Extremal Black Holes in Higher Derivative Gravity

References:

A.S. hep-th/0506177

+ hep-th/0505122 + work in progress

Related (complementary) work:

Kraus & Larsen, hep-th/0506176

Goldstein, Iizuka, Jena, Trivedi,

hep-th/0507095

Earlier related work:

Ooguri, Vafa, Strominger, hep-th/0405146

2

Dedicated to the memory of Prof. Amal Kumar Raychaudhuri, – the author of the Raychaudhuri equation, and a source of inspiration for generations of Indian scientists – who passed away on June 18, 2005.

Motivation:

In $\mathcal{N} = 2$ supergravity theory in four dimensions there are some nice results for extremal BPS black holes.

1. Near horizon values of the vector multiplet moduli fields are independent of the asymptotic values of these scalar fields.

2. These values are determined by extremizing an 'entropy function', whose value at the extremum gives the entropy S_{BH} of the black hole.

3. As a result S_{BH} in independent of the asymptotic values of these moduli.

→ **Attractor mechanism**

Ferrara, Kallosh, Strominger

Some of these properties generalize upon inclusion of a specific class of higher derivative terms in the theory known as the generalized prepotential.

Cardoso, de Wit, Mohaupt

Questions:

1. Do these properties survive when we add to the action a generic set of higher derivative terms?

2. How essential is supersymmetry?

3. Are there analogous properties in higher dimensions?

We shall examine these questions for spherically symmetric extremal black hole.

The $D = 4$ case:

We define extremal black hole to be a black hole with

1. near horizon geometry $AdS_2 \times S^2$

2. all other background fields respecting the $SO(2,1) \times SO(3)$ symmetry of $AdS_2 \times S^2$.

The entropy of an extremal black hole

$\equiv$ entropy of a non-extremal black hole in the extremal limit.

Thus we can use the formula for the entropy for a non-extremal black hole with bifurcate horizon.

Wald

Jacobson, Kang, Myers

Consider an arbitrary general coordinate invariant theory of gravity coupled to a set gauge fields $A_\mu^{(i)}$ and neutral scalar fields $\{\phi_s\}$.

The most general form of the near horizon geometry of an extremal black hole consistent with the symmetry of $AdS_2 \times S^2$:

$$ds^2 \equiv g_{\mu\nu}dx^\mu dx^\nu = v_1\left(-r^2dt^2 + \frac{dr^2}{r^2}\right) + v_2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

$$\phi_s = u_s$$

$$F_{rt}^{(i)} = e_i, \qquad F_{\theta\phi}^{(i)} = \frac{p_i}{4\pi}\sin\theta\,,$$

v_1, v_2: sizes of AdS_2 and S^2

u_s: scalar field values at the horizon.

$p_i/4\pi$: near horizon radial magnetic field

e_i: near horizon radial electric field

$$ds^2 = v_1\left(-r^2 dt^2 + \frac{dr^2}{r^2}\right) + v_2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

$$\phi_s \doteq u_s$$

$$F^{(i)}_{rt} = e_i, \qquad F^{(i)}_{\theta\phi} = \frac{p_i}{4\pi}\sin\theta\,,$$

Let $\sqrt{-\det g}\,\mathcal{L}$ be the Lagrangian density.

Define:

$$\boxed{f(\vec{u},\vec{v},\vec{e},\vec{p}) \equiv \int d\theta\, d\phi\, \sqrt{-\det g}\,\mathcal{L}}$$

$$q_i \equiv \frac{\partial f}{\partial e_i}, \qquad F(\vec{u},\vec{v},\vec{q},\vec{p}) \equiv 2\pi(e_i\, q_i - f(\vec{u},\vec{v},\vec{e},\vec{p}))$$

Thus $F/2\pi$ is the Legendre transform of f with respect to the variables e_i.

Results:

For an extremal black hole of electric charge $\vec{q}$ and magnetic charge $\vec{p}$,

1. the values of $\{u_s\}$, v_1 and v_2 are obtained by extremizing $F(\vec{u}, \vec{v}, \vec{q}, \vec{p})$ with respect to these variables.

$$\boxed{\frac{\partial F}{\partial u_s} = 0, \qquad \frac{\partial F}{\partial v_1} = 0\,, \qquad \frac{\partial F}{\partial v_2} = 0}$$

2. the near horizon electric field variables e_i are given by:

$$\boxed{e_i = \frac{1}{2\pi} \frac{\partial F(\vec{u}, \vec{v}, \vec{q}, \vec{p})}{\partial q_i}}$$

3. the black hole entropy S_{BH} is given by

$$\boxed{S_{BH} = F(\vec{u}, \vec{v}, \vec{q}, \vec{p})}$$

at the extremum of F with respect to $\vec{u}$, $\vec{v}$.

The results are derived using the equations of motion and Wald's formula for entropy in the presence of higher derivative terms in the action.

The derivation does not require the theory and/or the solution to be supersymmetric.

The only requirements are gauge and general coordinate invariance of the action.

Similar results hold in higher dimensions as well for extremal black holes with near horizon geometry $AdS_2 \times S^{D-2}$.

These results lead to a generalized 'attractor mechanism'

1. The single 'entropy function' $F(\vec{u}, \vec{v}, \vec{q}, \vec{p})$ determines

   – the near horizon values $\{u_s\}$ of the scalar fields,

   – the sizes v_1, v_2 of AdS_2 and S^2

   – the gauge field strengths $\{e_i\}$

2. If F has no flat directions then the extremization of F determines $\vec{u}$, $\vec{v}$ completely in terms of $\vec{q}$, $\vec{p}$.

   $\rightarrow$ $S_{BH} = F$ is independent of the asymptotic values of the scalar fields.

   If F has flat directions, then extremization of F does not determine all the parameters $\vec{u}$, $\vec{v}$.

   Location of $(\vec{u}, \vec{v})$ along the flat directions may depend on the asymptotic values of the scalar fields.

   But since F does not depend on the flat directions, $S_{BH} = F$ is still independent of the asymptotic values of the scalar fields.

Application

Consider heterotic string theory on $\mathcal{M}\times S^1\times\tilde{S}^1$.

$\mathcal{M}$: some compact space (*e.g.* $K3$, T^4 or some orbifolds of these)

We want to consider a black hole solution that carries

1. w units of F-string winding along S^1

2. n units of momentum along S^1

3. $\widetilde{W}$ units of H-monopole charge along $\tilde{S}^1$

($\widetilde{W}$ five-branes wrapped along $\mathcal{M}\times S^1$)

4. $\widetilde{N}$ units of Kaluza-Klein monopole charge along $\tilde{S}^1$.

We want to compute the entropy of the solution.

Convention:

$$\hbar = 1, \qquad c = 1, \qquad \alpha' = 16$$

x^9: the coordinate along S^1

x^8: the coordinate along $\tilde{S}^1$

x^8, x^9 are periodic with period $2\pi\sqrt{\alpha'} = 8\pi$.

Define 4-dimensional fields in terms of the ten dimensional fields $G^{(10)}_{MN}$, $B^{(10)}_{MN}$ and $\Phi^{(10)}$:

$$\Phi = \Phi^{(10)} - \frac{1}{4}\ln(G^{(10)}_{99}),$$

$$S = e^{-2\Phi}, \qquad T = \sqrt{G^{(10)}_{99}}, \qquad \tilde{T} = \sqrt{G^{(10)}_{88}},$$

$$G_{\mu\nu} = G^{(10)}_{\mu\nu} - (G^{(10)}_{99})^{-1} G^{(10)}_{9\mu} G^{(10)}_{9\nu} - (G^{(10)}_{88})^{-1} G^{(10)}_{8\mu} G^{(10)}_{8\nu},$$

$$A^{(1)}_\mu = \frac{1}{2}(G^{(10)}_{99})^{-1} G^{(10)}_{9\mu}, \qquad A^{(2)}_\mu = \frac{1}{2} B^{(10)}_{9\mu},$$

$$A^{(3)}_\mu = \frac{1}{2}(G^{(10)}_{88})^{-1} G^{(10)}_{8\mu}, \qquad A^{(4)}_\mu = \frac{1}{2} B^{(10)}_{8\mu},$$

$G_{\mu\nu}$: string metric

Interpretation

S: Inverse string coupling^2

T: size of S^1

$\widetilde{T}$: size of $\widetilde{S}^1$

$A_\mu^{(1)}$: couples to momentum along S^1

$A_\mu^{(2)}$: couples to winding along S^1

$A_\mu^{(3)}$: couples to momentum along $\widetilde{S}^1$

$A_\mu^{(4)}$: couples to winding along $\widetilde{S}^1$

Normalization of $A_\mu^{(i)}$ are such that:

$$\boxed{q_1 = \frac{1}{2} n,} \quad \boxed{q_2 = \frac{1}{2} w,} \quad \boxed{p_3 = 4\pi \widetilde{N},} \quad \boxed{p_4 = 4\pi \widetilde{W}}$$

The effective action:

$$\mathcal{S} = \frac{1}{32\pi}\int d^D x\,\sqrt{-\det G}\,S\,[R_G + S^{-2}\,G^{\mu\nu}\,\partial_\mu S\partial_\nu S$$

$$-T^{-2}\,G^{\mu\nu}\,\partial_\mu T\partial_\nu T - \tilde{T}^{-2}\,G^{\mu\nu}\,\partial_\mu \tilde{T}\partial_\nu \tilde{T}$$

$$-T^2\,G^{\mu\nu}\,G^{\mu'\nu'}\,F^{(1)}_{\mu\mu'}F^{(1)}_{\nu\nu'} - T^{-2}\,G^{\mu\nu}\,G^{\mu'\nu'}\,F^{(2)}_{\mu\mu'}F^{(2)}_{\nu\nu'}$$

$$-\tilde{T}^2\,G^{\mu\nu}\,G^{\mu'\nu'}\,F^{(3)}_{\mu\mu'}F^{(3)}_{\nu\nu'} - \tilde{T}^{-2}\,G^{\mu\nu}\,G^{\mu'\nu'}\,F^{(4)}_{\mu\mu'}F^{(4)}_{\nu\nu'}]$$

$+$ higher derivative terms

In this theory we look for a solution with near horizon geometry:

$$ds^2 = v_1\left(-r^2 dt^2 + \frac{dr^2}{r^2}\right) + v_2 d\Omega_2^2,$$

$$S = u_S, \qquad T = u_T, \qquad \tilde{T} = u_{\tilde{T}}$$

$$F^{(1)}_{rt} = e_1, \quad F^{(2)}_{rt} = e_2, \quad F^{(3)}_{\theta\phi} = \frac{p_3}{4\pi}, \quad F^{(4)}_{\theta\phi} = \frac{p_4}{4\pi}$$

If we ignore the higher derivative terms, then:

$$\begin{aligned}
&f(v_1, v_2, u_S, u_T, \tilde{u}_T, e_1, e_2, p_3, p_4) \\
&\equiv \int d\theta d\phi \sqrt{-\det G}\,\mathcal{L} \\
= \ &\frac{1}{8} v_1\, v_2\, u_S \Bigg[-\frac{2}{v_1} + \frac{2}{v_2} + \frac{2\, u_T^2\, e_1^2}{v_1^2} + \frac{2\, e_2^2}{u_T^2\, v_1^2} \\
&-2\, u_{\tilde{T}}^2 \frac{p_3^2}{16\pi^2 v_2^2} - 2\, u_{\tilde{T}}^{-2} \frac{p_4^2}{16\pi^2 v_2^2} \Bigg]
\end{aligned}$$

Taking the Legendre transform, we get

$$\begin{aligned}
&F(v_1, v_2, u_S, u_T, \tilde{u}_T, q_1, q_2, p_3, p_4) \\
= \ &\frac{\pi}{4} v_1\, v_2\, u_S \Bigg[\frac{2}{v_1} - \frac{2}{v_2} + \frac{8\, q_1^2}{u_T^2\, v_2^2\, u_S^2} + \frac{8\, u_T^2\, q_2^2}{u_T^2\, v_2^2\, u_S^2} \\
&+2\, u_{\tilde{T}}^2 \frac{p_3^2}{16\pi^2 v_2^2} + 2\, u_{\tilde{T}}^{-2} \frac{p_4^2}{16\pi^2 v_2^2} \Bigg]
\end{aligned}$$

Now extremize it with respect to v_1, v_2, u_S, u_T, $u_{\tilde{T}}$.

Result:

$$\boxed{\begin{aligned} & v_1 = v_2 = \frac{1}{4\pi^2} p_3 p_4 = 4\widetilde{N}\,\widetilde{W} \\ & u_S = 8\pi\sqrt{\frac{q_1 q_2}{p_3 p_4}} = \sqrt{\frac{n\,w}{\widetilde{N}\,\widetilde{W}}}, \\ & u_T = \sqrt{\frac{q_1}{q_2}} = \sqrt{\frac{n}{w}}, \quad u_{\widetilde{T}} = \sqrt{\frac{p_4}{p_3}} = \sqrt{\frac{\widetilde{W}}{\widetilde{N}}}, \\ & S_{BH} = \sqrt{q_1 q_2 p_3 p_4} = 2\pi\sqrt{nw\widetilde{N}\widetilde{W}} \end{aligned}}$$

(We have used

$$q_1 = \frac{1}{2}n, \quad q_2 = \frac{1}{2}w, \quad p_3 = 4\pi\widetilde{N}, \quad p_4 = 4\pi\widetilde{W})$$

The result for entropy agrees with the standard result for the entropy of four charge black holes in four dimensions.

e.g. Maldacena, Strominger, Witten

$$\boxed{v_1 = v_2 = 4\widetilde{N}\,\widetilde{W}, \qquad u_S = \sqrt{\frac{n\,w}{\widetilde{N}\,\widetilde{W}}}, \qquad u_T = \sqrt{\frac{n}{w}}, \quad u_{\widetilde{T}} = \sqrt{\frac{\widetilde{W}}{\widetilde{N}}}.}$$

The string coupling is

$$\boxed{u_S^{-1/2} \sim \left(\frac{\widetilde{N}\widetilde{W}}{nw}\right)^{1/4}}$$

We shall keep this small by taking

$$\boxed{nw >> \widetilde{N}\widetilde{W}}$$

and work with tree level effective action.

The sizes of AdS_2 and S^2 are

$$\boxed{\sqrt{v_1} \sim \sqrt{\widetilde{N}\widetilde{W}}, \qquad \sqrt{v_2} \sim \sqrt{\widetilde{N}\widetilde{W}}}$$

→ α' corrections give a power series expansion in $1/\widetilde{N}\widetilde{W}$.

Consider now the effect of adding the Gauss-Bonnet combination to the tree level heterotic string action:

$$\boxed{\Delta\mathcal{L} = C\,\frac{S}{16\pi}\left\{R_{G\mu\nu\rho\sigma}R_G^{\mu\nu\rho\sigma} - 4R_{G\mu\nu}R_G^{\mu\nu} + R_G^2\right\}}$$

For heterotic string theory $C = 1$.

$\rightarrow$ additional contribution to f:

$$\boxed{\Delta f = -2Cu_S \quad \rightarrow \quad \Delta F = 4\pi C u_S.}$$

We now redo the analysis using the corrected function $F + \Delta F$

Result:

$$\boxed{S_{BH} = 2\pi\sqrt{nw}\sqrt{\widetilde{N}\widetilde{W} + 4C} = 2\pi\sqrt{nw}\sqrt{\widetilde{N}\widetilde{W} + 4}}$$

for $C = 1$.

$$\widetilde{N}, \widetilde{W} = 0 \Rightarrow S_{BH} = 4\pi\sqrt{nw}$$

(Dabholkar, ...) !

(A.S. hep-th/0505122)

For heterotic string theory on $K3 \times S^1 \times \tilde{S}^1$, the microscopic entropy of these black holes can be computed by representing them as configuration of 5-branes in dual M-theory on $CY_3 \times S^1$

Maldacena, Strominger, Witten

Result

$$\boxed{S_{stat} = 2\pi\sqrt{nw}\,\sqrt{\tilde{N}\tilde{W}+4}}$$

in exact agreement with S_{BH}!

We should have needed the full set of α' corrections to reproduce the answer.

Is this an accident?

Notes:

1. The agreement at first order in a power series expansion in $(\tilde{N}\tilde{W})^{-1}$ was found by Maldacena, Strominger and Witten.

2. Cardoso, de Wit and Mohaupt considered a different set of terms in the action which are obtained by supersymmetrizing the Weyl tensor^2 term.

This also reproduced the same answer for the entropy.

Is this an accident?

Entropy is the
Legendre transform of
the "Lagrangian density