Supersymmetric Spacetime with torsion

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In their proposed compactification of superstring, Candelas, Horowitz, Strominger and Witten took the metric product of a maximal symmetric four dimensional spacetime M with a six dimensional Calabi-Yau vacua X as the ten dimensional spacetime; they identified the Yang-Mills connection with the SU(3) connection of the Calabi-Yau metric and set the dilaton to be a constant. To make this theory compatible with the standard grand unified field theory, Witten and Horava-Witten proposed to use higher rank bundles for strong coupled heterotic string theory so that the gauge groups can be SU(4) or SU(5). Mathematically, this approach relies on Uhlenbeck-Yau's theorem on constructing Hermitian-Yang-Mills connections over stable bundles.

A. Strominger analyzed heterotic superstring background with spacetime sypersymmetry and non-zero torsion by allowing a scalar "warp factor" to multiply the spacetime metric. He considered a ten dimensional spacetime that is the product $M \times X$ of a maximal symmetric four dimensional spacetime M and an internal space X; the metric on $M \times X$ takes the form

$$e^{2D(y)} \begin{pmatrix} g_{ij}(y) & 0 \\ 0 & g_{\mu\nu}(x) \end{pmatrix}, \qquad x \in X, \quad y \in M;$$

The connection on an auxiliary bundle is Hermitian-Yang-Mills over X:

$$F \wedge \omega^2 = 0, \quad F^{2,0} = F^{0,2} = 0.$$

Here ω is the hermitian form $\omega = \sqrt{-1} g_{i\bar{j}} dz^i d\bar{z}^j$.

In this system, the physical relevant quantities are

$$h = \frac{\sqrt{-1}}{2}(\bar{\partial} - \partial)\omega,$$

$$\phi = \frac{1}{8} \log \|\Omega\| + \phi_0,$$

and

$$g_{i\bar{j}}^0 = e^{2\phi_0} \|\Omega\|^{\frac{1}{4}} g_{i\bar{j}},$$

for a constant ϕ_0. The spacetime supersymmetry forces $D(y)$ to be the dilaton field.

In order for such ansatze to provide a supersymmetric configuration, one introduces a Majorana-Weyl spinor ϵ so that

$$\delta\phi_j^0 = \nabla_j^0\epsilon^0 + \frac{1}{48}e^{2\phi}(\gamma_j^0 H^0 - 12h_j^0)\epsilon^0 = 0,$$

$$\delta\lambda^0 = \nabla^0\phi\epsilon^0 + \frac{1}{24}e^{2\phi}h^0\epsilon^0 = 0,$$

$$\delta\chi^0 = e^{\phi}F_{ij}\Gamma^{0ij}\epsilon^0 = 0,$$

where ψ^0 is the gravitano, λ^0 is the dilatino, χ^0 is the gluino, ϕ is the dilaton and h is the Kalb-Ramond filed strength obeying

$$dh = \mathrm{tr}F \wedge F - \mathrm{tr}R \wedge R$$

Write

$$
\begin{aligned}
g_{MN} &= e^{-2\phi} g^{\circ}_{MN} \\
\varepsilon &= e^{-\phi/2} \varepsilon^{\circ} \\
\psi_M &= e^{-\phi/2} \left(\psi^{\circ}_M - \frac{1}{2} \Upsilon^{\circ}_M \lambda^{\circ} \right) \\
\lambda &= e^{\phi/2} \lambda^{\circ} \\
\Upsilon_M &= e^{\phi} \Upsilon^{\circ}_M .
\end{aligned}
$$

Then

$$
\begin{aligned}
\nabla_M \varepsilon - \frac{1}{4} H_M \varepsilon = 0 \\
(\not{\nabla} \phi) \varepsilon + \frac{1}{24} H \varepsilon = 0.
\end{aligned}
$$

Then there exist positive and negative chirality spinors $\eta_\pm$ that are H-covariantly constant. (The three form H_{MNP} defines a connection. Note that we assume ϕ depends only on K and the components of H tangent to the maximally symmetric spacetime vanish.)

We normalize

$$\eta_\pm^\dagger \eta_\pm = 1.$$

Then

$$J_m^n = \sqrt{1}\, \eta_+^\dagger \Gamma_m^n \eta_+$$

is an almost complex structure.

J_m^n is H-covariant constant

$$\nabla_m J_n^p - H_{sm}^p J_n^s - H_{mn}^s J_s^p = 0.$$

It is integrable.

The Kähler form is

$$\begin{aligned}\omega &= \frac{1}{2} J_m^n g_{np}\, dx^m \wedge dx^p \\ &= \sqrt{-1}\, g_{a\bar{b}}\, dz^a \wedge d\bar{z}^b\end{aligned}$$

and the torsion is

$$H = \frac{\sqrt{-1}}{2}(\bar{\partial} - \partial)\omega.$$

The holomorphic n form is given by

$$\Omega = e^{8\phi}\eta_-^\dagger\Gamma_{a_1a_2a_3}\eta_-\, dz^{a_1}\cdots dz^{a_n}.$$

It turns out that

$$\phi - \frac{1}{8}\ln\|\Omega\| \quad \text{is a constant.}$$

Since $\delta\lambda = 0$,

$$-8\nabla_m\phi + J_m^n\nabla_p J_n^p = 0$$

and

$$d^*\omega - \sqrt{-1}\,(\bar\partial - \partial)\ln\|\Omega\|_\omega = 0$$

and we arrive at the equation of Strominger.

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Assume that M is a complex 3 dimensional manifold so that

① there exists a Hermitian form J (ω)
so that $d(J \wedge J) = 0$ → $d(J^2) = 0$.
balanced

② Let V be a holomorphic bundle which is stable with respect to J. $c_1(V) = 0$ and

$$c_2(V) = c_2(M)$$

$\frac{dJ = 0}{\Updownarrow}$ Kahler

③ There is a non-vanishing holomorphic 3-form over M.

Conjecture: There is a unique solution of Strominger's system.

System has to be elliptic

Two ways to solve the Strominger system

I. Perturbation method (with Jun Li)

When the vector bundle is direct sum of tangent bundle plus trivial bundle and the manifold is Calabi-Yau, the Strominger system can be solved trivillay. (The hermitian Yang-Mills connection is simply the Levi-Civita connection plus trivial connection.)

Let E_s be smooth family of holomorphic vector bandles over X. Let h_0 be a Hermitian–Yang–Mills connection on E_0.

Then we like to extend h_0 to be a smooth family of Hermitian–Yang–Mills connection.

The interesting case is when h_0 is reducible.

Let (X, ω_0) be Kähler. Let (E_1, D_1'') and (E_2, D_2'') be degree zero and slope-stable vector bundles. Let h_1 and h_2 be the Hermitian metrics on E_1 and E_2 respectively. Then $h_1 \oplus th_2$ is still a Hermitian metric corresponds to the connection $D_0'' = D_1'' \oplus D_2''$.

$E \simeq T_M \oplus \mathcal{O} \to E'$

Suppose we are given a deformation of holomorphic structure D''_s of D''_0. Then Kodaira–Spensor identifies the first order deformation of D''_s at 0 to an element

$$k \in H^1(X, \varepsilon^* \otimes \varepsilon)$$

where ε is the sheaf of holomorphic section s of (E, D''_0). Therefore

$$k \in \oplus_{i,j=1}^{2} H^1(\varepsilon_i^* \otimes \varepsilon_j).$$

Theorem Suppose k_{12} and k_{21} are nonzero. Then there is a unique t so that for s sufficiently small $h_0(t) = h_1 \oplus e^t h_2$ extends to a smooth family of Hermitian–Yang–Mills metric on (E, D''_s).

The fourth equation of Strominger system is equivalent to

$$d\left(\|\Omega\|_\omega \omega^2\right) = 0.$$

balanced

Let $\mathcal{H}(X)$ be the cone of positive definite Hermitian form on X. Let $\mathcal{H}(E)_0$ be the space of determinant one Hermitian metric on the bundle E (i.e., the induced metric on $\wedge^r E \simeq \mathbb{C}_X$ is the constant one metric).

We define

$$
\begin{aligned}
&L = L_1 \oplus L_2 \oplus L_3 : \\
&\mathcal{H}(E)_0 \times \mathcal{H}(X) \longrightarrow \\
&\Omega^{3,3}(\mathrm{End}^0 E) \oplus \mathrm{Im}\, \sqrt{-1}\partial\bar{\partial} \oplus \mathrm{Im}\, d_0^*
\end{aligned}
$$

where

$$
\begin{aligned}
L_1(H,\omega) &= \sqrt{-1}F_H \wedge \omega^2 \\
L_2(H,\omega) &= \sqrt{-1}\partial\bar{\partial}\omega - \mathrm{tr}_E(F_H \wedge F_H) \\
&\qquad + \mathrm{tr}_T(R_g \wedge R_g) \\
L_3(H,\omega) &= *_0 d\left(\|\Omega\|_\omega \omega^2\right).
\end{aligned}
$$

We shall apply implicit function theorem to L.

Fix a determinant one Hermitian metric $\langle \, , \, \rangle$ on E. We can write other determinant one Hermitian metric on E by a unique positive definite $\langle \, , \, \rangle$-Hermitian symmetric endomorphism H of E satisfying $\det H = 1$.

Such spaces of H will be denoted by $\Gamma(\mathrm{End}_h^+ E)$. Identity $I \in \Gamma(\mathrm{End}_h^+ E)$.

The tangent space at I is $\Gamma(\mathrm{End}_h^0 E)$ traceless symmetric endomorphisms of E.

$$
\begin{aligned}
& \delta L_1(I,\omega_0)(\delta h,\delta\omega) \\
= \quad & D''D'_H\delta h + 2F_H \wedge \omega_0 \wedge \delta\omega \\
& \delta L_2(I,\omega_0)(\delta h,\delta\omega) \\
= \quad & \sqrt{-1}\partial\bar{\partial}(\delta\omega) - 2(\mathrm{tr}_E \delta F_I(\delta h) \wedge F_I) \\
& \quad + \mathrm{tr}_T \delta R_{g_0}(\delta g) \wedge R_{g_0} \\
& \delta L_3(I,\omega_0)(\delta h,\delta\omega) \\
= \quad & 2d_0^*(\delta\omega) - d_0^*((\delta\omega,\omega_0)\omega_0).
\end{aligned}
$$

Construction of irreducible solution to Strominger's system perturbatively.

Start with a Calabi–Yau manifold,

$$(E, D_0'') = C_X^{\oplus(r-3)} \oplus T_X,$$

the metric is identified with $I : E \longrightarrow E$.

For all $c > 0$, $(I, c\omega_0)$ is a solution to $L = 0$.

Let

$$W_1 = \Omega^{3,3}_{\mathbb{R}}(\mathrm{End}^0_h E)_{L^p_{k-2}}$$

$$W_2 = (\mathrm{Im}\,\sqrt{-1}\,\partial\bar{\partial})_{L^p_{k-2}} \oplus (\mathrm{Im}\, d_0^*)_{L^p_{k-1}}$$

$$V_0 = \Big\{ A \oplus aI_{T_X} \mid \quad A \in \mathrm{End}\,\mathbb{C}_X^{\oplus(r-3)} \text{ are constant matrices such that } A = A^{-t},\ \mathrm{tr}\,A + 3a = 0 \Big\}$$

$$V_1 = \omega_0^3 \otimes V_0.$$

Then $\exists\, C > 0$ such that for all $c > C$,

$$\delta L_1(I, c\omega_0) \oplus \delta L_2(I, c\omega_0) \oplus \delta L_3(I, c\omega_0)$$

$$: \quad \Gamma(\mathrm{End}^0_h E)_{L^p_k} \oplus \Omega^{1,1}(X) \longrightarrow W_1/V_1 \oplus W_2$$

is surjective.

Theorem. Let X be a Calabi–Yau 3-fold with ω a Ricci flat Kähler form. Let D''_s be a smooth deformation of holomorphic structure D''_0 on $E = \mathbb{C}_X \oplus T_X$. Suppose the associated cohomology classes $[C_{12}]$ and $[C_{21}]$ are non-zero. Then for sufficiently large c there is a family of pairs of Hamiltonian metrics and Hamiltonian forms (H_s, ω_s) for $0 \leq s < \varepsilon$ so that

1. $\omega_0 = c\omega$ and the harmonic part of ω_s is equal to $c\omega$.

2. The pair (H_s, ω_s) is a solution to Strominger's system for the holomorphic vector bundle (E, D''_s).

Let

$$\begin{aligned} D''_s &= D''_0 + A_3, A_s \in \Omega^{0,1}(\operatorname{End} E) \\ A_0 &= \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \in \Omega^{0,1}(\operatorname{End} E). \end{aligned}$$

We can assume C_{ij} are D''_0 harmonic. Since $H^1(X, \mathcal{O}_X) = 0$, $C_{11} = 0$.

In general, we consider the $r+3$ holomorphic vector bundle $\mathbb{C}_X^{\oplus r} \oplus T_X$. We also have

$$D_0'' = \begin{pmatrix} 0 & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

where

$$C_{12} = (\alpha_1, \ldots, \alpha_r)^t \in \Omega^{0,1}(T_X)^{\oplus j}$$
$$C_{21} = (\beta_1, \ldots, \beta_r) \in \Omega^{0,1}(T_X^*)^{\oplus j}$$
$$C_{22} \in \Omega^{0,1}(\mathrm{End}\, T_X).$$

Suppose $[\alpha_1], \ldots, [\alpha_r] \in H^1(X, T_X^*)$ are linearly independent and $[\beta_1], \ldots, [\beta_r] \in H^1(X, T_X^*)$ are linearly independent. Then the above theorem holds.

Example

Consider

$$X = \{z_0^5 + \cdots + z_x^5 = 0\} \subset \mathbf{P}^4$$

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & TX & \longrightarrow & T_X P^4 & \longrightarrow & \mathcal{O}_X(5) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & F & \longrightarrow & \mathcal{O}_X(1)^{\oplus 5} & \longrightarrow & \mathcal{O}_X(5) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & \mathcal{O}_X & = & \mathcal{O}_X & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Here F is the kernel of $\mathcal{O}_X(1)^{\oplus 5} \longrightarrow \mathcal{O}_X(5)$ and fill in

$$0 \longrightarrow \mathcal{O}_X \longrightarrow F \longrightarrow TX \longrightarrow 0$$

The above sequence is a non-split extension.

Making use of this element in $\mathrm{Ext}^1(T_X, \mathcal{O}_X)$ we can form a deformation of holomorphic structure D_t'' of that $C_{12} \neq 0$, $C_{21} \neq 0$.

Hence we have proved:

Theorem. Let X be a smooth quintic threefold and ω be any Kähler form on X. Then for large $c > 0$, there is a smooth deformation of $\mathbb{C}_X \oplus T_X$ so that for small s, there are pairs (H_s, ω_s) of Hermition metrics on E and Hermition forms ω_s on X. That solves Strominger's system.

For the Calabi–Yau manifold with three generation that I constructed:

$$X \subset \mathbf{P}^3 \times \mathbf{P}^3$$

given by

$$\sum x_i^3 = 0$$
$$\sum y_i^3 = 0$$
$$\sum x_i y_i = 0$$

quotiented by $\mathbb{Z}_3$. One can also construct irreducible solution to Strominger's system on $TX \oplus \mathbb{C}_X^{\oplus 2}$.

II. Construction method on the non-Kähler case (with Jixiang Fu)

The most common examples of non-Kaehler manifolds X are some T^2 bundles over Calabi-Yau varieties by K.Berk, M.Berk, K.Dasgupta, P.S.Green, E.Sharpe, G.L.Cardoso, G.Curio, G.Dall'Agata, E.Goldestein, S.Prokushkin, S.Gurrieri, J.Lious, A.Micu, D.Waldram, S.Kachru, M.B.Schulz, P.K.Tripathy. Because internal six manifold X is a complex manifold with a non-vanishing holomorphic three form Ω, at first we may considerthe T^2 bundles (X, ω, Ω) over complex surfaces (S, ω_S, Ω_S) with non-vanishing holomorphic 2-form Ω_S. According to the classification of complex surfaces by Enriques and Kodaira, such surfaces include K3 surface and complex torus

(Calabi-Yau) and Kodaira surface (non-Kahler).

If (X, ω, Ω) satisfies the Strominger's equation (4), then the above Lemma tells us that $d(\| \Omega \| \omega^2) = 0$. If we let $\omega' = \| \Omega \|^{\frac{1}{2}} \omega$, then $d\omega'^2 = 0$. Michelson called ω' balanced metric. Because holomorphic submersion π from X to complex surface S is proper, Michelson proved that S is also balanced (actually $\pi_* \omega'^2$ is the balanced metric). Note that when $\dim_{\mathbb{C}} = 2$, the conditions of being balanced and Kaehler are equivalent. So S is Kaehler. Then there is no solution to Strominger's equation on T^2 bundles over Kodaira surfaces (which is not Kähler) and we should only consider the case of K3 surface and complex torus.

We construct this solution on some torus bundles over K3 surface or complex torus provided by Goldstein and Prokushkin. Let (S, ω_S, Ω_S) be a K3 surface or complex torus with Kahler form ω_S and a non-vanishing holomorphic (2,0) form Ω_S. Let ω_1 and ω_2 are anti-self-dual (1,1) forms on (S, ω_S) such that $\frac{\omega_1}{2\pi}$ and $\frac{\omega_2}{2\pi}$ represent integral cohomology classes. Using these two forms, Goldstein and Prokushkin constructed the non-Kahler manifold X such that $\pi : X \to S$ is a holomorphic T^2 fibration over S with hermitian form $\omega_0 = \pi^*\omega_S + \frac{\sqrt{-1}}{2}\theta \wedge \bar{\theta}$ and holomorphic 3 form $\Omega = \Omega_S \wedge \theta$, where $\theta = \theta_1 + \sqrt{-1}\theta_2$, (θ_1, θ_2) is the connection on the principal toric bundle X over S such that $d\theta_j = \pi^*\omega_j$.

Goldestein and Prokushkin have studied the cohomology of this non-Kähler manifold X:

$$h^{1,0}(X) = h^{1,0}(S),$$

$$h^{0,1}(X) = h^{0,1}(S) + 1;$$

In particular

$$h^{0,1}(X) = h^{1,0}(X) + 1.$$

Moreover,

$$\begin{aligned}
b_1(X) &= b_1(S) + 1, \quad \text{when } \omega_2 = n\omega_1, \\
b_1(X) &= b_1(S), \qquad\quad \text{when } \omega_2 \neq n\omega_1; \\
b_2(X) &= b_2(S) - 1, \quad \text{when } \omega_2 = n\omega_1, \\
b_2(X) &= b_2(S) - 2, \quad \text{when } \omega_2 \neq n\omega_1
\end{aligned}$$

and

$$b_3(X) = 0.$$

Let L_1 and L_2 be holomorphic line bundles over S such that their first Chern class are $c_1(L_1) = [-\frac{\omega_1}{2\pi}]$ and $c_2(L_2) = [-\frac{\omega_2}{2\pi}]$ respectively. Then we can choose hermitian metrics h_1 and h_2 on L_1 and L_2 such that their curvature forms are $\sqrt{-1}\omega_1$ and $\sqrt{-1}\omega_2$. Let

$$E = L_1 \oplus L_2 \oplus T'S$$

and

$$H_0 = (h_1, h_2, \omega_S).$$

Then

$$F_{H_0} = \operatorname{diag}(\sqrt{-1}\omega_1, \sqrt{-1}\omega_2, R_S).$$

Let u be any smooth function on S and let

$$\omega_u = \pi^*(e^u \omega_S) + \frac{\sqrt{-1}}{2}\theta \wedge \bar{\theta}$$

Then $(V = \pi^* E, F_{\pi^* H_0}, X, \omega_u)$ satisfies the Strominger's equation (1), (2) and (4). So we should only need to consider the equation (3).

Because ω_1 and ω_2 are harmonic, locally we can write

$$\begin{aligned}
\omega_1 &= \bar{\partial}\phi = \bar{\partial}(\phi_1 dz_1 + \phi_2 dz_2) \\
\omega_2 &= \bar{\partial}\psi = \bar{\partial}(\psi_1 dz_1 + \psi_2 dz_2)
\end{aligned}$$

where (z_1, z_2) is the local coordinate on S. Let

$$A = \begin{pmatrix} \phi_1 + \sqrt{-1}\psi_1 \\ \phi_2 + \sqrt{-1}\psi_2. \end{pmatrix}$$

Using matrix A we can calculate the curvature R_u of metric ω_u and $R_u \wedge R_u$.

Theorem $(V = \pi^* E, F_{\pi^* H_0}, X, \omega_u)$ is the solution of Strominger's system if and only if the function u of S satisfies the equation

$$\triangle e^u \cdot \frac{\omega_S^2}{2!} + \partial\bar{\partial}(e^{-u}\mathrm{tr}(\bar{\partial}A \wedge \partial A^* \cdot g^{-1})) + \partial\bar{\partial}u \wedge \partial\bar{\partial}u = 0 \tag{5}$$

where g is the metric corresponding to Kahler form ω_S. In particular, when $\omega_2 = n\omega_1$, $n \in \mathbb{Z}$, $(V, F_{\pi^* H}, X, \omega_u)$ is the solution to Strominger's system if and only if smooth function u on S satisfies the following equation:

$$\triangle \left(e^u + \frac{(1+n^2)}{4} \parallel \omega_1 \parallel^2 e^{-u} \right) - 8 \frac{\det(u_{i\bar{j}})}{\det(g_{i\bar{j}})} = 0 \tag{6}$$

Actually we can prove that

$$-\sqrt{-1}\operatorname{tr}(\bar{\partial} A \wedge \partial A^* \cdot g_S^{-1})$$

is a well-defined real (1,1)-form on S. In particular, when $\omega_2 = n\omega_1$, $n \in \mathbb{Z}$,

$$-\sqrt{-1}\operatorname{tr}(\bar{\partial} A \wedge \partial A^* \cdot g_S^{-1}) = \frac{1+n^2}{4} \parallel \omega_1 \parallel_{\omega_S}^2 \omega_S.$$

Let $f = \frac{(1+n^2)}{4} \parallel \omega_1 \parallel^2$ and let

$$g'_{i\bar{j}} = (e^u - fe^{-u})g_{i\bar{j}} - 4u_{i\bar{j}} > 0$$

then we can rewrite the equation (7) as

$$\frac{\det g'_{i\bar{j}}}{\det g_{i\bar{j}}} = (e^u - fe^{-u})^2 + 2(e^u + fe^{-u}) \mid \nabla u \mid^2 + 2e^{-u} \triangle f - 4e^{-u} \nabla u \cdot \nabla f$$

Zeroth order estimate

The estimate of $\inf u$ can be derived from elliptic condition and normalization. Timing elliptic condition $e^u - fe^{-u} > \triangle u$ by pe^{-pu} and integrating

$$\int_S \mid \nabla(e^{-u})^{\frac{p}{2}} \mid^2 \leq \frac{p}{4} \int_S (e^{-u})^{p-1}$$

Using Moser iteration and Poincare inequality,

$$\exp(-\inf u) = \parallel e^{-u} \parallel_\infty \leq C_0 \parallel e^{-u} \parallel_2^B$$

where

$$B = \prod_{\beta=1}^{\infty}(1 - \frac{1}{2^\beta})$$

The estimate $\sup u$ can be derived from the equation. Define the elliptic operator $P = 2g'^{i\bar{j}} \frac{\partial^2}{\partial z_i \partial \bar{z}_j}$. We have two methods to calculate

$$\int_S P(e^{pu}) \frac{\det g'_{i\bar{j}}}{\det g_{i\bar{j}}} \frac{\omega^2}{2!}.$$

Combining, we can get that for all $p \geq 2$,

$$\int_S \mid \nabla (e^u)^{\frac{p}{2}} \mid^2 \leq C_0 p \int_S e^{(p-2)u}$$

Then using the Moser iteration and Poincare inequality, we can get the estimate of $\sup u$.

Gradient estimate

Let

$$v(u) = e^{4\sup u - 2u} + e^{2u - 4\sup u}.$$

We compute

$$P(\ln \mid \nabla u \mid^2 + \ln v(u)) \frac{\det g'_{i\bar{j}}}{\det g_{i\bar{j}}}$$

at the point q where $\ln \mid \nabla u \mid^2 + \ln v(u)$ achieves its maxiumum. Then we can get

$$\mid \nabla u \mid^2 \leq C_1 \frac{\left(e^{4\sup u - 2\inf u} + e^{2\inf u - 4\sup u}\right)}{\left(e^{2\sup u} + e^{-2\sup u}\right)}$$

Second order estimate

By elliptic condition, $(e^u - fe^{-u})g_{i\bar{j}} - 4u_{i\bar{j}}$ is positive definite. So to get a second order estimate of u it sufficient to have an upper bound estimate of $e^u - fe^{-u} - \triangle u$. At the point where $e^u - fe^{-u} - \triangle u$ achieves its maximum, we have

$$P(e^u - fe^{-u} - \triangle u)\frac{\det g'_{i\bar{j}}}{\det g_{i\bar{j}}}$$
$$\geq (\triangle u)^2 - C_2 \triangle u - C_2.$$

Then inf $\triangle u$ follows.

Third order estimate

Let

$$
\begin{aligned}
T &= g^{i\bar{j}} g^{k\bar{l}} u_{ik} u_{\bar{j}\bar{l}} \\
U &= g'^{i\bar{r}} g'^{s\bar{j}} g'^{k\bar{t}} u_{i\bar{j}k} u_{\bar{r}s\bar{t}} \\
Q &= g'^{i\bar{j}} g'^{k\bar{l}} g'^{p\bar{q}} u_{ikp} u_{\bar{j}\bar{l}\bar{q}} \\
W &= g'^{i\bar{j}} g'^{k\bar{l}} g'^{p\bar{q}} g'^{r\bar{s}} u_{i\bar{l}pr} u_{\bar{j}k\bar{q}\bar{s}} \\
Y &= g'^{i\bar{j}} g'^{k\bar{l}} g'^{p\bar{q}} g'^{r\bar{s}} u_{i\bar{l}p\bar{s}} u_{\bar{j}k\bar{q}r}
\end{aligned}
$$

Let

$$
\begin{aligned}
Z = {} & (\kappa_1 - \triangle u) U + \kappa_2 (m - \triangle u) T \\
& + \kappa_3 \mid \bigtriangledown u \mid^2 T + \kappa_4 T
\end{aligned}
$$

where all κ_i are positive constants and m is a fixed constant such that $\kappa_1 - \triangle u > 1$ and $m - \triangle u > 0$.

At the point where Z achieves the maximum, we get

$$\begin{aligned} P(Z) \geq\ & \frac{m_2}{4}U^2 + \frac{m_2}{2}\kappa_1 UT \\ & + \frac{m_1}{2}k_2\kappa_1 T^2 + \frac{m_3}{2}\kappa_1 Q - C_3 \end{aligned}$$

where $k_2 = \frac{\kappa_2}{\kappa_1 - \triangle u}$, m_i denote constant which depend on f, S and u up to second order derivations. Above inequality gives the estimates of the quantity $\sup_S U$ and $\sup_S T$. This in turn gives the estimates of $u_{i\bar{j}k}$ and u_{ij}.

Because $\deg L_i = 0$ and TS is slope stable with respect to the Hodge class ω_S, we can prove

There is a family D''_s of deformations of holomorphic structures of E so that its k-th order for $k < m$ Kodaira-Spencer class κ all vanish while its m-th order Kodaira-Spencer class has non-vanishing summands in $H^1(L_i^\vee \otimes TS)$ and $H^1(TS^\vee \otimes L_i)$.

Fix the metric H_0 as the reference metric on E over S. Then for any hermitian metric H on E, we can define a smooth endomorphism h on E by

$$\langle s, t\rangle_H = \langle s \cdot h, t\rangle_{H_0}.$$

Under this isomorphism, we define $\mathcal{H}(E)_0$ be the space of all hermitian metric on E whose corresponding endomorphism has determinant one. Let $\mathcal{C}(\omega_S) = \{e^{\phi}\omega_S\}$ be the space of all hermitian metrics on S which are conformal to ω_S. Let $\mathcal{H}_0(S) = \left\{\psi\frac{\omega^2}{2!} \mid \int_S \psi\frac{\omega^2}{2!} = 0\right\}$.

We define the operator

$$\mathbf{L_s} = \mathbf{L_{s,1}} \oplus \mathbf{L_{s,2}} : \quad \mathcal{H}_0(E) \times \mathcal{C}(\omega_S) \\ \to \Omega^4_{\mathbb{R}}(\mathrm{End}^0 E) \oplus \mathcal{H}_0(S)$$

by

$$\begin{aligned} \mathbf{L_{s,1}}(h, e^{\phi}\omega_S) &= e^{\phi} h^{-\frac{1}{2}} F_{s,h} h^{\frac{1}{2}} \wedge \omega_S \\ \mathbf{L_{s,2}}(h, e^{\phi}\omega_S) &= \sqrt{-1}\partial\bar{\partial}(e^{u+\phi}\omega_S) \\ &\quad +\partial\bar{\partial}(e^{-u-\phi}\mathrm{tr}(\bar{\partial}A \wedge \partial A^* \cdot g^{-1}_{\omega_S})) \\ &\quad +\partial\bar{\partial}(u+\phi) \wedge \partial\bar{\partial}(u+\phi) \\ &\quad -\frac{1}{2}(\mathrm{tr}F_{s,h} \wedge F_{s,h} - \mathrm{tr}F_{s,H_0} \wedge F_{s,H_0}) \end{aligned}$$

If $(h, e^{\phi}\omega_S) \in \ker \mathbf{L_s}$, then

$$\left(V = \pi^* E, \pi^* D''_s, \pi^* h, \pi^*(e^{u+\phi}\omega_S) + \frac{\sqrt{-1}}{2}\theta \wedge \bar{\theta}\right)$$

is the solution of Strominger's system. Because $(V = \pi^* E, \pi^* D''_0, \pi^* H_0, \omega_u)$ is our reducible solution, $(I, \omega_S) \in \ker \mathbf{L_0}$.

Using the perturbation method, we can prove

Theorem Let (E, H_0, S, ω_S) be as before. Fix its holomorphic structure D_0''. Then there is a smooth deformation D_s'' of (E, D_0'') so that there are hermitian metric H_s on E and hermitian metric $e^{u+\phi_s}\omega_S$ on S such that

$$\left(V = \pi^* E, \pi^* D_s'', \pi^* H_s, \pi^*(e^{u+\phi_s}\omega_S) + \frac{\sqrt{-1}}{2}\theta \wedge \bar{\theta}\right)$$

are the irreducible solutions to strominger's system on X and so that $\lim_{s\to 0}\phi_s = 0$ and $\lim_{s\to 0} H_s$ is a regular reducible hermitian Yang-Mills connection on $E = L_1 \oplus L_2 \oplus TS$.