STRINGS 2006

The M5-Brane Elliptic Genus

- MODULARITY & BPS STATES

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work in progress with

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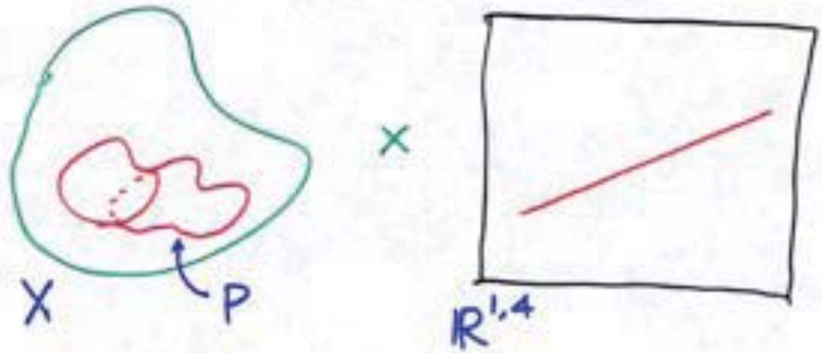


X Calabi - Yau 3-fold

M5-brane wrapped on 4-cycle $P \subset X$

(1+1)-diml (0,4) CFT

D4 - D2 - D0
bound states (IIA)

$\mathcal{N}=2$ BPS BHs

Modified Elliptic Genus

$$Z(\tau,\bar\tau,y) = \mathrm{Tr}_R\, \tfrac{1}{2} F^2 (-)^F q^{L_0} \bar q^{\bar L_0} e^{2\pi i y q}$$

- modular (Jacobi) form
weight $(-\frac{3}{2}, \frac{1}{2})$

The (0, 4) CFT

Maldacena, Strominger, Witten

$$(c_L, c_R) = (6D + c_2 \cdot P,\ 6D + \tfrac{1}{2} c_2 \cdot P)$$

$$D \equiv D_{ABC}\, p^A p^B p^C, \qquad D_{AB} \equiv D_{ABC}\, p$$

flux of self-dual 3-form on M5

$\Downarrow$

conserved currents j_A

$\rightarrow$ charges Q_A

- induced M2/D2 charge

- $\mathrm{Tr}_R (-)^F = 0$, due to degeneracy $\Psi_0^{\pm\pm}\cdots|0\rangle_R$

need to insert F^2

$\mathrm{Tr}_R\, F^2 (-)^F q^{L_0} \bar{q}^{\bar{L}_0}$ receives contribution from states with

$$\bar{L}_0 |\psi\rangle = \frac{(p^A Q_A)^2}{12D} |\psi\rangle.$$

- index of $\mathcal{A}_{K^*,\infty}$
- supersymmetry nonlinearly realized
  $\longleftrightarrow$ susic D4-D2-D0 bd states

General Structure of

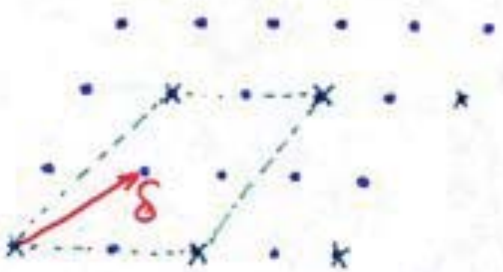
$$Z(\tau,\bar\tau,y) = \mathrm{Tr}_R \tfrac{1}{2} F^2 (-)^F q^{L_0} \bar q^{\bar L_0} e^{2\pi i y^A Q_A}$$

- $$Z(\tau,\bar\tau,y) = \sum_\delta Z_\delta(\tau)\, \theta_\delta(\tau,\bar\tau,y)$$

$$\theta_\delta(\tau,\bar\tau,y) = \sum_{\substack{q_A = 6 D_{AB} k^B + \delta_A \\ k^A \in \mathbb{Z}}} e^{-\frac{2\pi i \bar\tau}{12} D^{AB} q_A q_B}$$

$$\times\, e^{\frac{2\pi i}{12}(\tau-\bar\tau)\frac{(p^A q_A)^2}{D} + 2\pi i y^A q_A} \; (-)^{6 D_{AB} p^A k^B + p^A \delta_A}$$

Freed - Witten anomaly



• : lattice of D2-charges

x : lattice of $H^2(X;\mathbb{Z}) \hookrightarrow H^2(P;\mathbb{Z})$

- (θ_δ) form a representation of $SL(2,\mathbb{Z})$
- Z is a Jacobi form of weight $(-\frac{3}{2}, \frac{1}{2})$

$\Rightarrow$ know modular rep of $(Z_\alpha(\tau))$

polar terms of $Z_\alpha(q = e^{2\pi i \tau})$

$$\Big\Downarrow \text{ generalized Rademacher expansion}$$

$Z_\alpha(\tau)$ determined !

EXAMPLE

X : the quintic

$\cup$

P : hyperplane section

$(p^A = 1, \; D = \frac{5}{6})$

$$Z(\tau, \bar{\tau}, y) = \sum_{\kappa=0}^{4} Z_\kappa(\tau) \, \bar{\theta}_\kappa(\bar{\tau}, y)$$

- $$\bar{\theta}_\kappa(\bar{\tau}, y) = \sum_{n \in \mathbb{Z}} e^{-\pi i \bar{\tau} 5 (n + \frac{\kappa}{5} + \frac{1}{2})^2} \times e^{2\pi i y (5n + \kappa + \frac{1}{2})} (-)^{n+\kappa}$$

- $Z_\kappa = Z_{5-\kappa}$

- $\chi(P) = 55, \quad (b_+, b_-) = (9, 44)$

  P not a spin manifold,

  flux $F = \frac{J}{2}$ + integral form

$$\begin{pmatrix} Z_0(\tau) \\ Z_1(\tau) \\ Z_2(\tau) \end{pmatrix} \quad \text{weight } -\tfrac{3}{2} \text{ modular rep.}$$

$$T = e^{-2\pi i \frac{55}{24}} \begin{pmatrix} 1 & & \\ & \omega^3 & \\ & & \omega^2 \end{pmatrix}$$

$$S = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 & 2 \\ 1 & \omega+\omega^{-1} & \omega^2+\omega^{-2} \\ 1 & \omega^2+\omega^{-2} & \omega+\omega^{-1} \end{pmatrix}$$

$$\omega = e^{2\pi i/5}$$

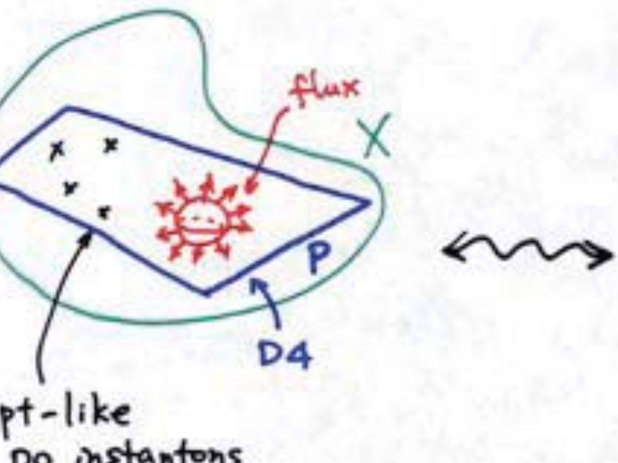
- By a naive direct counting of D4-D2-D0 bound states

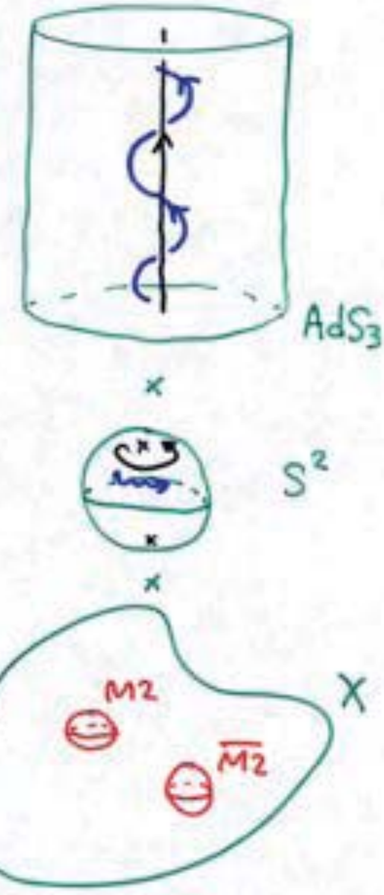
$$Z_0(q) = q^{-\frac{55}{24}} \Big(\boxed{5 - 800q + 58900q^2} \leftarrow \text{polar terms} + 5755150q^3 + \cdots \Big)$$

$$Z_1(q) = q^{-\frac{55}{24}+\frac{1}{10}} \Big(\boxed{8625 q^{\frac{3}{2}}} - 1150000 q^{\frac{5}{2}} + \cdots \Big)$$

$$Z_2(q) = q^{-\frac{55}{24}+\frac{2}{5}} \Big(-1218500 q^2 + 447327750 q^3 + \cdots \Big)$$

Descriptions of D4-D2-D0 bound states





Roughly:

pt-like instantons $\longleftrightarrow$ gravitons

flux

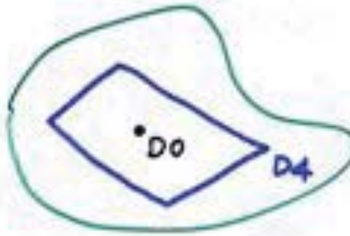
$F = \Sigma C_i - \Sigma C_j'$ $\longleftrightarrow$ M2 wrapped on C_i

$\overline{\text{M2}}$ wrapped on C_j'

- "pure" D4   induced D0 charge $-\frac{55}{24}$

  moduli space $\mathbb{P}^4 \rightarrow \chi(\mathbb{P}^4) = 5$ states

- D4 + 1 D0



  moduli space

$$\begin{array}{ccc} \mathbb{P}^3 & \rightarrow & \mathcal{M} \\ & & \downarrow \\ & & X \end{array}$$

  # states (counted w/ $(-)^F$)

$$= \chi(\mathbb{P}^3) \cdot \chi(X) = -800$$

- D4 + 2 D0

  moduli space

$$\begin{array}{ccc} \mathbb{P}^2 & \rightarrow & \mathcal{M} \\ & & \downarrow \\ & & \mathrm{Sym}^2(X) \end{array} \quad \cup \quad \begin{array}{ccc} \mathbb{P}^3 & \rightarrow & \mathcal{M}' \\ & & \downarrow \\ & & X \end{array}$$

  ↖ subtlety ignored

$$\# \text{ states} = \chi(\mathbb{P}^2)\, \chi(\mathrm{Sym}^2(X)) + \chi(\mathbb{P}^3)\chi(X)$$

$$= 58900$$

- D4 + 3 D0
  - D0's as point-like instantons
  - D0's dissolve into flux

$$F = C_1 - C_1'$$

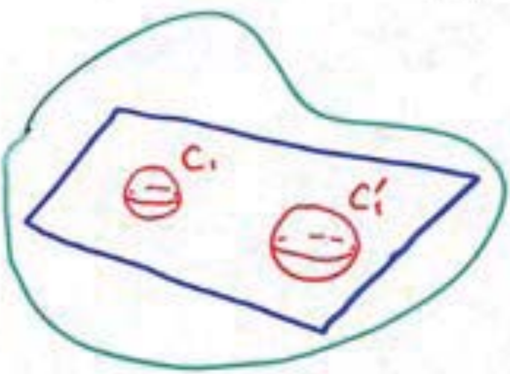
deg 1 rat'l curves

$$C_1 \cdot C_1 + \underbrace{C_1 \cdot J}_{=1} = 2\overset{=0}{g} - 2$$

$$\Rightarrow C_1 \cdot C_1 = -3$$

induced D0 charge

$$-\frac{1}{2} F^2 = \underline{3}$$



2 deg 1 curves generically determine a hyperplane

$$\Downarrow$$

2875 × 2874 states

total # (w/ $(-)^F$) states

$$= 5735150$$

Rademacher expansion

$$Z'_\alpha(\tau) = \eta(\tau)^3 Z_\alpha(\tau)$$
$$= q^{\Delta_\alpha} \sum_{n=0}^{\infty} F_\alpha(n)\, q^n$$

weight 0
modular rep.

- $n > -\Delta_\alpha$,

$$F_\alpha(n) = 2\pi \sum_{k=1}^{\infty} \sum_{\beta} k^{-2}\ Kl(n,\alpha;m,\beta|k)$$

$$\times \sum_{m<-\Delta_\beta} F_\beta(m) \cdot \left[2\pi(-\Delta_\beta - m)\right] \times \tilde{I}_1\left[\frac{4\pi}{k}\sqrt{(-\Delta_\beta - m)(n+\Delta_\alpha)}\right.$$

- $\tilde{I}_1(z) = \frac{2}{z} I_1(z)$

- $Kl(n,\alpha;m,\beta|k)$

  Kloosterman sum

  depends on T, S - matrix

Naive counting of D4-D2-D0 bd states

$\Rightarrow$

$$Z_0'(q) = q^{-\frac{13}{6}}\left(\boxed{5 - 815\,q + 61300\,q^2} + 5578475\,q^3 + \cdots\right)$$

polar terms

$$Z_1'(q) = q^{-\frac{13}{6}+\frac{1}{10}}\left(\boxed{8625\,q^{\frac{3}{2}}} - 1175875\,q^{\frac{5}{2}} + \cdots\right)$$

$$Z_2'(q) = q^{-\frac{13}{6}+\frac{2}{5}}\left(-1218500\,q^2 + 450983250\,q^3 + \cdots\right)$$

Rademacher expansion

order 1	order 6	naive counting
5.2957×10^6	5.62842×10^6	5578475
-1.07549×10^6	-1.16931×10^6	-1175875
-1.12900×10^6	-1.21105×10^6	-1218500
4.45088×10^8	4.45642×10^8	450983250

agree within 1%!

LESSONS

- Modularity of the M5 elliptic genus encodes surprising structure of enumerative geometry
  - "non-perturbative" version of Gromov-Witten invariants ...