SUPERSYMMETRIC BLACK HOLE PARTITION FUNCTIONS

Strings 05

.. with G.L. Cardoso, J. Käppeli, T. Mohaupt

BPS $\Leftrightarrow$ extremal black holes

& N = 2, 4 supersymmetry

- String/M theory microstates

  $\rightarrow$ entropy (statistical/microscopic)

- black hole solutions $\rightarrow$ area

'96 Strominger Vafa $\longrightarrow$ entropy (macroscopic)

area law

$$S_{macro} = \frac{A}{4G_N}$$

associated with the first law of black hole mechanics

$N=2$ black hole solutions (charged)

solitonic interpolation

$$\boxed{\text{HORIZON}} \longleftrightarrow \boxed{\text{SPATIAL } \infty}$$

$N=2$ susy — $N=1$ susy BPS/extremal — $N=2$ susy

attractor mechanism
'96 Ferrara, Kallosh, Strominger

moduli fixed by charges

moduli: integration constants

- Wilsonian action encoded in holomorphic homogeneous function $F(Y)$
- subject to e/m duality
- chiral superspace densities ("F-terms")

chiral fields

$Y^I + \cdots + \theta^2 F_{\mu\nu}{}^I + \cdots$ vector multiplet (reduced chiral)

$\mathcal{I} + \cdots \quad \cdots \quad \theta^4 R^2 + \cdots$ (conf) supergravity (Weyl multiplet)^2

CORRECTIONS / EXTENSIONS

① subleading in charges $\Leftrightarrow$ R^2 - terms

Wilsonian — Wald entropy

② nonholomorphic terms $\Leftrightarrow$ non-Wilsonian (nonlocal)

(Cardoso, dW, Mohaupt '98-'99)

attractor equations at horizon

$$Y^I - \bar{Y}^I = i\, p^I \qquad \text{magnetic}$$
$$F_I - \bar{F}_I = i\, q_I \qquad \text{electric}$$

e/m duality

$$F_I \equiv \frac{\partial F}{\partial Y^I}$$

$\Upsilon = -64$ Weyl^2 multiplet

$$F(Y, \Upsilon, \bar{Y}, \bar{\Upsilon})$$

R^2-terms

non-holomorphic

$$\frac{\text{Area}}{G_N} = 4\pi\ |Z|^2 \quad \propto \sqrt{Q^4}$$

$$|Z|^2 = p^I F_I - q_I Y^I$$

$$\mathcal{S}_{macro} = \pi \Big\{ \underset{\text{area}}{|Z|^2} - 256\ \underbrace{\mathrm{Im}\ F_\Upsilon}_{\text{subleading}} \Big\}_{\Upsilon = -64}$$

(CdWM '98)

Wald entropy: deviates from area law

(1-st law)

M theory on $CY_3 \times S^1$

$$F(Y,\Upsilon) = -\frac{1}{6}\frac{C_{ABC}Y^AY^BY^C}{Y^0} - \frac{c_{2A}\,Y^A}{24\cdot 64\ Y^0}\ \Upsilon$$

$$\mathcal{S} = 2\pi\sqrt{\frac{1}{6}|\hat{q}_0|\left(C_{ABC}\,p^Ap^Bp^C + \underset{\text{subleading}}{c_{2A}\,p^A}\right)}$$

microscopic: Maldacena, Strominger, Witten
Vafa '97

macroscopic: CdWM '98

area law:

$$\frac{A}{4G_N} = \frac{c_R}{c_L}\ \mathcal{S}_{macro}(p,q)$$

$c_{L,R}$: L-R central charge

Heterotic black holes ($\approx$ type II on $K3 \times T^2$)

$\mathcal{N}=2$ lowest order $\quad F(Y) = -\dfrac{Y^1}{Y^0}\, Y^a \eta_{ab} Y^b$

$$a,b = 2,\ldots,n$$

symmetry: $\left[\dfrac{SU(1,1)}{U(1)}\right]_S \times \left[\dfrac{SO(2,n-1)}{SO(2)\times SO(n-1)}\right]_T$ $\quad$ $2(n+1)$ charges

$$\downarrow$$

extension to $\mathcal{N}=4$ $\quad \left[\dfrac{SO(6,22)}{SO(6)\times S(22)}\right]_T$ $\quad$ $n=23$ $+\,2\cdot 4$ charges

$$S_{macro}(p,q) = \frac{A}{4G_N} = \pi\sqrt{q^2p^2-(q\cdot p)^2}$$

$T \times S$ invariant

(p^2, q^2 <u>not</u> positive definite)

in $\mathcal{N}=4$ extension: 2 types of BPS states!

- $\frac{1}{4}$ - BPS $\quad$ 'dyonic' $\quad q^2p^2-(p\cdot q)^2 > 0$

- $\frac{1}{2}$ - BPS $\quad$ 'electric' $\quad q^2p^2-(p\cdot q)^2 = 0$

<u>zero</u> classical area
related to perturbative
string states
(Dabholkar, Harvey '89)

microscopic

- **¼-BPS states**

dyonic degeneracies

$$d(p,q) = \oint_{3\text{-cycle}} d\Omega \; \frac{e^{i\pi(Q,\Omega Q)}}{\Phi_{10}(\Omega)}$$

Ω: period matrix
$g=2$ R-surface

formally S-duality invariant

(Dijkgraaf, Verlinde, Verlinde '96)
(Shi, Strominger, Yin '05)

contains electric states (subtleties)

- **½-BPS states**

related to perturbative string states

$$d(q) = \oint_{1\text{-cycle}} d\sigma \; \frac{e^{i\pi\sigma}}{\eta^{24}(\sigma)}$$

discrepancy with vanishing of the classical area

macroscopic

include both R^2-terms and nonholomorphic corrections (for S-duality)

$$F(Y,\bar{Y},\Upsilon,\bar{\Upsilon}) = -\frac{Y^1}{Y^0}\, Y^a \eta_{ab} Y^b$$

$$+ \frac{i}{64\pi}\, \Upsilon \log \eta^{12}(S) + \frac{i}{128\pi}(\Upsilon+\bar{\Upsilon}) \log (S+\bar{S})^6$$

(CdWM '99)

($Y^1/Y^0 = iS$ dilaton)

yields:

$$\mathcal{S}_{macro} = -\pi \left[\frac{q^2 - i\, p\cdot q\,(S-\bar{S}) + p^2 |S|^2}{S+\bar{S}} \right]$$

$$-2 \log \left[(S+\bar{S})^6\, |\eta(S)|^{24} \right]$$

attractor eq: $\boxed{\dfrac{\partial \mathcal{S}_{macro}}{\partial S} = 0}$

- 1/4-BPS emerges in saddle-point approximation from DVV formula (including $e^{-2\pi S}$ terms) (Cardoso, dW, Käppeli, Mohaupt, '04)
- 1/2-BPS $q^2 p^2 - (p\cdot q)^2 = 0$ "subleading becomes leading"

$$S+\bar{S} \approx \sqrt{|q^2|/2} \quad \text{weak string coupling}$$

$$\mathcal{S}_{macro} \approx 4\pi \sqrt{|q^2|/2} - \underset{\underset{vs}{12}}{6} \log |q^2|$$

27/4

partition functions, statistical ensembles

$$S_{macro}(p,q) = \mathcal{F}(p,\phi) - q_I \phi^I$$
$$\frac{\partial \mathcal{F}}{\partial \phi^I} = q_I$$
$\Big\}$ Legendre transform

with $\mathcal{F}(p,\phi) = 4\pi \, \mathrm{Im} \, F(Y,\Upsilon) \Big/_{\substack{\Upsilon = -64 \\ Y^I = \frac{\phi^I}{2\pi} + \frac{i p^I}{2}}}$

(Ooguri, Strominger, Vafa '04)

→ ensemble over electric charges q_I with fixed p^I and given electric potentials ϕ^I

partition function

$$Z_{BH} = e^{\mathcal{F}} = \Big| \underbrace{e^{-2\pi i F}}_{\text{topological string}} \Big|^2 = \sum_{\{q\}} d(q,p)\, e^{q_I \phi^I}$$

→ inverse Laplace transform

$$d(q,p) \sim \int d\phi \; \big| e^{-2\pi i F} \big|^2 \, e^{-q_I \phi^I}$$

i.e. microscopic black hole degeneracies from topological string amplitudes!

integral representations for $d(q,p)$

large q → Laplace transform
→ Legendre transform

i.e. saddle point approximation yields

$$d(q,p) \approx e^{S_{macro}(q,p)}$$

but:

- is the integral well-defined/convergent
- does it respect the putative symmetries
- should one include non-holomorphic terms
- how far can one go: nonperturbative, inverse powers of charges, ... ?

Verlinde, Sen, Dabholkar, Denef, Moore, Pioline, Ooguri, Vafa

etc

Variational principles

$$\Sigma = -i(\bar{Y}^I F_I - Y^I \bar{F}_I) + p^I(F_I + \bar{F}_I) - q_I(Y^I + \bar{Y}^I)$$

(consistent w.r.t. e/m duality)

$$\delta\Sigma = i(Y^I - \bar{Y}^I - i p^I)\, \delta(F_I + \bar{F}_I) - i(F_I - \bar{F}_I - i q_I)\, \delta(Y^I + \bar{Y}^I)$$

$\delta\Sigma = 0 \Rightarrow$ attractor eqs.

$$\Sigma\Big|_{\text{attractor}} = \frac{1}{\pi}\, \mathcal{S}_{\text{macro}}(p,q)$$

(Behrndt et al '96)
(see also: Ooguri, Vafa, Verlinde '05)

'reduction'

$$\Sigma\Big|_{Y-\bar{Y}-ip=0} = \frac{1}{\pi}\left[\mathcal{F}(\phi,p) - q_I\,\phi^I\right]$$

$\to$ OSV proposal

further reductions possible for heterotic b.h.

generalization with $\mathcal{R}^2$-terms and/or non holomorphic corrections

Integral representation(s) and saddle point appr.

$$d(p,q) \sim \int dY\, d\bar{Y}\; e^{\pi \Sigma(Y,\bar{Y},p,q)}$$ complex ('phase space')

$\downarrow$

$$d(p,q) \sim \int d(Y+\bar{Y})\; e^{\mathcal{F} - q\cdot\phi}$$ real (OSV)

$\downarrow$ (for heterotic)

$$d(p,q) \sim \int d^2S\; e^{S_{macro}(S,\bar{S},p,q)}$$ dilatonic

$\downarrow$

$$e^{S_{macro}(p,q)}$$

up to integration measure/fluctuation determinants, which should be large but slowly varying (happens in saddle point appr. dyonic/DVV case)

$\boxed{\text{strategy}}$ one can determine the form of Σ and the measure in the complex representation

(see also, Verlinde and Ooguri, Vafa, Verlinde) in principle there exists a natural measure which ensures that the appropriate symmetries are satisfied.

For the complex integral:

$$\prod_{I,J} dY^I \, d\bar{Y}^J \, \left| \det(\mathrm{Im}\, F_{KL}) \right|$$

- can be extended with nonholomorphic terms
- $S \times T$ duality invariant

upon saddle-point approximations (when appropriate) $\rightarrow$ measure for real and dilatonic integrals (partial cancellations against fluctuation dets $\rightarrow$ $\boxed{\exp(S_{macro}(p,q))}$

subtleties: saddle points (at intermediate stages) $\neq$ attractor values !

and: for ½-BPS measure vanishes in the real (OSV) integral up to nonperturbative ($e^{-2\pi S}$) and/or nonholomorphic terms.

specifically: evaluate complex representation

$$\int \frac{d^2S}{(S+\bar{S})^2}\left\{\int d^2z\, \mathcal{M}_n(S,\bar{S},z,\bar{z}) \quad \exp\left(\pi\Sigma_{\rm eff}\right)\right\}$$

$$\pi\Sigma_{\rm eff} = \frac{1}{1-z-\bar{z}}\left[|z|^2 + \mathrm{Re}\left(e^{2i\alpha}\,\bar{z}(1-\bar{z})(S+\bar{S})\frac{\partial}{\partial S}\right)\right]$$

$$\underbrace{\left[\pi\,\frac{q^2-2ip\cdot q\,(S-\bar{S})-p^2|S|^2}{S+\bar{S}} - 2\,\frac{1-z-\bar{z}}{|z|^2}\log\left[(S+\bar{S})\,|\eta(S)|^{24}\right]\right]}_{S,T\text{ - inv}}$$

note

$$z=1 \Leftrightarrow \text{attractor value}\begin{cases} Y^0-\bar{Y}^0 = ip^0 \\ Y^1-\bar{Y}^1 = ip^1\end{cases}$$

$$\downarrow$$

$$\pi\Sigma_{\rm eff} = S_{macro}(S,\bar{S},p,q)$$

subtlety phase $e^{2i\alpha} = \dfrac{p^1-iSp^0}{p^1+i\bar{S}p^0}$

independence of α $\to$ $S\times T$ duality inv !

partially checked.

To conclude

$$\int \frac{d^2 S}{(S+\bar{S})^2} \quad e^{S_{macro}(S,\bar{S},p,q)}$$

saddle point $\rightarrow$ $e^{S_{macro}(p,q)}$ as before (without nonholo terms)

fluctuation determinant

- electric $\sqrt{\frac{(S+\bar{S})^3}{|q^2|/2}} \approx \sqrt{S+\bar{S}}$

$$\Rightarrow \exp\left(-\tfrac{3}{2}\ln(S+\bar{S}) - 6\log|q^2|\right)$$

$$\rightarrow \exp\left(-\tfrac{27}{4}\log|q^2|\right)$$

z-integral/measure $\quad z \sim O(1)$

S duality vital for this 'refinement'

- dyonic $\frac{(S+\bar{S})^2}{\sqrt{p^2q^2-(p\cdot q)^2}}$ $\rightarrow$ cancels

  $\sqrt{p^2q^2-(p\cdot q)^2}$ $\rightarrow$ cancels against leading order z-integral (cf. DVV saddle point)

  S×T duality

Somewhat misleading, but .....

Things seem to work, but the z- and S- integrals become nontrivially entangled!