g>0; Aganagic, Saulina, Vafa, H.O. 2004

ψ_{top} Topological string amplitudes on this CY was recently computed to all order in the perturbative expansion.

Bryan, Pandharipande 2004

$\Omega(p,q)$ N D4 branes on $\mathcal{L}_{-m} \to \Sigma_g$

$\Downarrow$

U(N) gauge theory on the worldvolume

What does it mean to compute the entropy of the 4d black hole when the CY is non-compact?

Potential problem:

$$\left(M_{PL}^{(4d)}\right)^2 = \left(\mathrm{vol}\,(CY_3)\right) \times \left(M_{PL}^{(10d)}\right)^8$$

$$\to \infty \quad \text{for a non-compact } CY_3$$

4d black hole entropy $\sim \dfrac{M_{BH}^2}{\left(M_{PL}^{(4d)}\right)^2}$

In our case $M_{BH} \to \infty$
since $\mathcal{L}_{-m}$ is also non-compact.

One can think of our computation as <u>a smooth limit of a well-defined compact case</u>, where both the black hole mass and the 4d Planck mass go to infinity but their ratio remains finite.

The gauge theory partition function

$$Z_{D_4} = S^{2-2g} \sum_{\mathcal{R}} (\dim_q \mathcal{R})^{2-2g} \, q^{\frac{p}{2} C_2} e^{i\theta C_1}$$

$\mathcal{R} = [\mathcal{R}_1, \cdots, \mathcal{R}_N]$: representation of $U(N)$

$$C_2 = \sum_i \mathcal{R}_i (\mathcal{R}_i - 2i + 1) + N \mathcal{R}_i$$

$$C_1 = \sum_i \mathcal{R}_i$$

$$\dim_q \mathcal{R} = \prod_{i<j} \frac{[\mathcal{R}_i - \mathcal{R}_j - i + j]_q}{[i-j]_q}$$

$$\left(\text{where } [m]_q = q^{m/2} - q^{-m/2} \right)$$

$$S = q^{\rho^2} \prod_{i<j} [i-j]_q \quad , \quad \vec{\rho} \text{ : Weyl vector}$$

$$q = e^{-\lambda}$$

The large N gauge theory partition function is factorized:

$$Z_{D4} = \sum_{k=-\infty}^{\infty} \int dU_1 \cdots dU_{2g-2}$$

$$\psi_{top}(\lambda, t + km\lambda ; U_1, \cdots, U_{2g-2})$$

$$\times \ \overline{\psi}_{top}(\lambda \quad \bar{t} - km\lambda ; U_1, \quad , U_{2g-2})$$

ψ_{top} is the topological string wave-function

for $\mathcal{L}_{-m} \oplus \mathcal{L}_{2g-2+m} \rightarrow \Sigma_g$

with extra D3 branes inserted at (2g-2) points on the Riemann surface on the base.

$U_1, \cdots, U_{2g-2}$ are holonomies on the D3 branes.

$\in U(\infty)$

t : Kähler moduli associated to Σ_g.

$$\tau_{n,i} \equiv \mathrm{tr}\, U_i^{\,n}$$

can be interpreted as Kahler moduli associated to non-renormalizable deformation of the Calabi-Yau toward the asymptotic infinity. (Or, these parameters remember how the local model is embedded in a compact Calabi-Yau.)

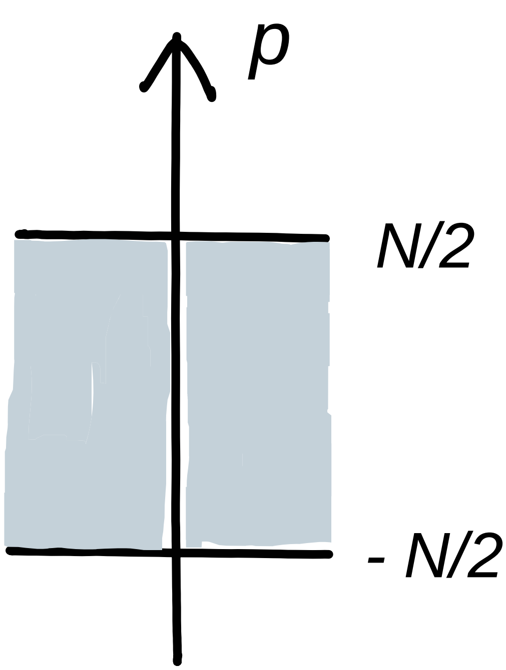
Aganagic, Neitzke, and Vafa 2005

When g=1, the U(N) YM partition function Z_{D_4} can be expressed in terms of
N non-relativistic free fermions on a circle.

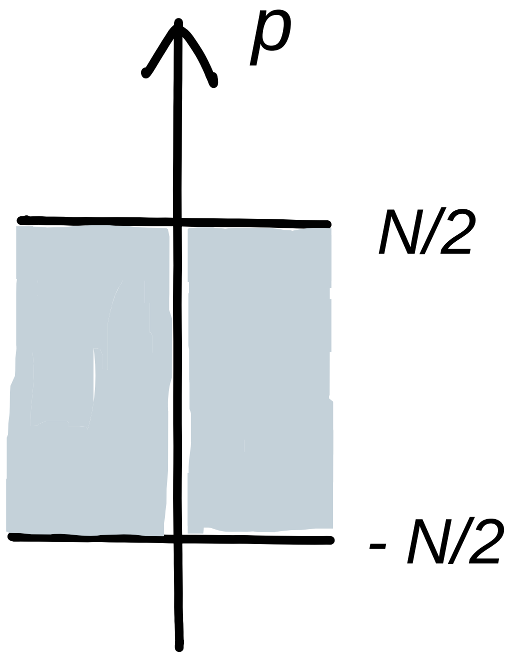
$$Z_{D_4} = \underbrace{\mathrm{tr}}_{N \text{ fermion Fock space}} \left[q^{-mH} e^{i\theta P} \right]$$

$$H = \frac{1}{2} \sum_{i=1}^{N} p_i^2 \,, \qquad P = \sum_{i=1}^{N} p_i$$

$$p_i = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \cdots$$



When *N* is large,
the two fermi surfaces
are decoupled.

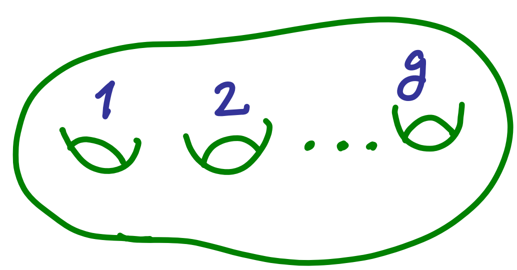


In the large N limit, the two fermi surfaces are decoupled.

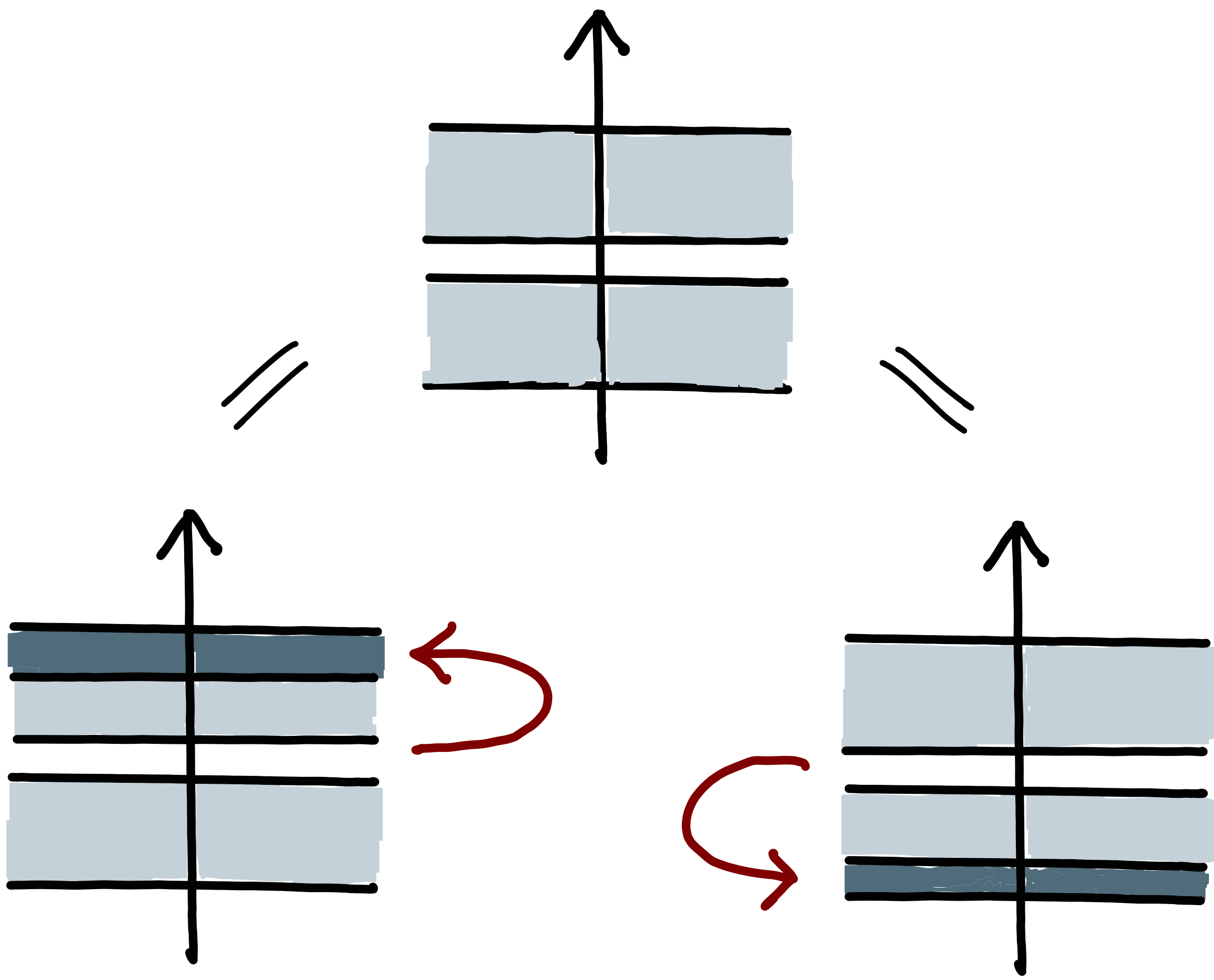
Fluctuations of each fermi surface are described by free relativistic fermions.

The $1/N$ expansion of the non-relativistic free fermion partition function correctly reproduces the absolute value square of the perturbative string amplitude.

$$\left| \exp\left(\sum_{g=0}^{\infty} \frac{1}{N^{2g-2}} \; \square \right) \right|^2$$



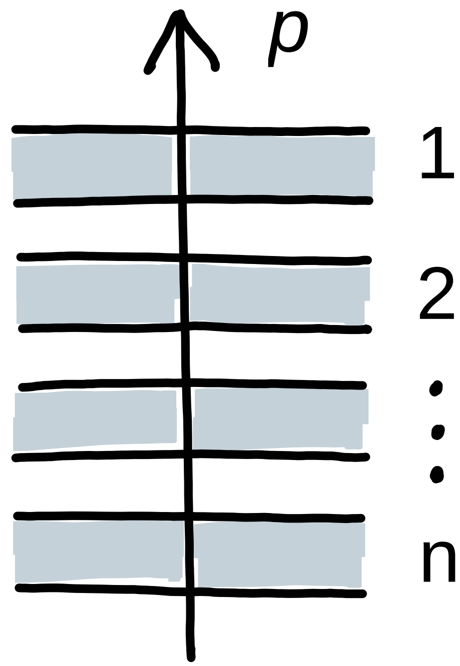
For finite *N*, the two fermi surfaces are entangled by excitations that are non-perturbative in *1/N*.



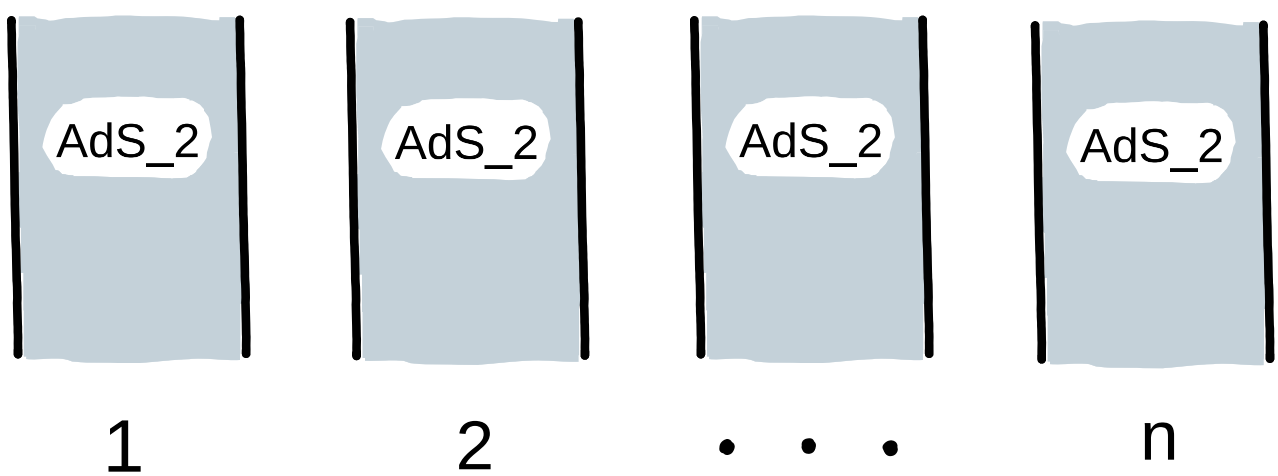
These non-perturbative states would be over-counted if we ignore the entanglement of the two fermi surfaces.

Dijkgraaf, Gopakumar, Vafa + H.O. 2005

Configurations with 2n fermi surfaces



is dual to n disjoint universes



weighted by the Catalan number of planar binary trees with n branches (remembering how baby universe have been created from the parent universe).

If Topological String Theory is an answer, what is the question?

Topological string theory can be used to derive superpotential terms for the superstring theory compactified on CY3.

In particular, string loop corrections to entropies of extremal black holes in four dimensions can be computed to all order in the string perturbation theory.

In examples, quantum corrected entropies obtained in this way agree with independent computation using gauge theories on branes.

There are important non-perturbative corrections, suggesting that unitarity of quantum gravity can be maintained after we sum over topologies.

What is Topological String Theory?

There are powerful mathematical methods
to compute topological string ampliutdes
to all order in the string perturbation theory.

For a given CY3, a non-perturbative defintion
of topological string theory can be given
in terms of the gauge theory on branes
wrapping cycles of the CY3.

A background independent formulation is
desired. Perhaps a better understanding of the
topological M theory in 7 dimensions will help.

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