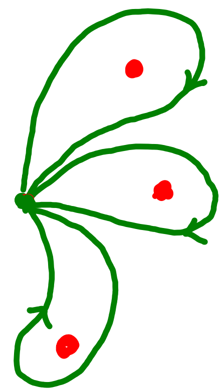
MONODROMY AND THE TOPOLOGICAL STRING



TO APPEAR

WITH V. BOUCHARD AND
A. KLEMM

SYMMETRY IS KEY TO ANY ATTEMPT TO UNDERSTAND A THEORY

SYMMETRY IN THE TOPOLOGICAL STRING

A-MODEL		B-MODEL
COUNTING OF HOLOMORPHIC MAPS TO X	MIRROR $\Longleftrightarrow$ SYMMETRY	VARIATIONS OF COMPLEX STRUCTURES OF Y

MANY PROPERTIES OF THE THEORY,

IN PARTICULAR THE SYMMETRIES,

BECOME MORE TRANSPARENT WHEN

MODULI OF X, Y ARE ALLOWED

TO VARY :

THE B-MODEL IS IDEALLY

SUITED FOR THIS.

RECALL SOME BASIC FACTS

ABOUT THIS THEORY

GIVEN A SYMPLECTIC BASIS OF $H_3(Y)$

$$A^I \cap B_J = \delta^I{}_J$$

AND A CHOICE OF A 3-FORM ω,

THE PERIODS

$$X^I = \oint_{A^I} \omega \quad , \quad P_I = \oint_{B_I} \omega$$

ARE NOT INDEPENDENT:

$$P_I = \frac{\partial}{\partial X^I} \underline{F_0}$$

GENUS ZERO TOPOLOGICAL STRING AMPLITUDE

$\Rightarrow$ $H^3(Y)$ = PHASE SPACE

WITH SYMPLECTIC FORM $dP_I \wedge dX^I$

WITTEN

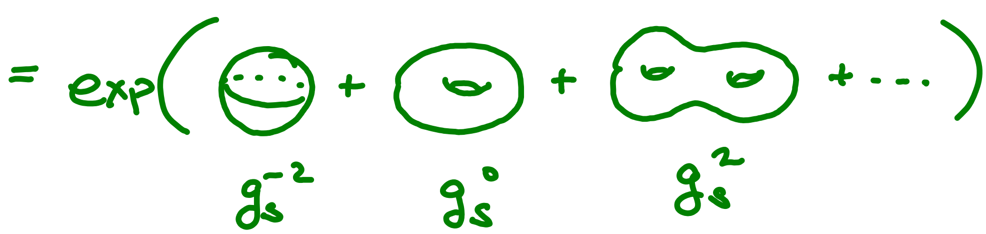
IN THE QUANTUM THEORY, p_I, x^J BECOME OPERATORS

$$[p_I, x^J] = g_s^2 \, \delta_I^{\ J}$$

AND, TOPOLOGICAL STRING PARTITION FN, A WAVE FUNCTION

$$Z(x) = \langle x | Z \rangle$$

$$Z(x) = \exp\left(\sum_{g=0} g_s^{2g-2} \mathcal{F}_g(x) \right)$$



$$p \, Z(x) = g_s^2 \frac{\partial}{\partial x} Z \sim \frac{\partial}{\partial x} \mathcal{F}_0 \quad \text{as } g_s \to 0$$

DIFFERENT CHOICES OF BASIS FOR $\mathcal{H}_3(Y)$ ARE RELATED BY SYMPLECTIC TRANSFORMATIONS

$$\begin{pmatrix} p \\ x \end{pmatrix} \to \begin{pmatrix} \tilde{p} \\ \tilde{x} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} p \\ x \end{pmatrix}$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2n, \mathbb{Z}).$$

THIS ACTS ON THE WAVE FUNCTION

$$\tilde{Z}(\tilde{x}) = \int dx\, e^{-\frac{S(x,\tilde{x})}{g_s^2}} Z(x)$$

$$S(\tilde{x},x) = -\frac{1}{2} x^T (C^{-1}D) x + x^T C^{-1} \tilde{x} - \frac{1}{2} \tilde{x}^T (AC^{-1}) \tilde{x}$$

FOR C INVERTIBLE.

FOR A SUBGROUP

$$\Gamma \subset Sp(2n, \mathbb{Z})$$

THIS IS A SYMMETRY: THE CHANGE

OF BASIS CAN BE UNDONE BY PICKING

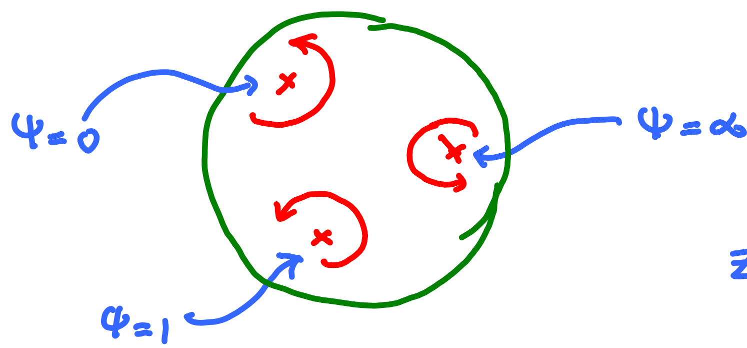
A DIFFERENT ω.

EXAMPLE:

THE MIRROR OF QUINTIC CY 3-FOLD:

$$\sum_{i=1}^{5} x_i^5 - 5\psi \prod_{i=1}^{5} x_i = 0, \quad \text{IN } \mathbb{P}^4$$

MODULI SPACE:



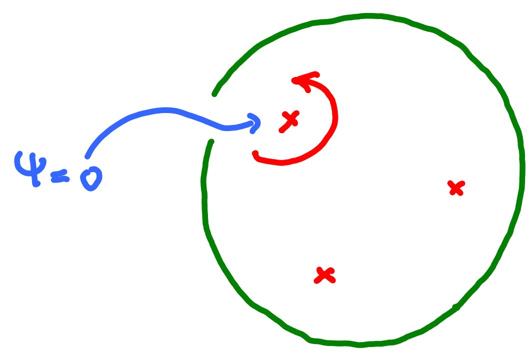
$$z = -\frac{1}{\psi^5}$$

THE SYMMETRY GROUP Γ IS GENERATED BY MONODROMIES

$$M_0, \quad M_1, \quad M_\infty$$

IN TERMS OF A NATURAL BASIS OF PERIODS

$$\Pi = \begin{pmatrix} P_1 \\ P_0 \\ X^1 \\ X^0 \end{pmatrix} \sim \begin{pmatrix} \text{4-cycle} \\ \text{6-cycle} \\ \text{2-cycle} \\ \text{0-cycle} \end{pmatrix}$$

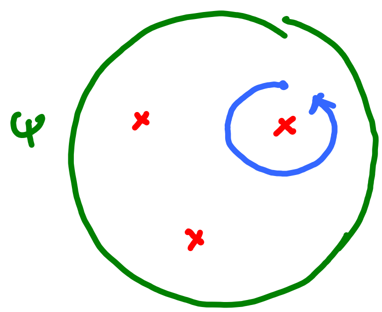


$$\Pi(\alpha\psi) = M_0\,\Pi(\psi)\,, \qquad \alpha^5 = 1$$

$$M_0 = \begin{pmatrix} -19 & -3 & 5 & -3 \\ 31 & -4 & -8 & -5 \\ -80 & -11 & 21 & -11 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \qquad M_0^5 = \mathrm{id}$$

WHAT DOES THIS MEAN FOR THE PARTITION FUNCTION $Z(x)$?

Z IS, AT LEAST LOCALLY, A FUNCTION ON THE MODULI SPACE $Z(X^I(\psi)) = Z(\psi)$



$$\psi \to M_\Gamma \psi$$

$$Z(\psi) \to Z(M_\Gamma \cdot \psi)$$

ON THE OTHER HAND:

$$\Pi(\psi) \to M_\Gamma \Pi(\psi)$$

$$Z(\psi) \to \int e^{-S_M} Z(\psi)$$

So

$$\boxed{Z(M_\Gamma \psi) = \int e^{-S_M} Z(\psi)}$$

WORKING PERTURBATIVELY IN g_s^2,

CHOICE OF CONTOUR DOES NOT ENTER.

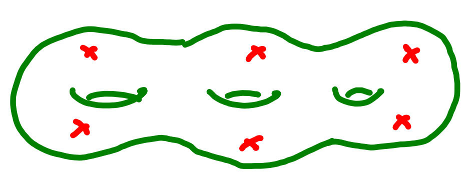
EXPANDING ABOUT THE SADDLE POINT:

INVERSE PROPAGATOR $$\Delta_{IJ} = (\tau + C^{-1}D)_{IJ}, \qquad M_\Gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

VERTICES $$\partial_{I_1}\cdots\partial_{I_N}\tilde{F}_g(\phi),$$

WHERE $$\tau_{IJ} = \frac{\partial}{\partial x^I} P_J$$

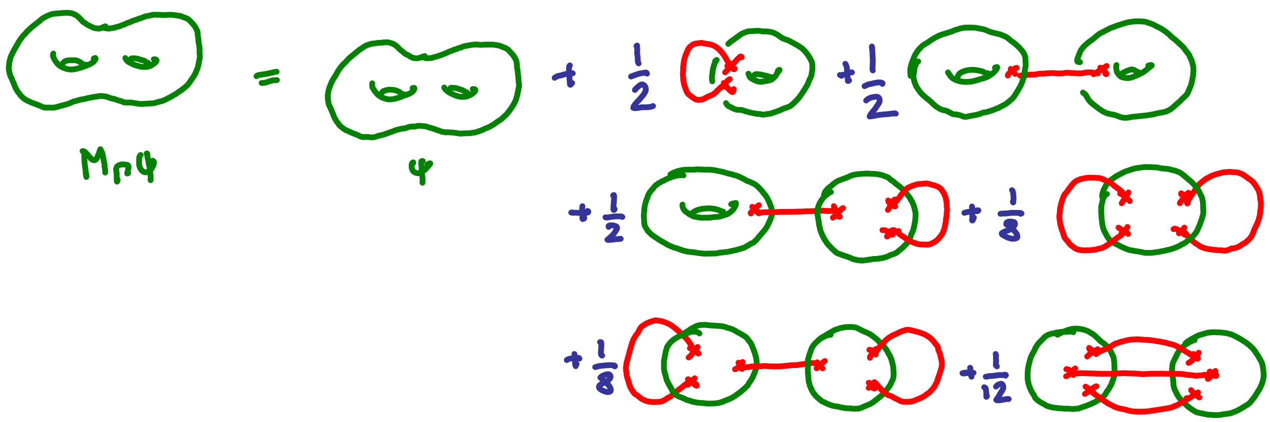
Δ^{IJ},



$$\Delta^{IK}\Delta_{KJ} = \delta^I_J$$

GENUS TWO:

$$\mathcal{F}_2(M_\Gamma \varphi) = \mathcal{F}_2(\varphi) + \Delta^{IJ}\left(\tfrac{1}{2}\partial_I\partial_J \mathcal{F}_1 + \tfrac{1}{2}\partial_I \mathcal{F}_1 \partial_J \mathcal{F}_1\right) + \cdots$$



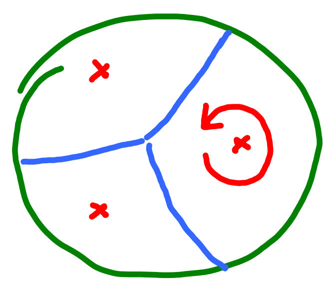
CORRESPONDING TO ALL POSSIBLE (STABLE)

DEGENERATIONS OF GENUS g RIEMANN SURFACE

NON-TRIVIAL MONODROMY AROUND A SINGULAR POINT CORRESPONDS TO CHOOSING A-CYCLES THAT ARE NOT WELL DEFINED :

$$X \to Cp + DX$$

$$\det C \neq 0$$



CONVERSELY, NEAR A SINGULAR PT THE "GOOD" VARIABLES ARE THOSE WITH NO NONTRIVIAL MONODROMY

THERE IS ANOTHER CHOICE OF POLARIZATION AVAILABLE:

PICK A BACKGROUND COMPLEX STRUCTURE Ω ON Y.

THAN, ANY $\omega \in \mathcal{H}^3(Y, \mathbb{C})$

$$\boxed{\omega = \varphi \Omega + z^i D_i \Omega + \bar{z}^i \overline{D_i \Omega} + \bar{\varphi} \overline{\Omega}}$$

COVARIANT W/ RESPECT TO KAHLER CONNECTION: $A_i = \partial_i K$, $\bar{A}_{\bar{i}} = \bar{\partial}_{\bar{i}} K$

USE z^i, φ as COORDINATES ON $\mathcal{H}^3(Y)$

$$\Rightarrow \qquad \hat{Z}(z^i;\varphi) = \langle z^i, \varphi | Z \rangle$$

DOES NOT REQUIRE A CHOICE OF SYMPLECTIC BASIS OF A- AND B-CYCLES.

CORRESPONDINGLY,

$$\hat{Z}(z^i;\varphi)$$

IS MONODROMY INVARIANT, AND WELL DEFINED ALL OVER THE MODULI SPACE. HOWEVER IT DEPENDS ON THE CHOICE OF BACKGROUND Ω, AND NOT HOLOMORPHICALLY.

MOREOVER, $\hat{Z}$ AND Z ARE CLOSELY RELATED.

$$\hat{Z}(z,\varphi) = \int e^{-\hat{S}(z,\varphi;x)/g_s^2}\, Z(x)$$

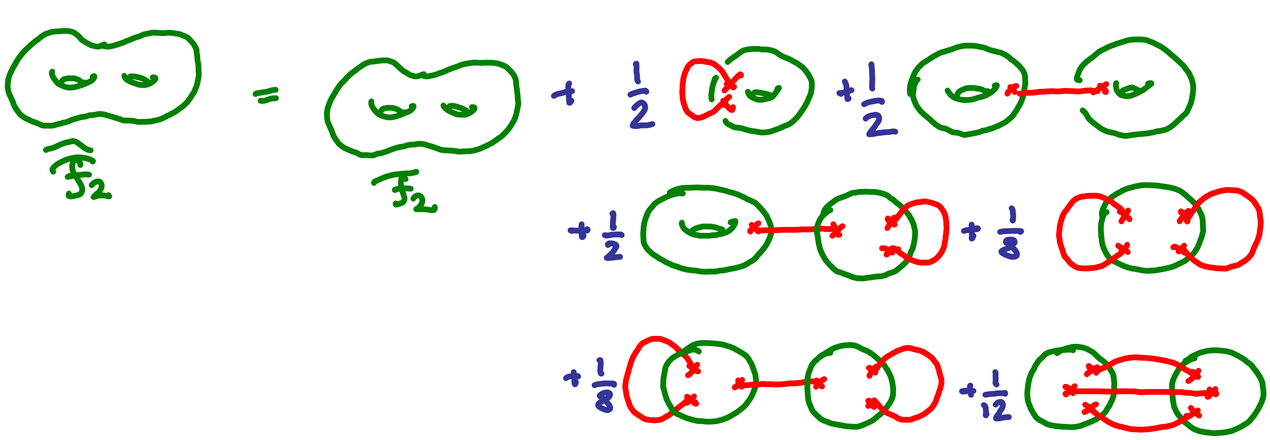
WAVE FN IN HOL. POLARIZATION

$\langle z,\varphi | x\rangle$

WAVE FN IN SYMPLECTIC POLARIZ.

EXPANDING ABOUT SADDLE POINT AT $\omega = \Omega$:

$$\begin{aligned}\hat{F}_2(x,\bar{x}) = F_2(x) &+ \left(\frac{1}{\tau-\bar\tau}\right)^{IJ}\left(\frac{1}{2}\partial_I\partial_J F_1 + \frac{1}{2}\partial_I F_1 \partial_J F_1\right)\\
&+ \left(\frac{1}{\tau-\bar\tau}\right)^{IJ}\left(\frac{1}{\tau-\bar\tau}\right)^{KL}\left(\frac{1}{2}\partial_I F_1\, \partial_{JKL} F_0 + \frac{1}{8}\partial_{IJKL} F_0\right)\\
&+ \left(\frac{1}{\tau-\bar\tau}\right)^{IJ}\left(\frac{1}{\tau-\bar\tau}\right)^{KL}\left(\frac{1}{\tau-\bar\tau}\right)^{MN}\left(\frac{1}{8}\partial_I\partial_J\partial_K F_0\, \partial_L\partial_M\partial_N F_0 + \frac{1}{12}\partial_I\partial_K\partial_M F_0\, \partial_J\partial_L\partial_N F_0\right)\end{aligned}$$



THUS, HAVE A TRADE-OFF.

THE PRICE OF HOLOMORPHICITY OF $\mathcal{F}_g$ IS THAT THEY FAIL TO BE PRECISELY MODULAR.

THE PRICE OF MODULARITY OF $\hat{\mathcal{F}}_g$ IS THAT THEY FAIL TO BE HOLOMORPHIC!

IN EITHER CASE, THE ANOMALY COMES FROM BOUNDARIES OF MODULI SPACES OF RIEMANN SURFACES.

TO SUMMARIZE:

FREE ENERGY IN HOLOMORPHIC POLARIZATION

i) $\hat{F}_g(x,\bar{x})$ IS INVARIANT UNDER Γ

ii) $\hat{F}_g(x,\bar{x})$ IS "ALMOST" HOLOMORPHIC

= FINITE POWER SERIES IN $(\tau-\bar{\tau})^{-1}$

FREE ENERGY IN REAL POLARIZATION

iii) $\tilde{F}_g(x)$ IS HOLOMORPHIC, BUT NOT MODULAR

iv) $\tilde{F}_g(x)$ IS THE CONSTANT PART OF THE SERIES EXPANSION OF $\hat{F}_g(x,\bar{x})$ IN $(\tau-\bar{\tau})^{-1}$

FORMS OF THIS TYPE WERE STUDIED BY KANEKO AND ZAGIER:

FORMS SATISFYING i) AND ii) (W/ ARBITRARY WEIGHT)

= ALMOST HOLOMORPHIC MODULAR FORMS OF Γ!

MOREOVER, FOR EVERY ALMOST HOLOMORPHIC MODULAR FORM CAN DEFINE THE ASSOCIATED QUASI-MODULAR FORM OF Γ, SATISFYING iii) AND iv).

$$F_g = \lim_{\bar{z} \to \infty} \hat{F}_g$$

ONE CAN MAKE THE SYMMETRY MANIFEST AS FOLLOWS. NOTE THAT:

$$M_n: \quad \frac{1}{\tau-\bar{\tau}} \to (C\tau+D)\frac{1}{\tau-\bar{\tau}}(C\tau+D)^T - C(C\tau+D)^T$$

$$M_n = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2n,\mathbb{Z})$$

FIND $E^{IJ}(\tau)$, SUCH THAT

$$\underline{\hat{E}^{IJ}(\tau,\bar{\tau}) = E^{IJ}(\tau) + \left(\frac{1}{\tau-\bar{\tau}}\right)^{IJ}}$$

TRANSFORMS AS

$$\underline{\hat{E}(\tau,\bar{\tau}) \to (C\tau+D)\,\hat{E}\,(C\tau+D)^T}$$

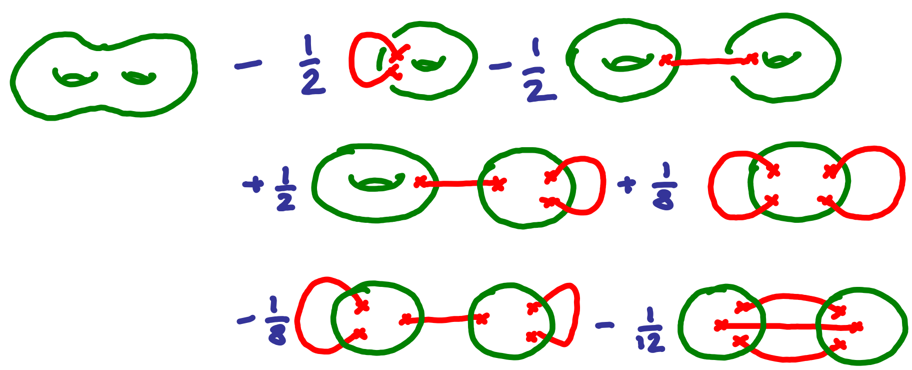
($E^{IJ}(\tau)$ ~ SECOND EISENSTEIN SERIES $E_2(\tau)$ OF $SL(2,\mathbb{Z})$)

THAN ONE CAN SHOW THAT WHILE $\hat{\mathcal{F}}_g(x,\bar{x})$ IS NOT HOLOMORPHIC, AND $\mathcal{F}_g(x)$ NOT MODULAR, AT EACH GENUS, $\exists$ COMBINATIONS OF THEM ARE <u>BOTH MODULAR AND HOLOMORPHIC</u>.

$$h_2(x) = \mathcal{F}_2(x) - E^{IJ}\left(\tfrac{1}{2}\partial_I\partial_J\mathcal{F}_1 + \tfrac{1}{2}\partial_I\mathcal{F}_1\,\partial_J\mathcal{F}_1\right) + \dots$$

$$= \hat{\mathcal{F}}_2(x,\bar{x}) - \hat{E}^{IJ}\left(\tfrac{1}{2}\partial_I\partial_J\hat{\mathcal{F}}_1 + \tfrac{1}{2}\partial_I\hat{\mathcal{F}}_1\,\partial_J\hat{\mathcal{F}}_1\right) + \dots$$

$\underline{h_2(x)}$ =



WEIGHT ZERO MODULAR FORM OF Γ

TURNING THIS AROUND,

$$\mathcal{F}_g(x) \quad , \quad \hat{\mathcal{F}}_g(x,\bar{x})$$

ARE FIXED, RECURSIVELY, IN TERMS OF LOWER GENUS AMPLITUDES, UP TO A WEIGHT ZERO MODULAR FORM OF Γ, $h_g(x)$.

h_g = RATIONAL FN ON MODULI SPACE

(EG, QUINTIC $\quad h_g = \dfrac{P(\psi)}{Q(\psi)}$

P, Q FINITE POLYNOMIALS)

THIS (RE)-DERIVES SOME ASPECTS OF BCOV FROM THE PERSPECTIVE OF SYMMETRY ALONE.

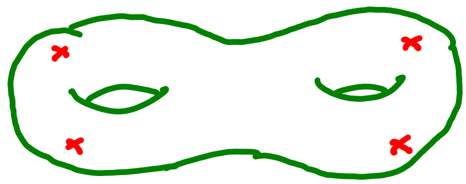
THIS WAS ALSO ANTICIPATED BY DIJKGRAAF ('93.)

FOR NON-COMPACT CY MANIFOLDS

BASED ON RIEMANN SURFACES

OF GENUS g.

$$Y: \quad uv = H(y,z) \quad \in \mathbb{C}^4$$



$H(y,z) = 0$

Γ IS THE MODULAR GROUP

OF THE RIEMANN SURFACE

$g>0 \Rightarrow$ SIEGEL (ALMOST) MODULAR

FORMS

$\Gamma \in Sp(2g, \mathbb{Z})$

- $\mathcal{N}=2$, $d=4$ GAUGE THEORIES

$$\mathcal{H}_{SU(N)}: \quad \underline{y^2 - P^2(x, u_i) + \Lambda^{2n}} = 0$$

SEIBERG–WITTEN CURVE

genus $= N-1$

TOPOLOGICAL STRING COMPUTES

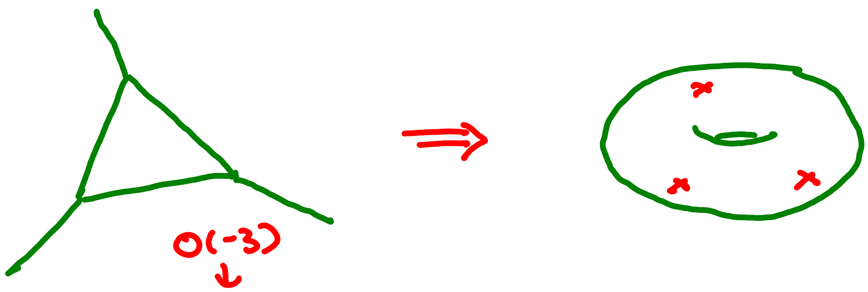
$$\int d^4x\, d^4\theta \; \mathcal{F}_g \; \mathcal{W}^{2g}$$

GRAVIPHOTON SUPERFIELD

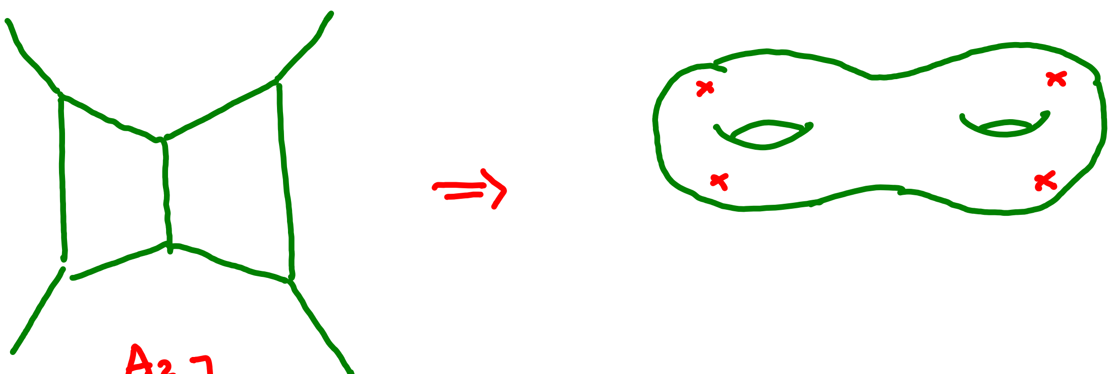
$$\langle a^i | Z \rangle = \exp\Big(\sum_g \mathcal{F}_g(a^i)\, g_s^{2g-2}\Big)$$

IS A WAVE FUNCTION, (PARTIALLY) DETERMINED BY MONODROMIES.

- RIEMANN SURFACES APPEAR AS MIRRORS OF LOCAL TORIC CALABI-YAU MANIFOLDS



$$\mathcal{O}(-3) \to \mathbb{P}^2$$



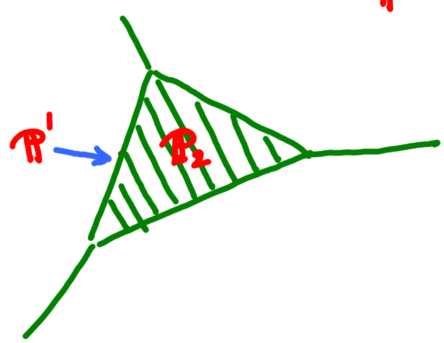
$$A_2 \to \mathbb{P}^1$$

$$\text{A-MODEL} \xRightarrow[\text{SYMMETRY}]{\text{MIRROR}} \text{B-MODEL}$$

EXAMPLE:

A-MODEL ON LOCAL $\mathbb{P}^2$:

$$X = \begin{array}{c} \mathcal{O}(-3) \\ \downarrow \\ \mathbb{P}^2 \end{array}$$



THE MIRROR IS AN ELLIPTIC CURVE

$$Y: \quad x_1^3 + x_2^3 + x_3^3 - 3\psi\, x_1 x_2 x_3 = x_3^3\, u v$$

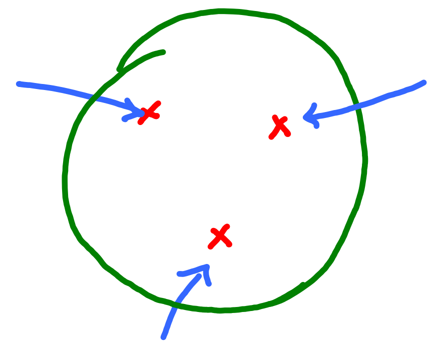
WITH 1-FORM

$$\omega = \log\left(\frac{x_2}{x_3}\right) \frac{dx_1}{x_1}$$

THE MODULI SPACE OF COMPLEX STRUCTURES HAS 3 SINGULAR POINTS

$\mathbb{C}^3/\mathbb{Z}_3$ ORBIFOLD POINT $\psi = 0$

$\psi = \infty$: LARGE RADIUS



CONIFOLD PT $\psi = 1$

$$\begin{pmatrix} t_D \\ t \end{pmatrix} \sim \begin{pmatrix} \text{4 BRANE} \\ \text{2 BRANE} \end{pmatrix}$$

AROUND WHICH THE PERIODS $\begin{pmatrix} t_D \\ t \end{pmatrix}$ UNDERGO MONODROMY

$$M_0 = \begin{pmatrix} -2 & -1 \\ 3 & 1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}$$

$$M_0^3 = \mathbb{1} \qquad M_\infty = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

GENERATES $\Gamma = \Gamma_0(3) \subset SL(2, \mathbb{Z})$

THE LARGE RADIUS BASIS $\begin{pmatrix} t_D \\ t \end{pmatrix}$

NON-TRIVIAL MONODROMY AROUND

ORBIFOLD POINT $\underline{\psi = 0}$

GOOD VARIABLES CORRESPOND TO

TWISTED SECTORS

$$\begin{pmatrix} \sigma_D \\ \sigma \end{pmatrix} = \begin{pmatrix} h^{2,2} \text{ twisted sector} \\ h^{1,1} \text{ twisted sector} \end{pmatrix}$$

AT ORBIFOLD PT

WHICH DIAGONALIZE THE MONODROMY

$$M_0 : \begin{pmatrix} \sigma_D \\ \sigma \end{pmatrix} \to \begin{pmatrix} \alpha^2 \sigma_D \\ \alpha \sigma \end{pmatrix} .$$

$$\boxed{\begin{pmatrix} t_D \\ t \end{pmatrix} = \tilde{M} \begin{pmatrix} \sigma_D \\ \sigma \end{pmatrix}}$$

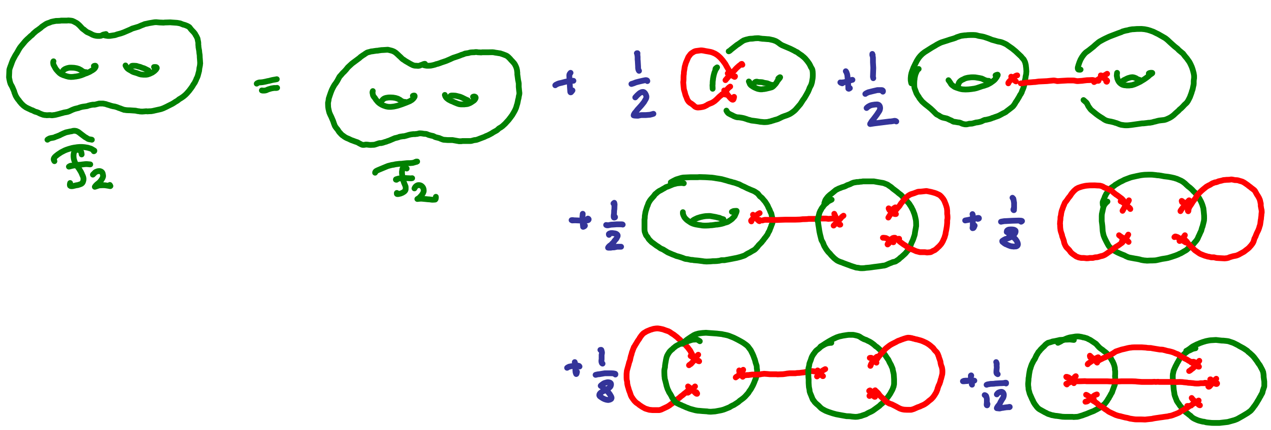
ORBIFOLD $\longrightarrow$ $\mathcal{F}_g^{ORB}(G)$, $\mathcal{F}_g^{\infty}(t)$ $\longleftarrow$ LARGE RADIUS

ARE RELATED, NOT BY

ANALITIC CONTINUATION,

BUT BY FEYNMAN GRAPH

EXPANSION $(g>0)$.



⇒ PREDICTIONS FOR GROMOV WITTEN-INVARIANTS OF $\mathbb{C}^3/\mathbb{Z}_3$, FROM LARGE RADIUS RESULTS

BRYAN ET AL.

AGUAN ET AL.

$$F_g^{ORB}(\sigma) = \sum_{n=1}^{\infty} \frac{N_n^g}{(3n)!} \sigma^{3n}$$

$$N_n^g = \langle \underbrace{O_\sigma O_\sigma \cdots\cdots O_\sigma}_{3n} \rangle_g$$

PRELIMINARY

g	$k=1$	2	3	4	5
1	$\frac{14}{3^3}$	$\frac{7351}{3^5}$	$\frac{22350580}{3^7}$	$\frac{66684490405}{3^8}$	$\frac{438923520132482}{3^9}$
2	$\frac{551}{2^4\cdot 3^4\cdot 5}$	$\frac{894877}{2^4\cdot 3^7}$	$\frac{10958155769}{2^4\cdot 3^8\cdot 5}$	$\frac{180394575808481}{2^4\cdot 3^{10}\cdot 5}$	$\frac{1018418892959991523629 29}{2^4 3^{13} 5}$
3	$\frac{395}{2^4 3^5 7}$	$\frac{194225051}{2^4 3^9 5\cdot 7}$	$\frac{1014517916687}{2^4 3^{10} 5\cdot 7}$	$\frac{2854004364880 1999}{2^4 3^{12} 5\cdot 7}$	$\frac{136407434717214391 6301}{2^4 3^{14} 5\cdot 7}$
4	$\frac{5197}{2^7 3^6 5^2}$	$\frac{476551195}{2^8 3^{11}}$	$\frac{1510891054573}{2^4 3^{12} 5}$	$\frac{1779852684752415167}{2^8 3^{13} 5^2}$	$\frac{36007367997312400904783}{2^7 3^{16} 5}$
5	$\frac{3065581}{2^9 3^8 5^2 7\cdot 11}$	$\frac{1211044328623}{2^9 3^{11} 5^2 7\cdot 11}$	$\frac{170831915058817}{2^9 3^9 5\cdot 7\cdot 11}$	$\frac{4258897269824407933 63}{2^9 3^{15} 5^2 7}$	$\frac{4341743582103698228305 73599}{2^9 3^{17} 5^2}$

- COMPACT CALABI-YAU

$$\Gamma \subset Sp(2b_2+2, \mathbb{Z})$$

A NEW CLASS OF MODULAR FORMS

ARIZES SINCE

$$(\tau-\bar{\tau})_{IJ}$$

IS NOT POSITIVE DEFINITE,

BUT HAS SIGNITURE

$$(1, b_2)$$

NEW THEORY OF

$\Longrightarrow$ "LORENTZIAN" MODULAR FORMS

THE B-MODEL HAS A LARGER SYMMETRY GROUP.

QUANTUM SYMMETRY OF B-MODEL:

$\Gamma \subset$ ω - PRESERVING DIFFEOMORPHISMS

(3,0) FORM ON CY

FOR LOCAL CY $\Rightarrow$ W_∞ ALGEBRA
WHICH FIXES THE AMPLITUDES

FOR COMPACT CY $\Rightarrow$?