Topological Strings and Crystal Melting Revisited

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Strings 2009, Rome

Based on the papers with M. Yamazaki:

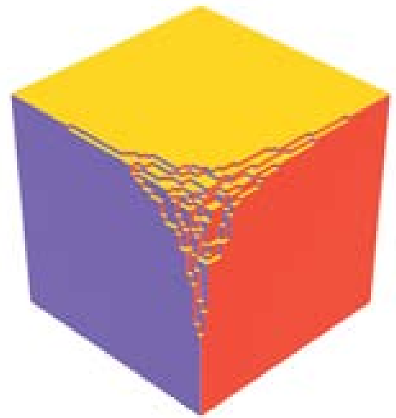
arXiv:0811.2801:
Crystal Melting and Toric Calabi-Yau Manifolds,

arXiv:0902.3996:
Emergent Calabi-Yau Geometry,

and the work in progress with
M. Aganagic, C. Vafa and M. Yamazaki
on the wall crossing phenomena.

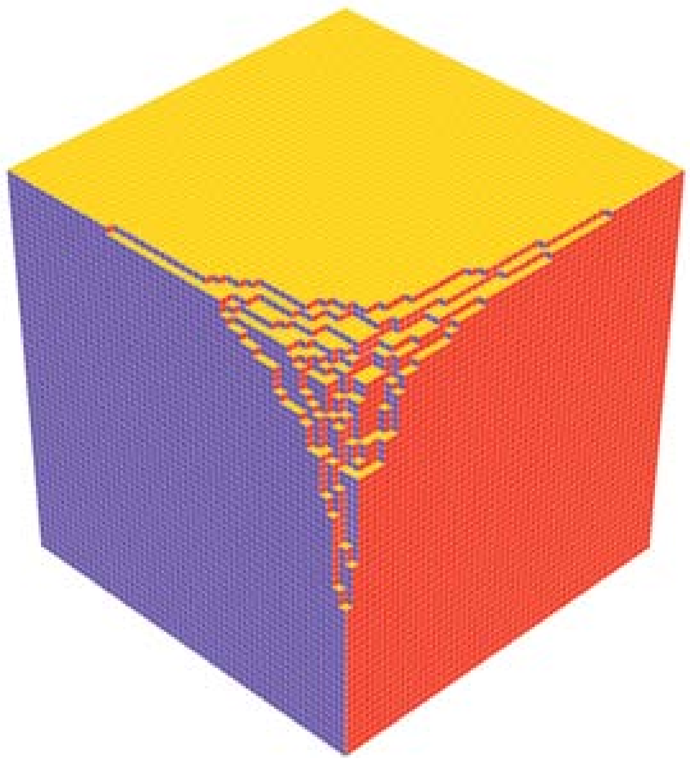
Topological Strings and Crystal Melting, *circa* 2003

Okounkov, Reshetikhin and Vafa showed that Z_{top} in C^3 can be expressed as a sum over molten crystals in 3 dimensions.



$$Z_{crystal} = \sum_{n} \Omega(n)\, e^{-g_s n}$$

This has been generalized to the topological vertex.



What does each crystal configuration mean in the physical superstring theory?

Topological String counts BPS States.

Topological String: $Z_{top} = \exp\left(\sum_g g_s^{2g-2} F_g \right)$

In the A-model, F_g counts $\Sigma_g \xrightarrow[\text{holomorphic}]{} CY_3$

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The holomorphic anomaly equations determine it recursively.

For the quintic, computation up to *g=51* is possible.

Huang, Klemm, Quackenbush ('06)

For toric CY3's, the topological vertex computes the partition function for all genera.

Aganagic, Klemm, Marino, Vafa ('03)

Gopakumar-Vafa:

$$Z_{BPS} = Z_{top}$$

OSV:

$$Z_{BPS} = |Z_{top}|^2$$

Gopakumar-Vafa:

$$Z_{BPS} = Z_{top}$$

We will start with this.

We will end with this.

$$Z_{BPS} = |Z_{top}|^2$$

Gopakumar-Vafa: $M2$ brane on $CY_3 \times \mathbb{R}^{4,1}$

N_{s_L, n_2} = # BPS states with

- $M2$ charge $n_2 \in H_2(CY_3)$
- (s_L, s_R) : $SU(2)_L \otimes SU(2)_R$ spin on $\mathbb{R}^{4,1}$.

$$Z_{top} = \prod_{s, n_2} \left(\prod_{m=1}^{\infty} (1 - q^{m+s} Q^{n_2})^m \right)^{N_{s, n_2}}$$

Donaldson-Thomas:

$1\, D_6 + n_0\, D_0 + n_2\, D_2$ on $CY_3 \times \mathbb{R}^{3,1}$

Gopakumar-Vafa:

$$M_2 \text{ brane on } CY_3 \times \mathbb{R}^{4,1}$$

Donaldson-Thomas:

$$1\, D_6 + n_0\, D_0 + n_2\, D_2 \text{ on } CY_3 \times \mathbb{R}^{3,1}$$

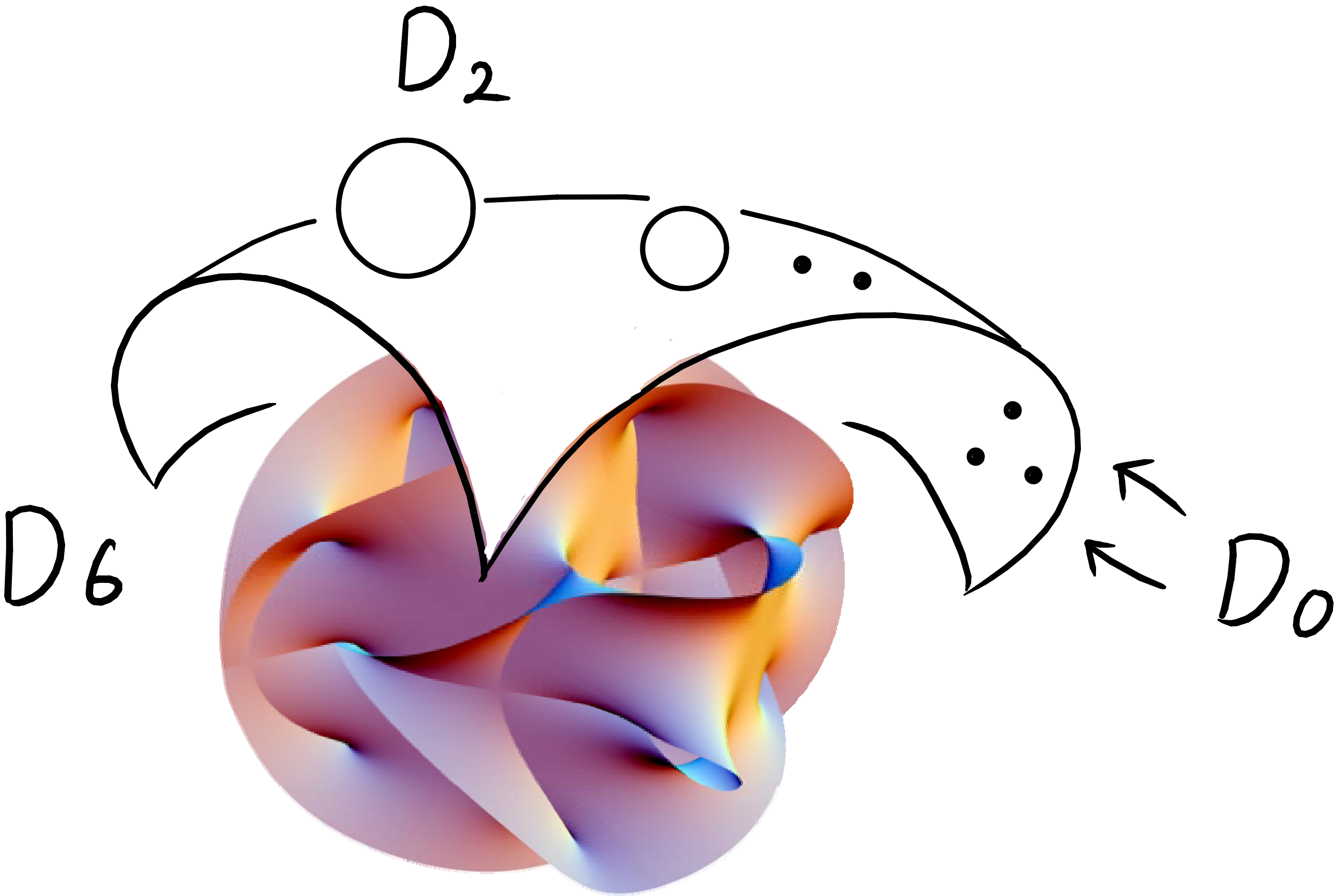
$$Z_{GV}^{(5d)} \sim Z_{DT}^{(4d)}$$

(1) Conjectured by Iqbal, Nekrasov, Okounkov, Vafa ('04).

(2) Explained by Dijkgraaf, Vafa, Verlinde ('06),
using the *4d - 5d* connection of Gaiotto, Strominger, Yin ('05).

(3) Proven mathematically by Maulik, Oblomkov, Okounkov, Pandharipande for toric CY3' ('08).

There is another way to count BPS states.


D2
D6
D0

D brane bound states

$$\Updownarrow$$

Crystal Melting

The low energy effective theory of D branes wrapping cycles of a **toric CY** is described by the gauge theory characterized by a **quiver diagram** and a **superpotential**.

Douglas, Moore ('96) **.....** Feng, Franco, Hanany, He, Imamura, Kennaway, Martelli, Sparks, Vafa, Vegh, Wecht, Yamazaki, **.....** ('06-'08)

Calabi-Yau Geometry $\longrightarrow$ **Combinatorial data for the quiver gauge theory**

Toric CY3: $\mathbb{C}^{N+3} / U(1)^{\otimes N}$

Mirror of Toric CY3: $uv + F(x, y) = 0$

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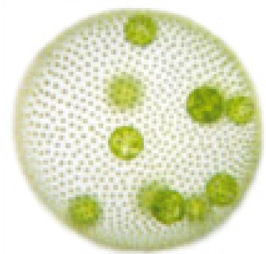
Mirror of Toric CY3: $uv + F(x,y) = 0$

Amoeba and Alga:



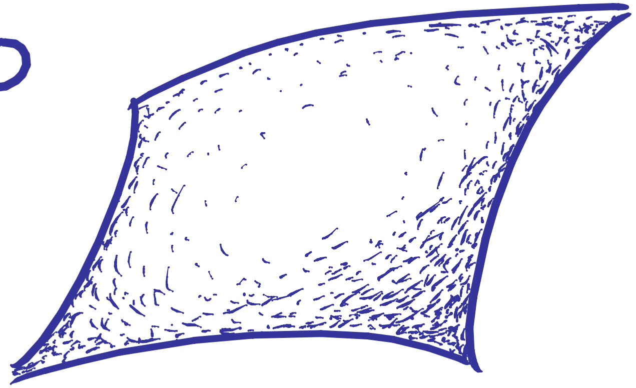
$$\text{Amoeba} = \{ (r,\rho) : \exists (\theta,\phi),\ F(e^{r+i\theta}, e^{\rho+i\phi}) = 0 \}$$

$$\text{Alga} = \{ (\theta,\phi) : \exists (r,\rho),\ F(e^{r+i\theta}, e^{\rho+i\phi}) = 0 \}$$

Example : $F(x, y) = x + y + 1$

$F(x, y) = 0$

in $\mathbb{C}^2$



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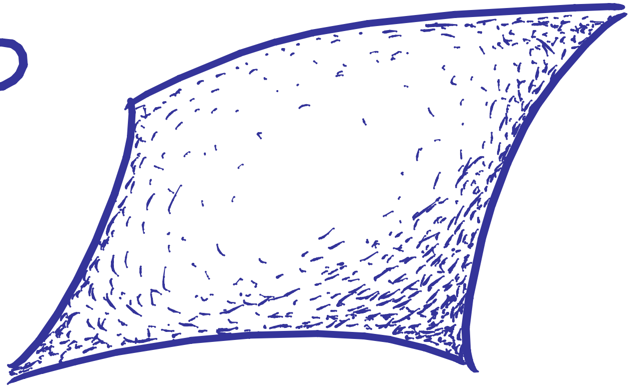
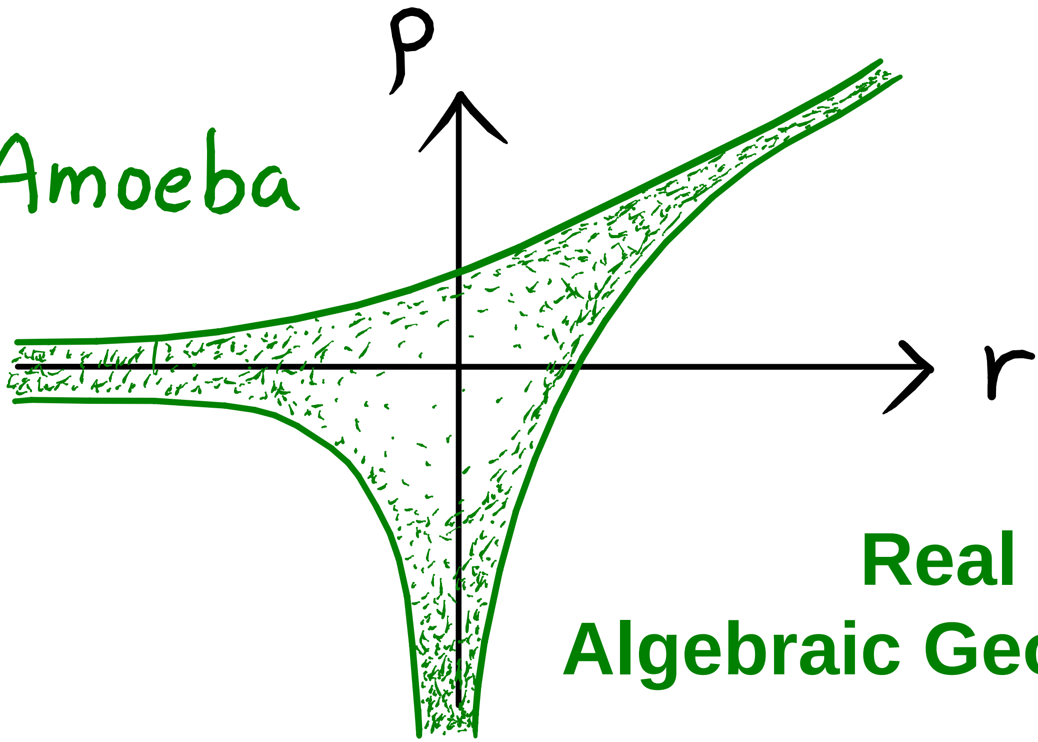


$\Downarrow$

Amoeba



Real
Algebraic Geometry

Example : $F(x,y) = x + y + 1$

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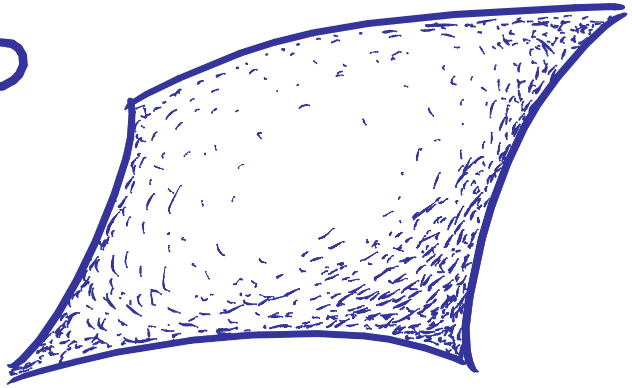
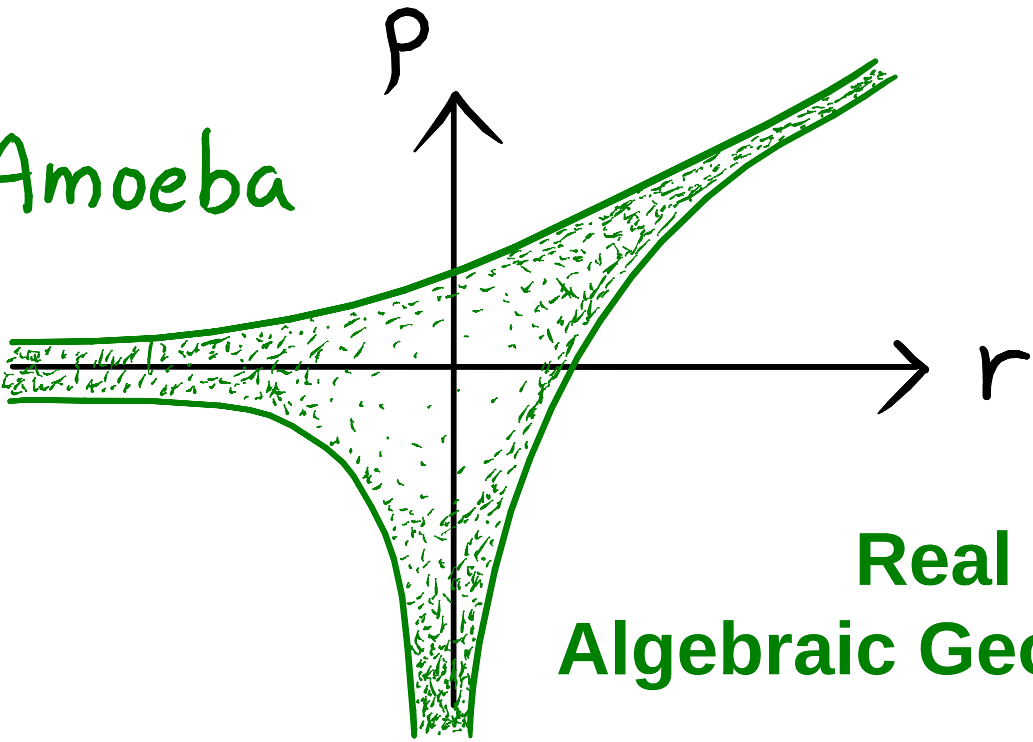
in $\mathbb{C}^2$

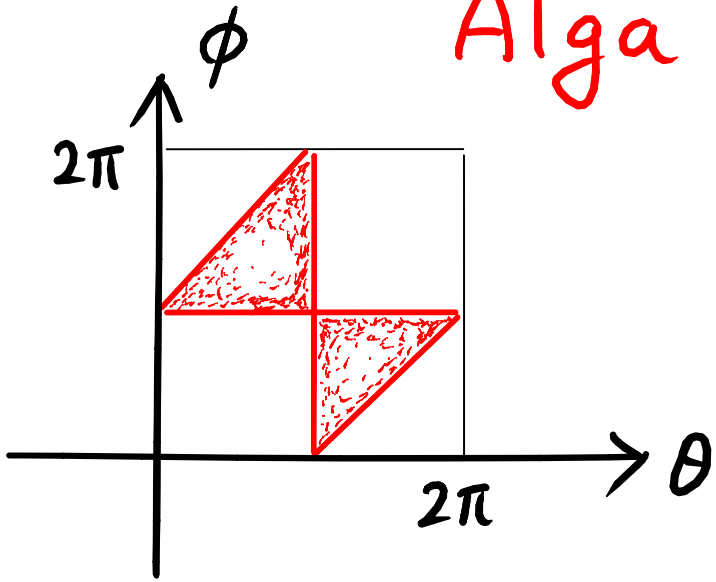


$\Downarrow$ $\qquad\qquad\qquad\qquad$ $\Rightarrow$

Amoeba



Alga

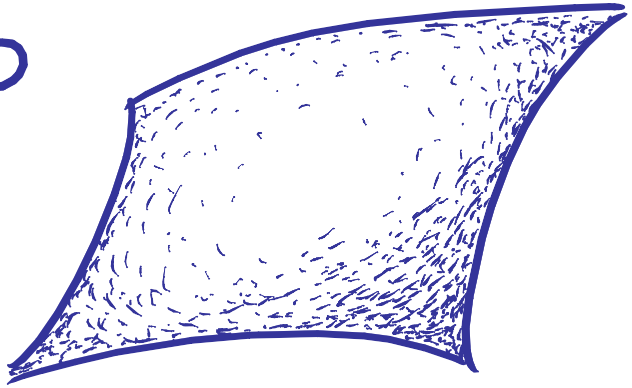


Real Algebraic Geometry

Example : $F(x,y) = x + y + 1$

$F(x,y) = 0$

in $\mathbb{C}^2$



$\Downarrow$ $\Rightarrow$

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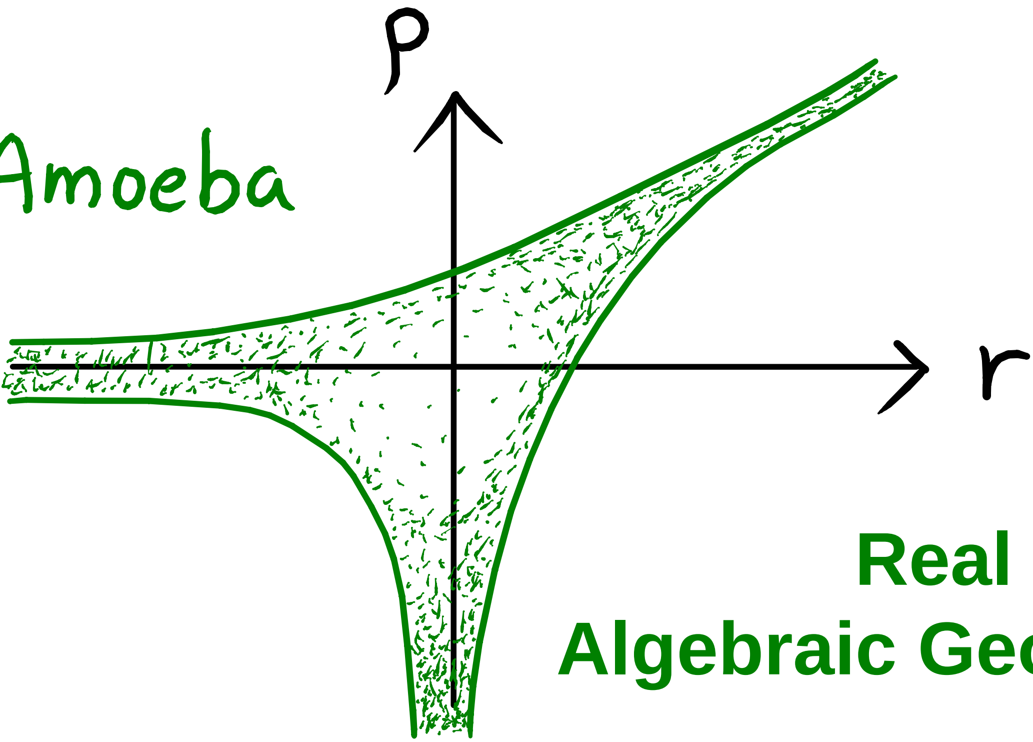
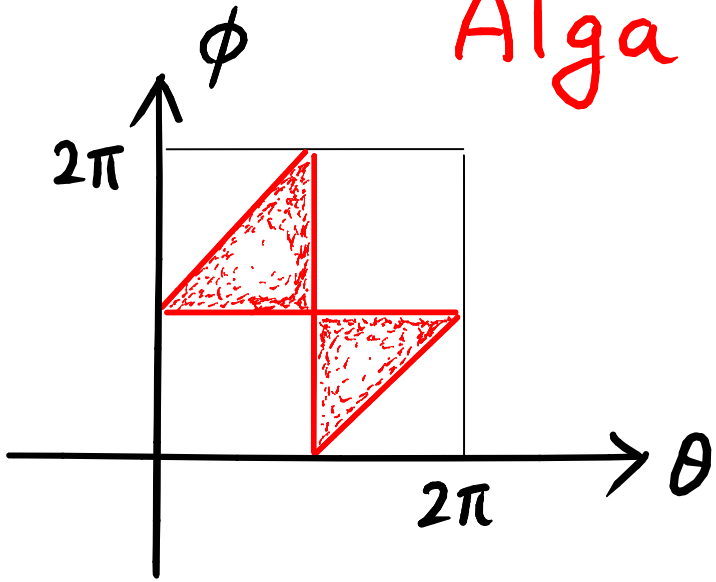


Real
Algebraic Geometry

Alga



Imaginary
Algebraic Geometry

The **Alga** determines the quiver diagram and the superpotential of the gauge theory on D branes in the toric CY3.

The **Amoeba** counts the number of bound states of D branes for large charges.

The **Alga** determines the quiver diagram and the superpotential of the gauge theory on D branes in the toric CY3.

The **Amoeba** counts the number of bound states of D branes for large charges.

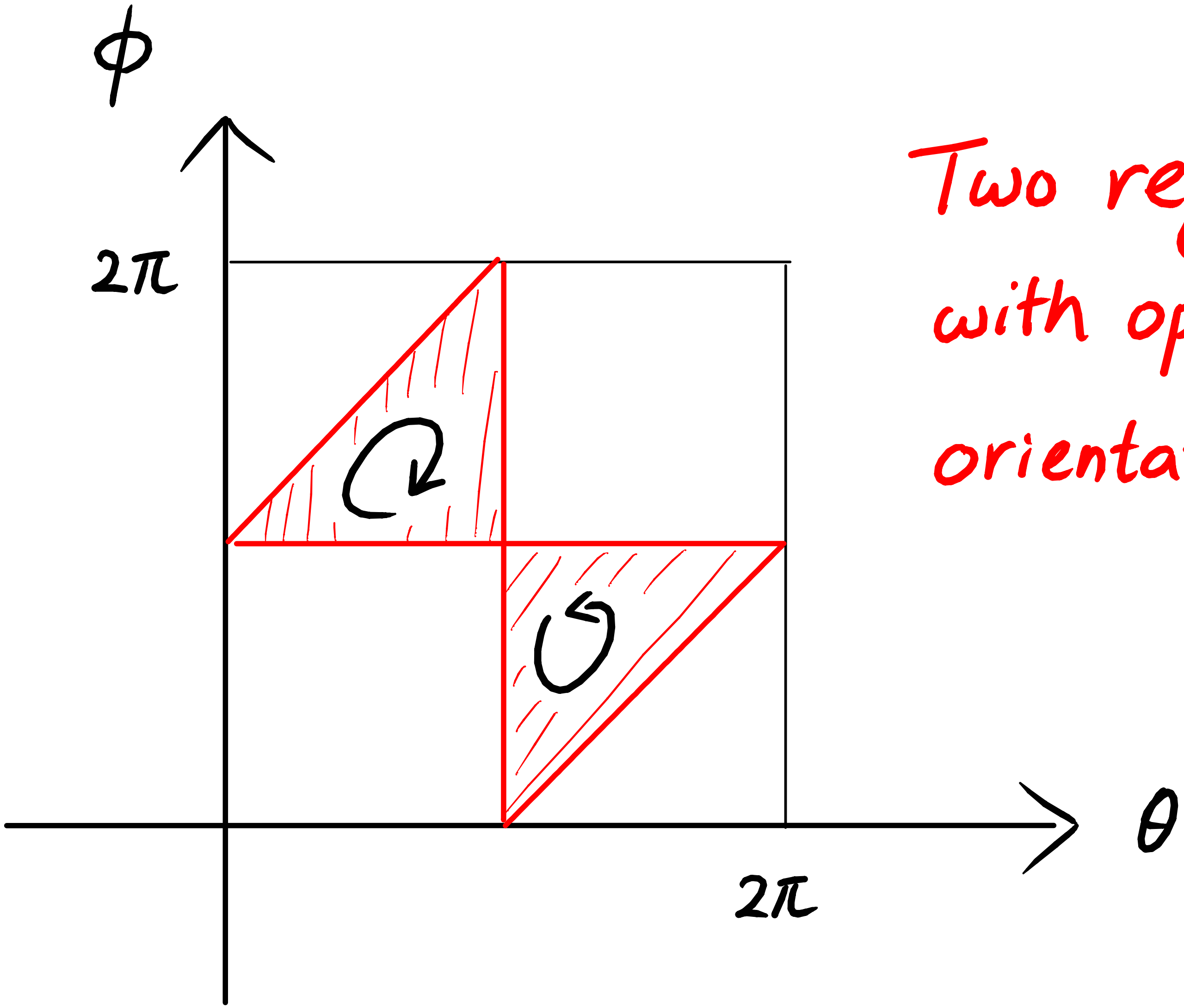
The Alga is the Question.
The Amoeba is the Answer.

The Alga determines the Lagrangian

(the quiver diagram
and the superpotential)

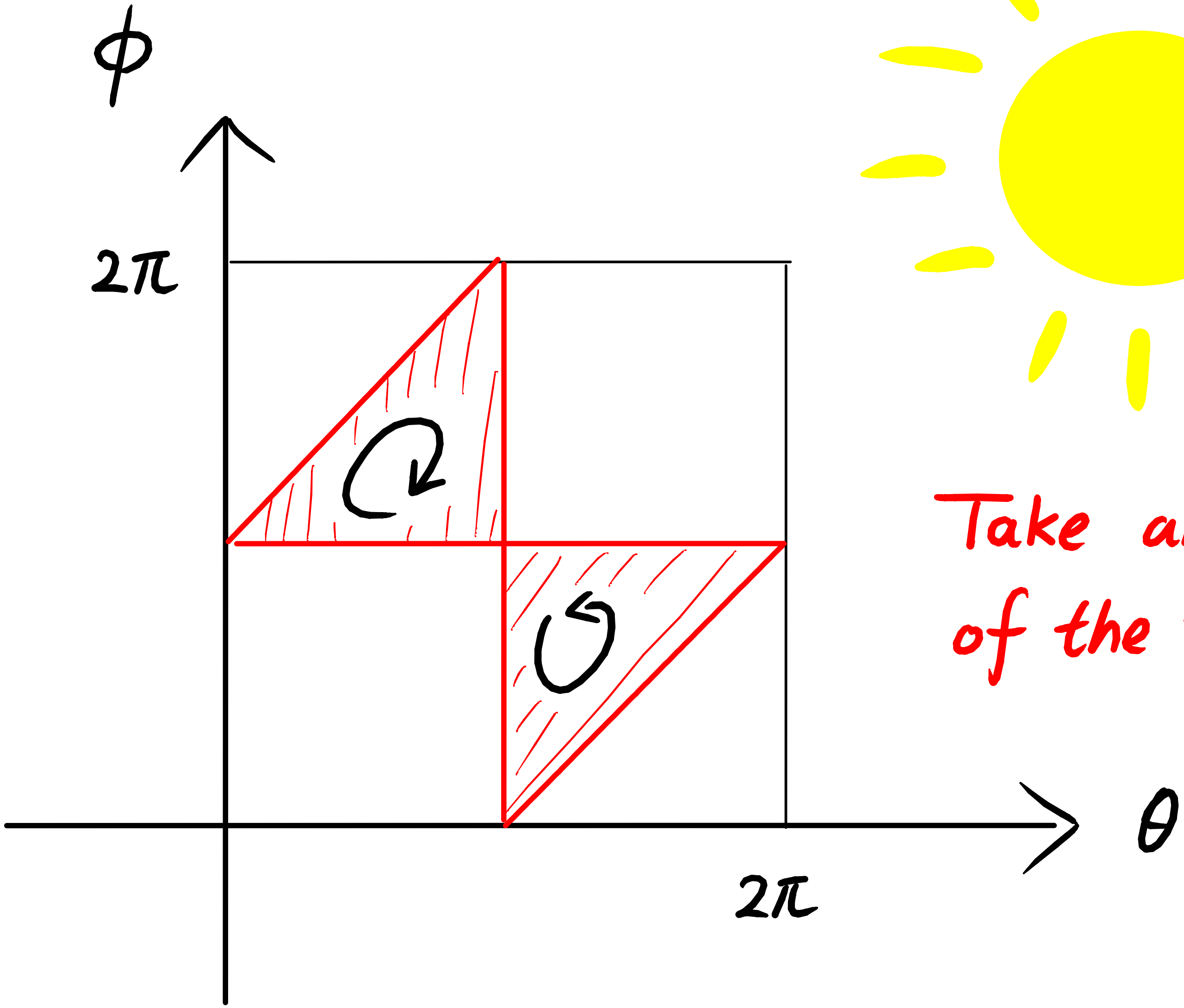
e.g. $\quad F(x, y) = x + y + 1$

mirror of $\mathbb{C}^3$



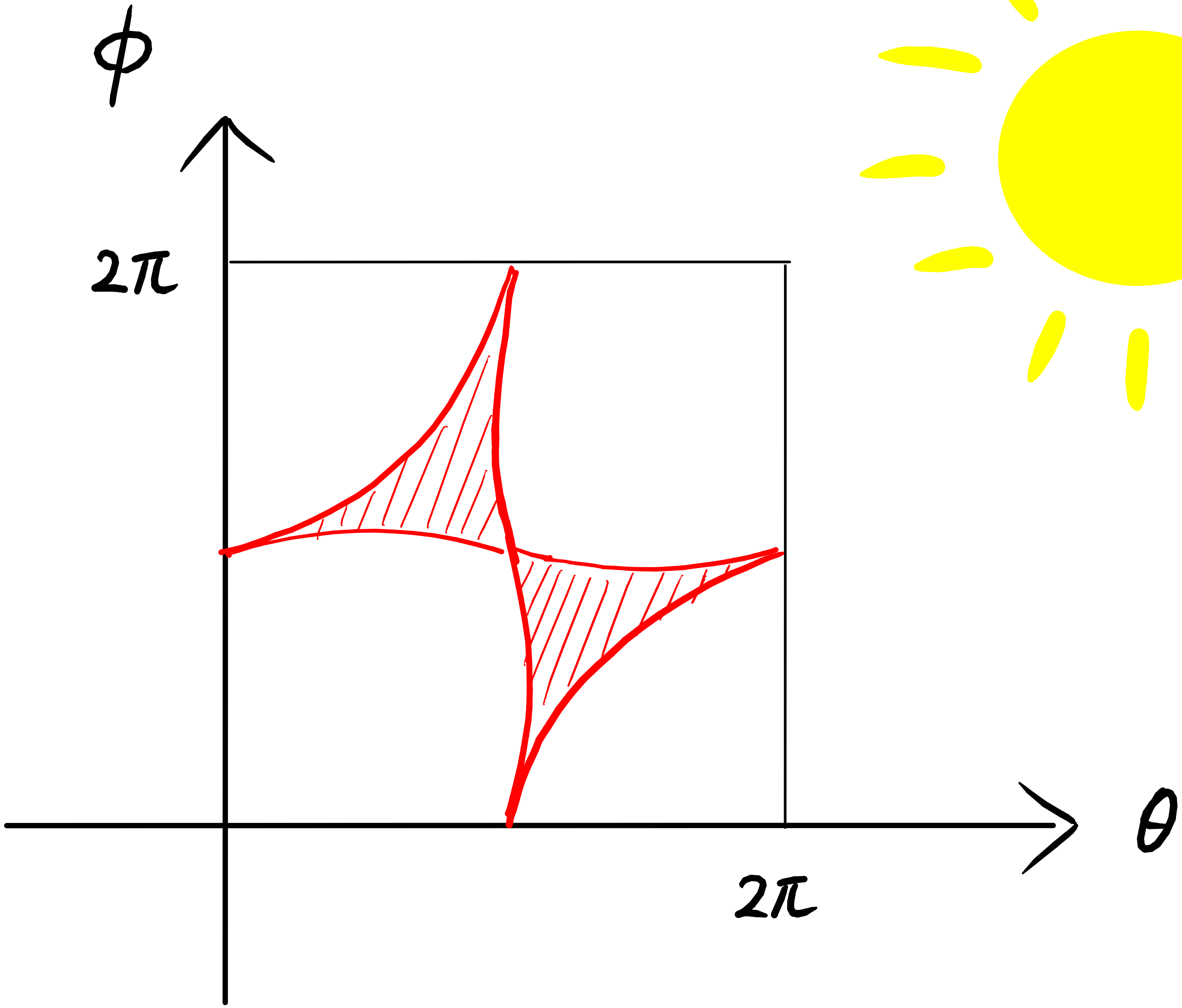
Two regions with opposite orientations

e.g. $\quad F(x, y) = x + y + 1$

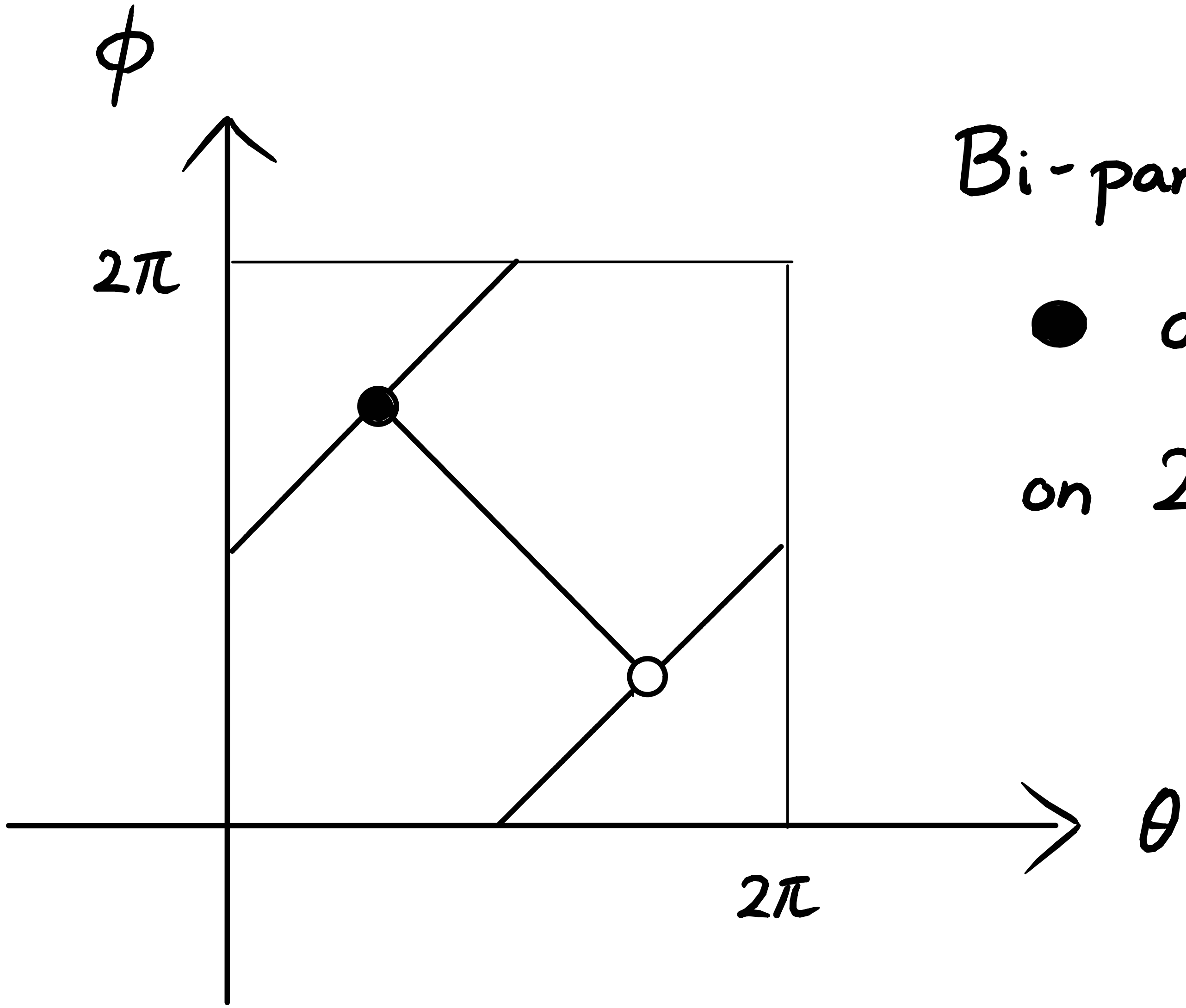


Take an analogue of the tropical limit.

e.g. $F(x, y) = x + y + 1$



e.g. $F(x, y) = x + y + 1$

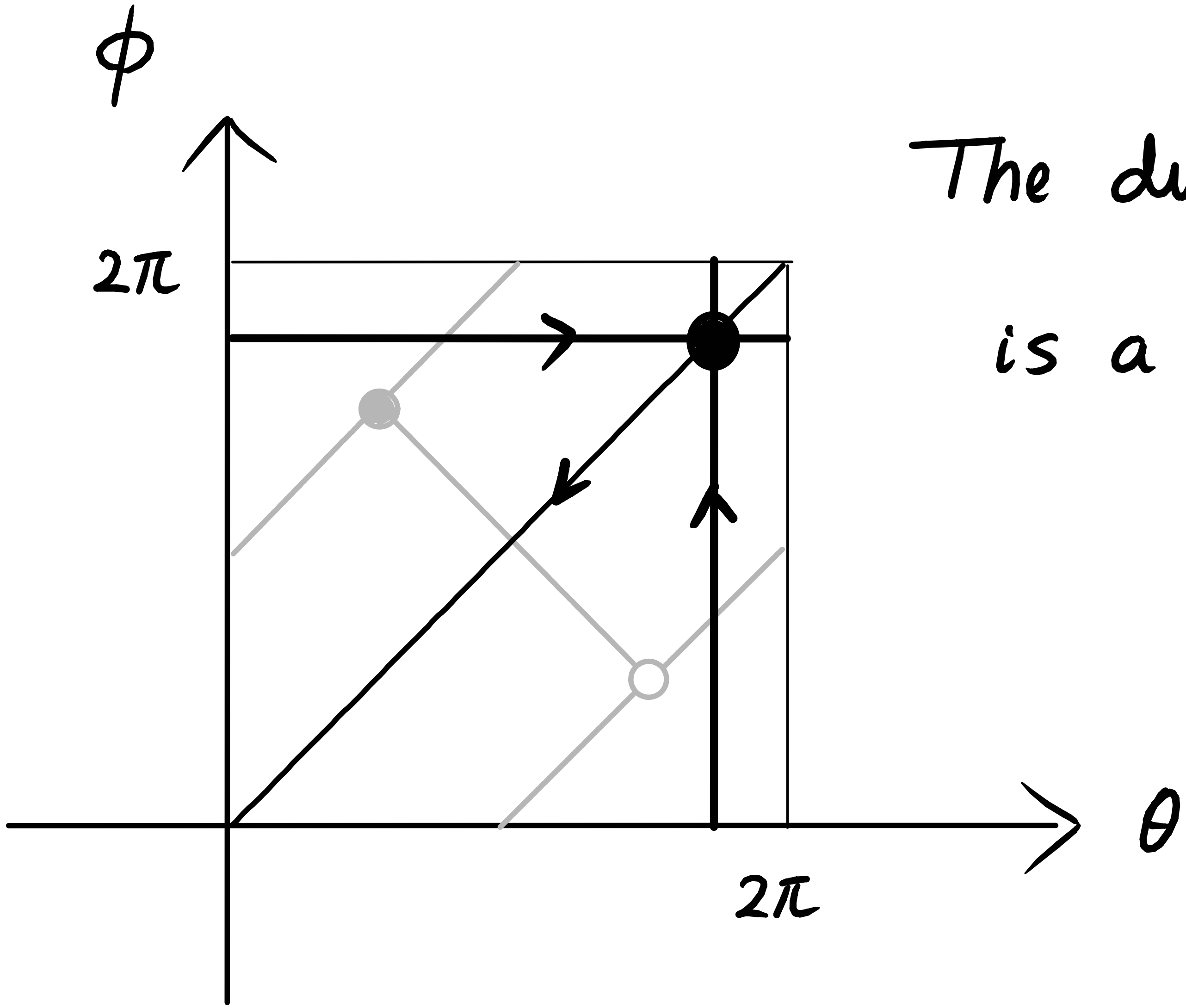


Bi-partite graph

● and ○

on 2d torus.

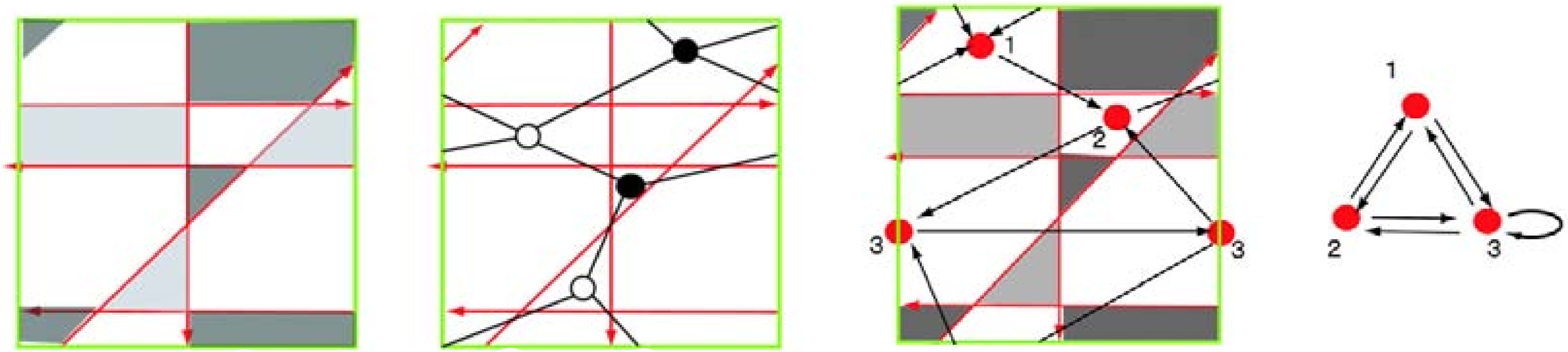
e.g. $F(x,y) = x + y + 1$



The dual graph is a quiver on T^2.

Superpotential = $\sum \pm$ (ordered product of bi-fundamentals around alga)

e.g., suspended pinched point singularity



$$W = \mathrm{tr}(X_{21}X_{12}X_{23}X_{32} - X_{23}X_{33}X_{32} + X_{33}X_{31}X_{13} - X_{31}X_{12}X_{21}X_{13})$$

Each bound state counted
by the Witten index

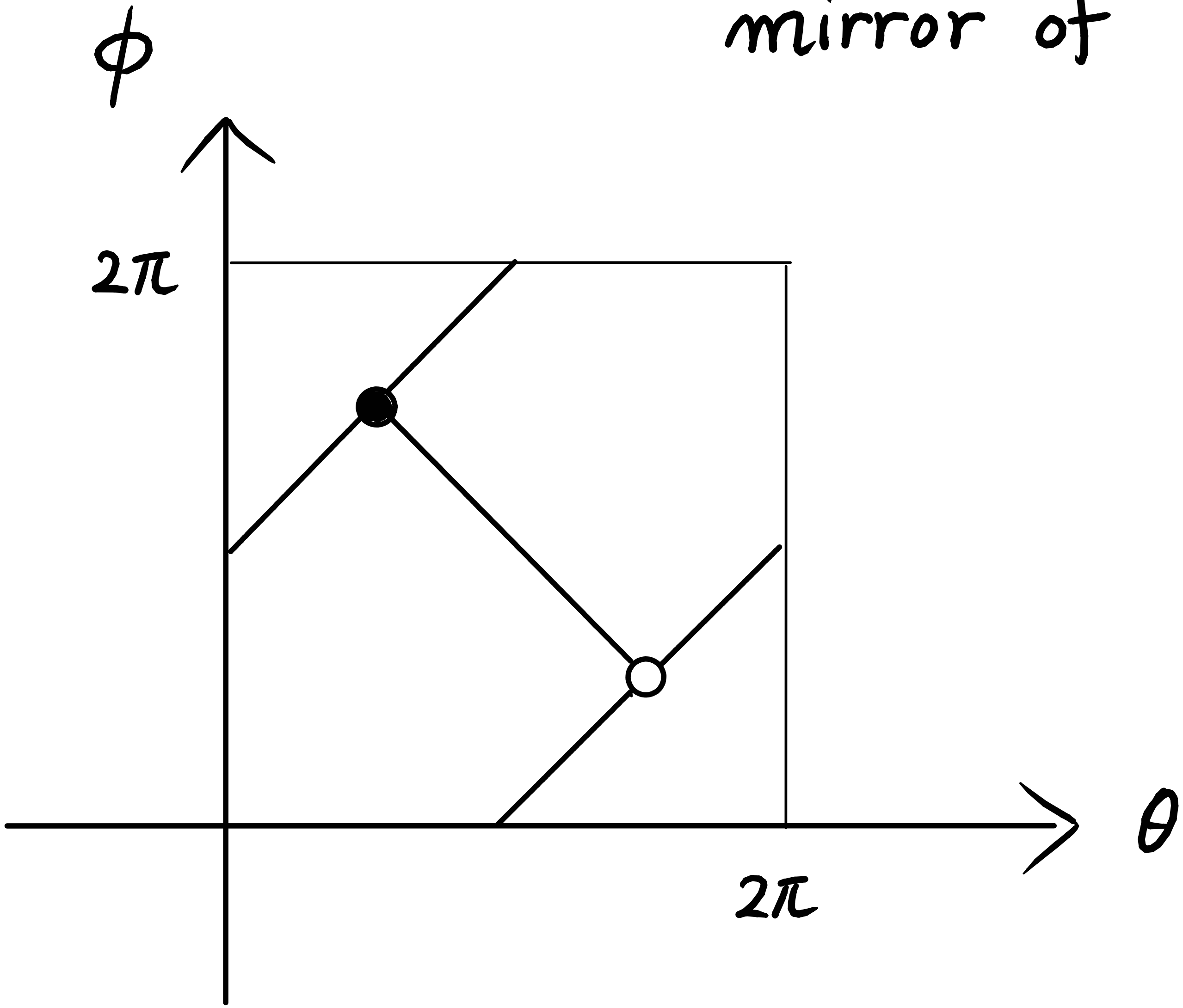
corresponds to

a perfect matching of
the periodic bi-partite graph.

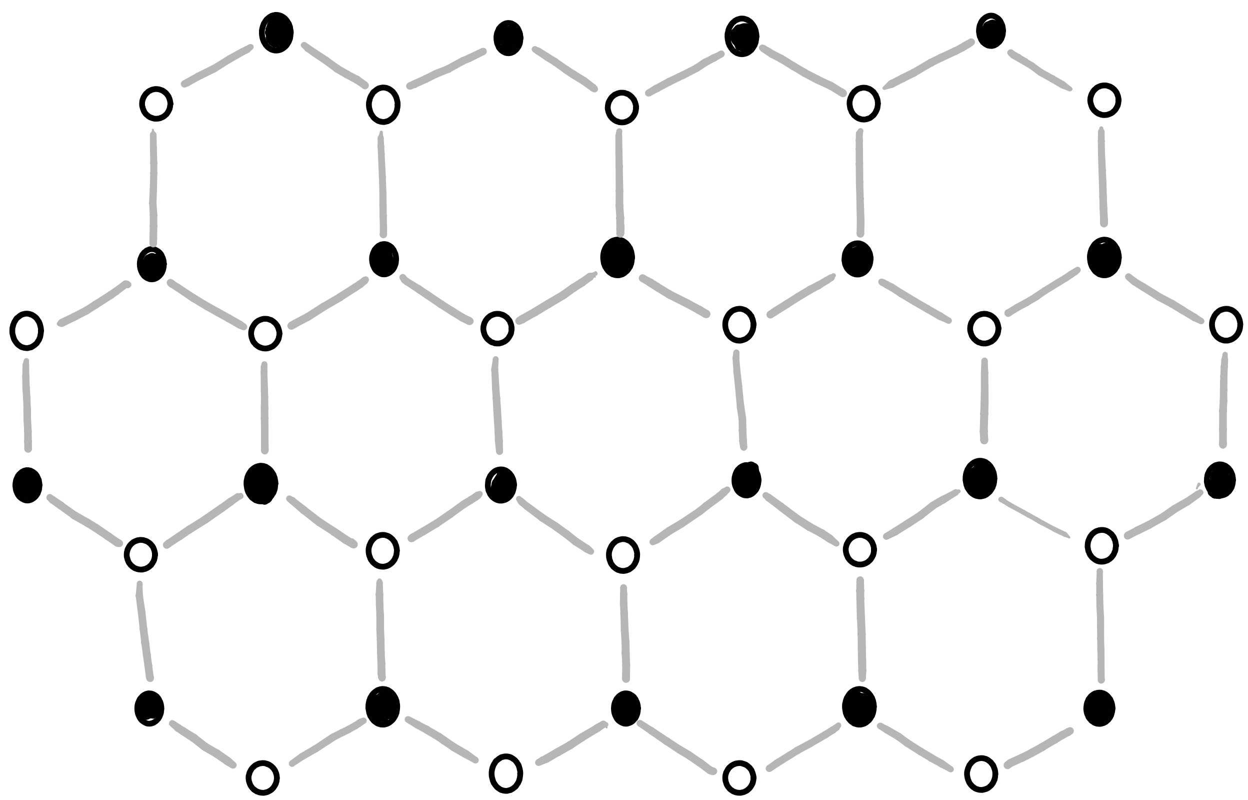
Szendroi [0705.3419]
Mozgovoy, Reineke [0809.0117]
Yamazaki + H.O. [0811.2801]

A simple example: $F(x, y) = x + y + 1$

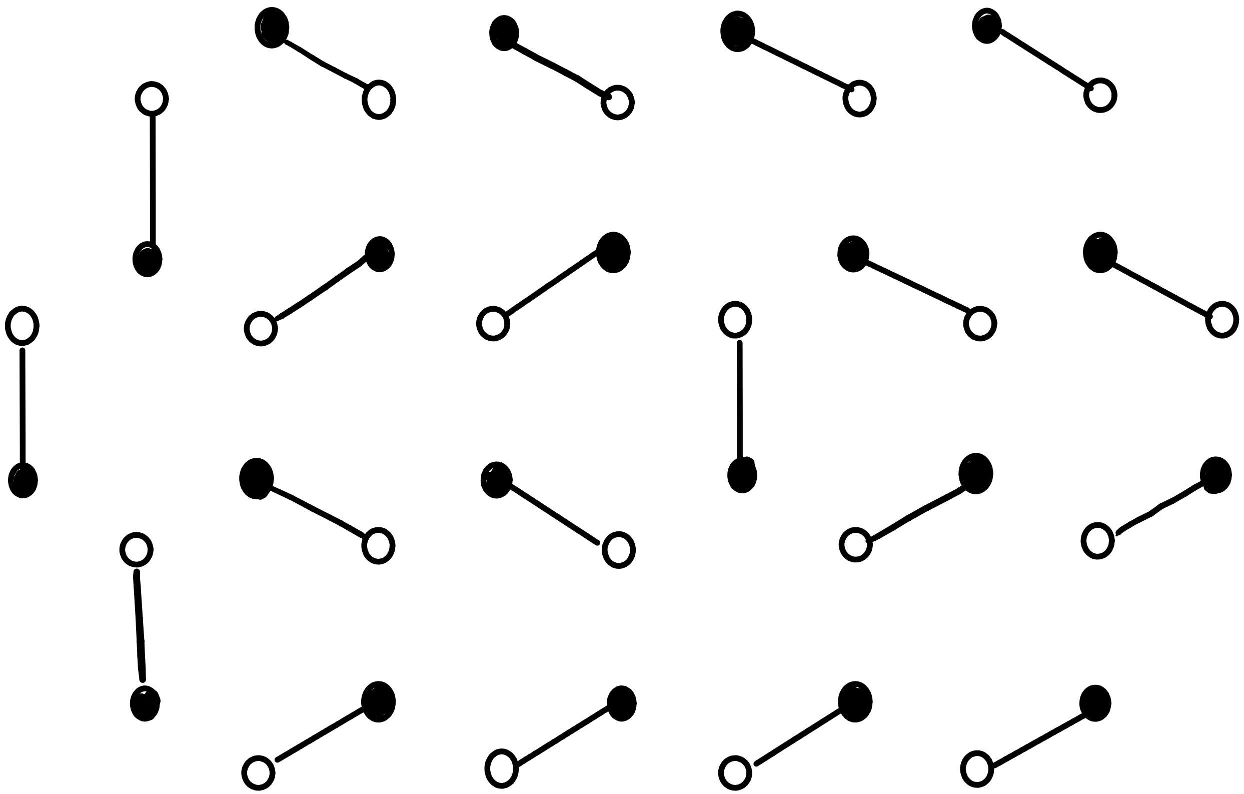
mirror of $\mathbb{C}^3$



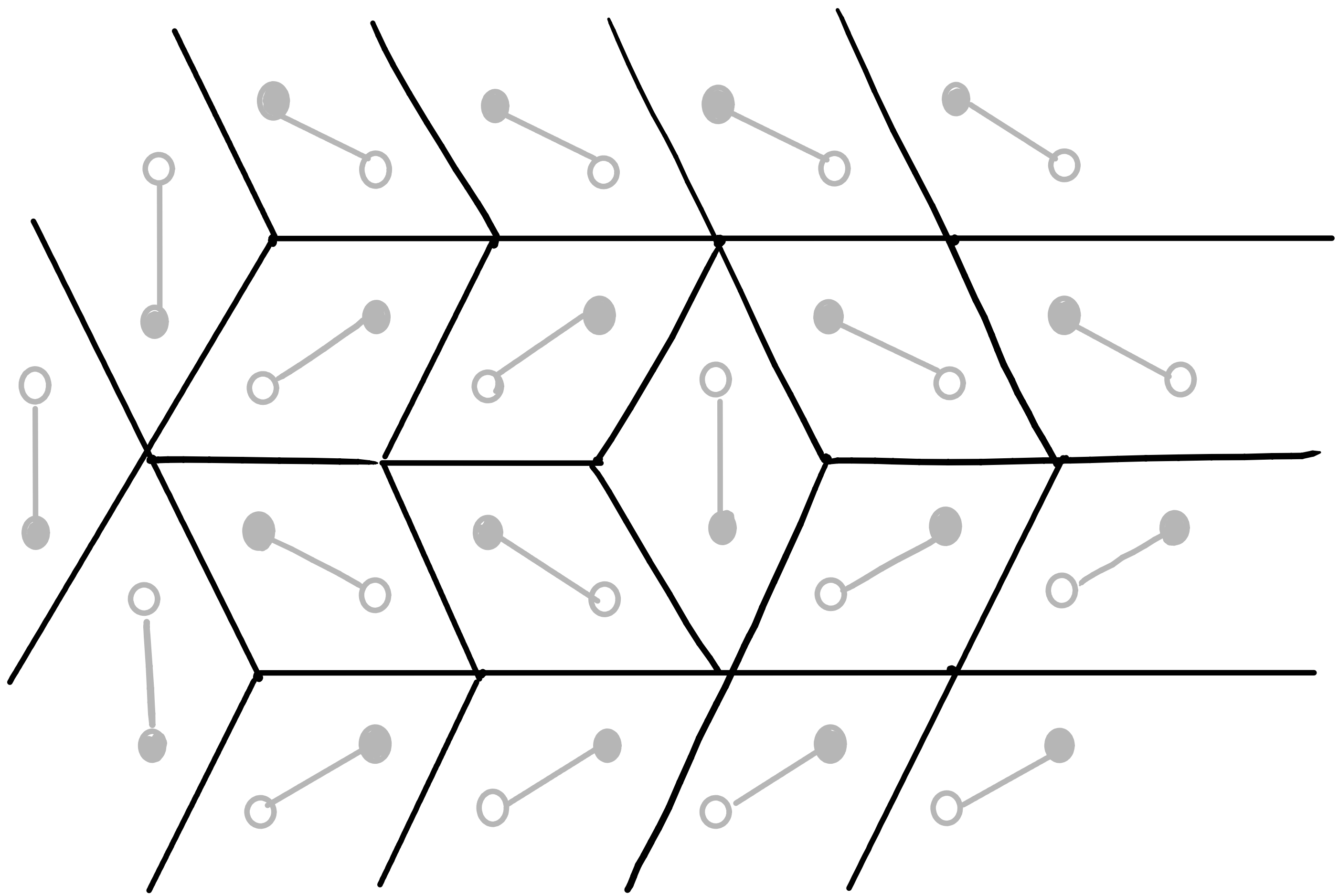
bi-partite graph on a torus



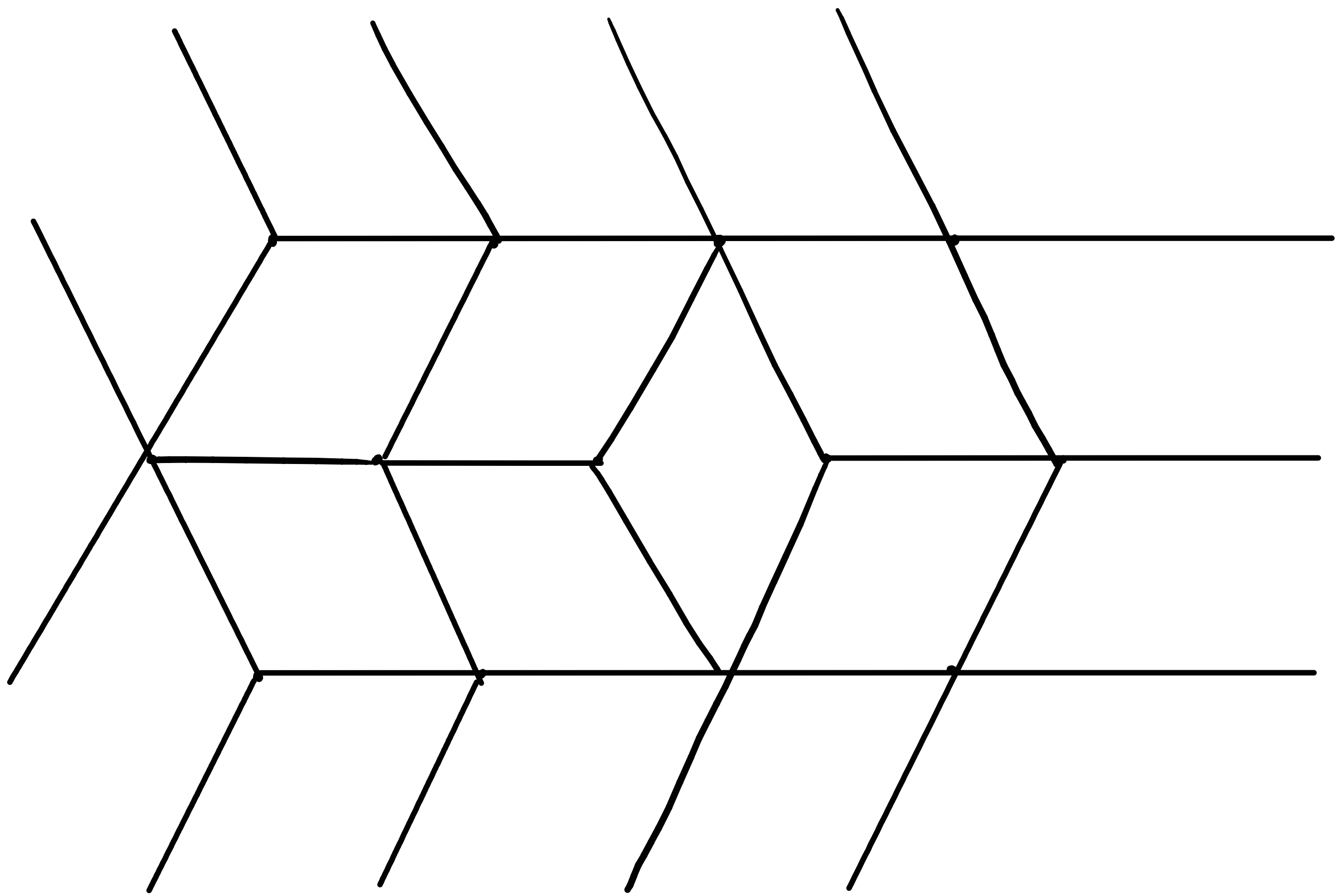
bi-partite graph in the universal covering



a perfect matching



Its dual graph looks
like a crystal corner.



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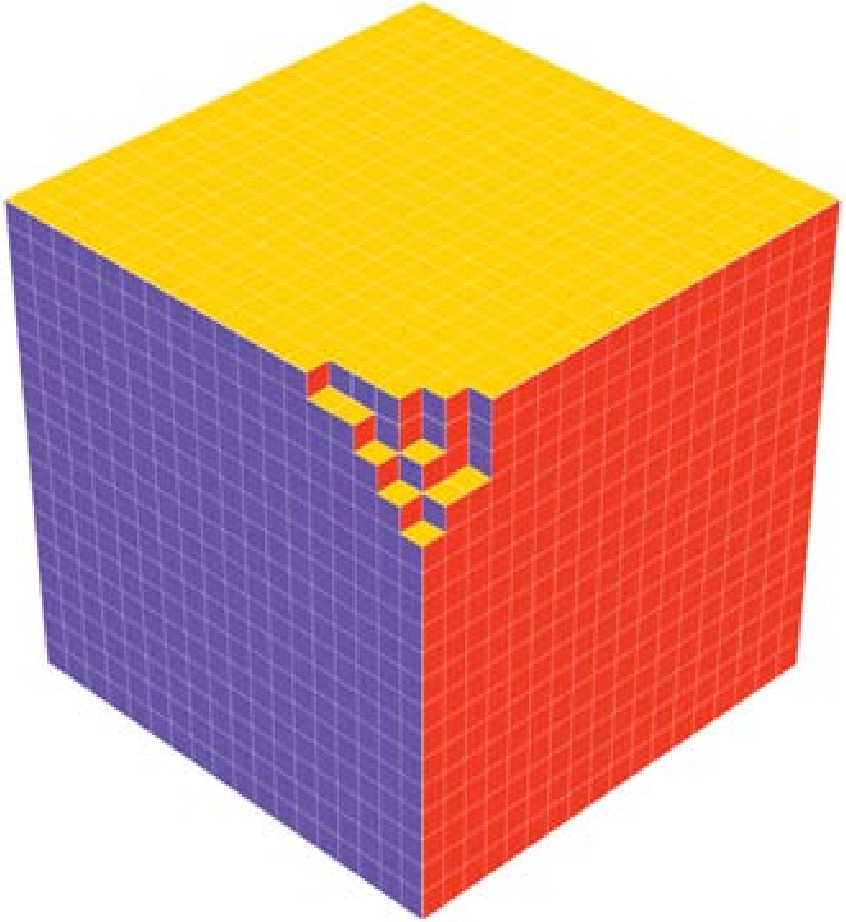
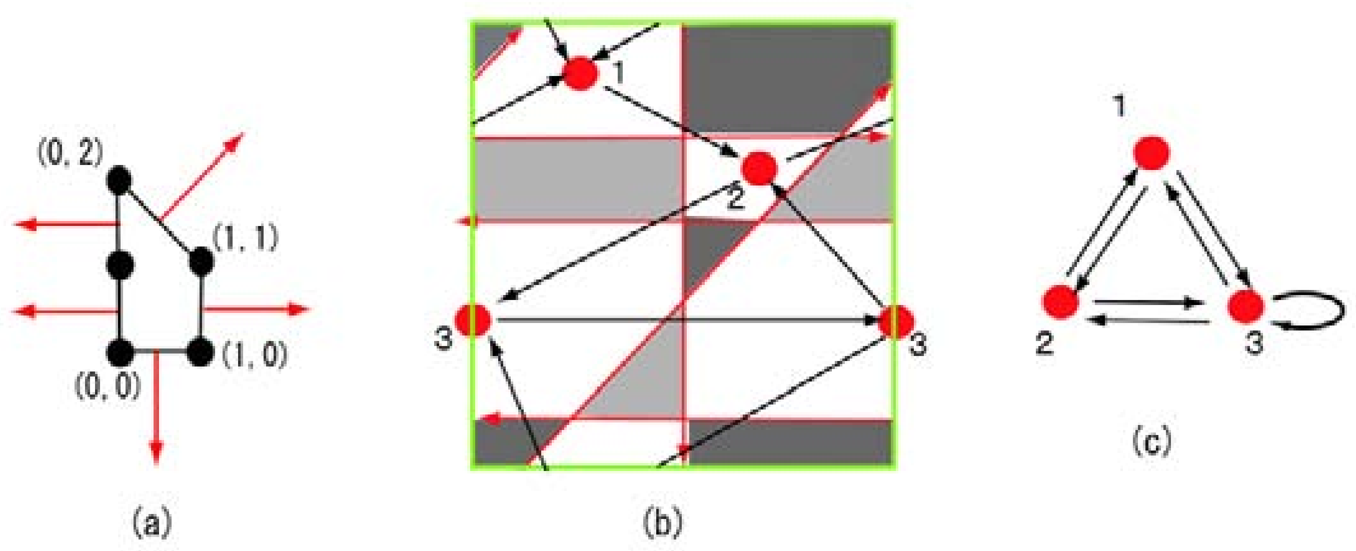


Each molten crystal represents
D0 brane bound state in $\mathbb{C}^3$

This generalizes to an **arbitrary toric CY3**.

Yamazaki + H.O. [0811.2801]

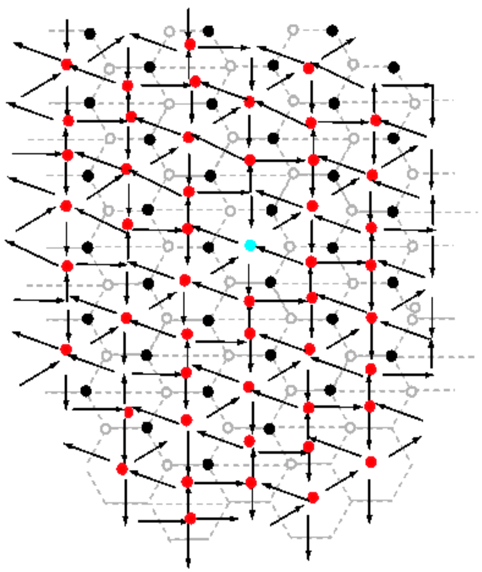


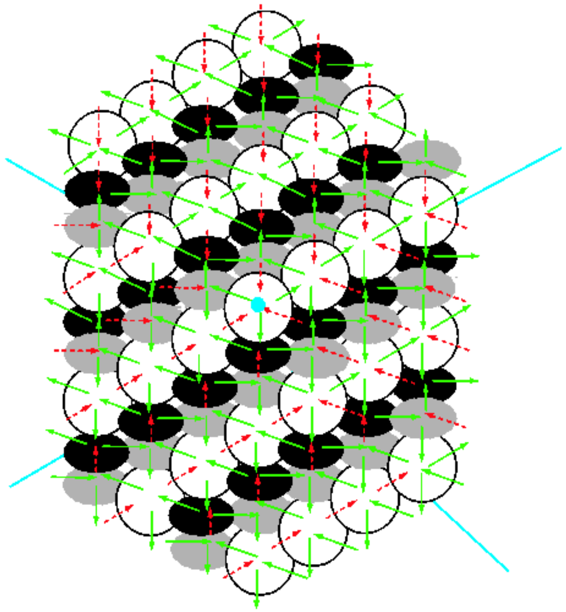
The crystal consists of **atoms** of different types corresponding to **nodes of the quiver** diagram.

The **edges** of the quiver determine the **chemical bonds**.

This generalizes to an **arbitrary toric CY3**.

Yamazaki + H.O. [0811.2801]





The crystal consists of **atoms** of different types corresponding to **nodes of the quiver** diagram.

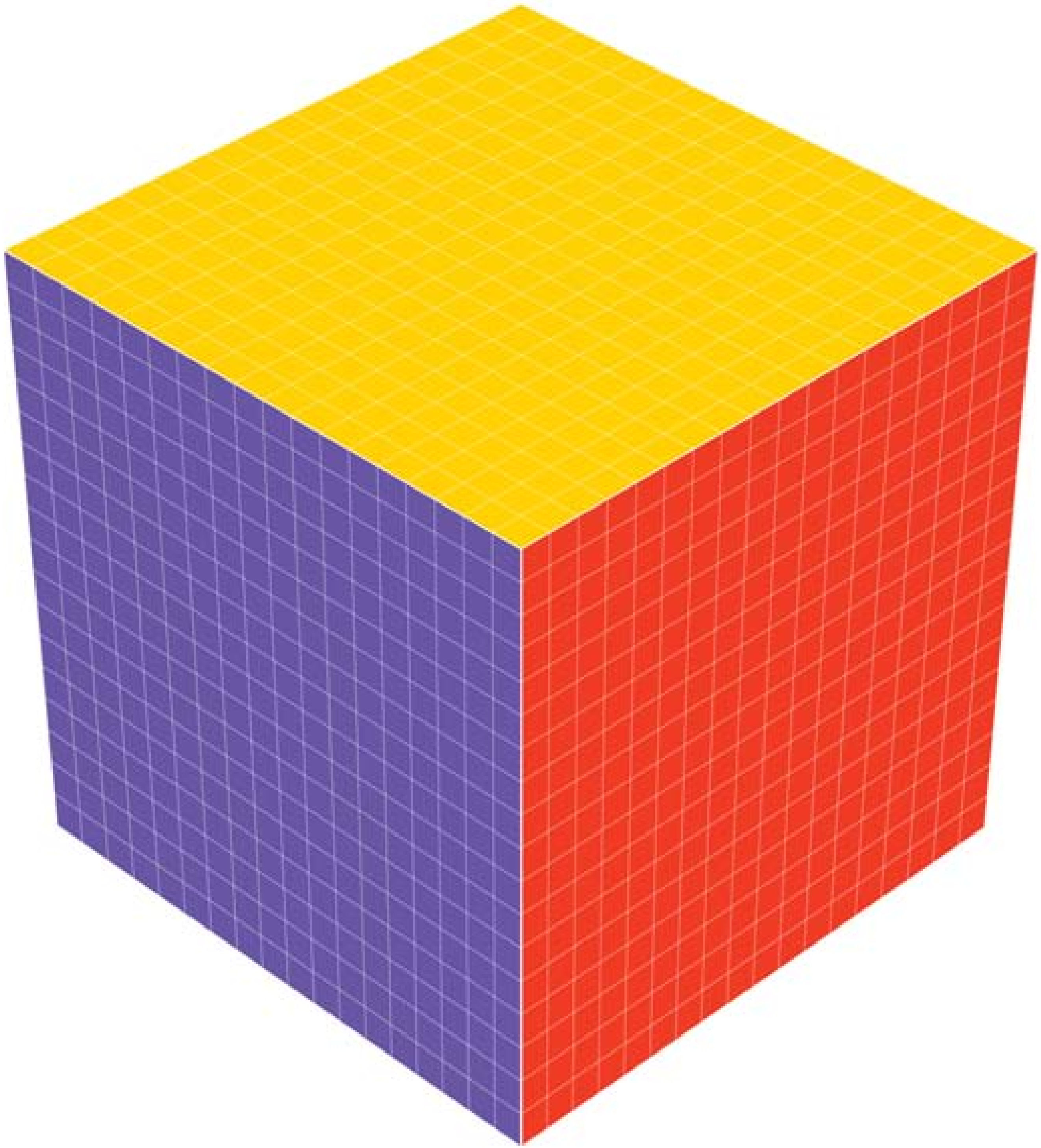
The **edges** of the quiver determine the **chemical bonds**.

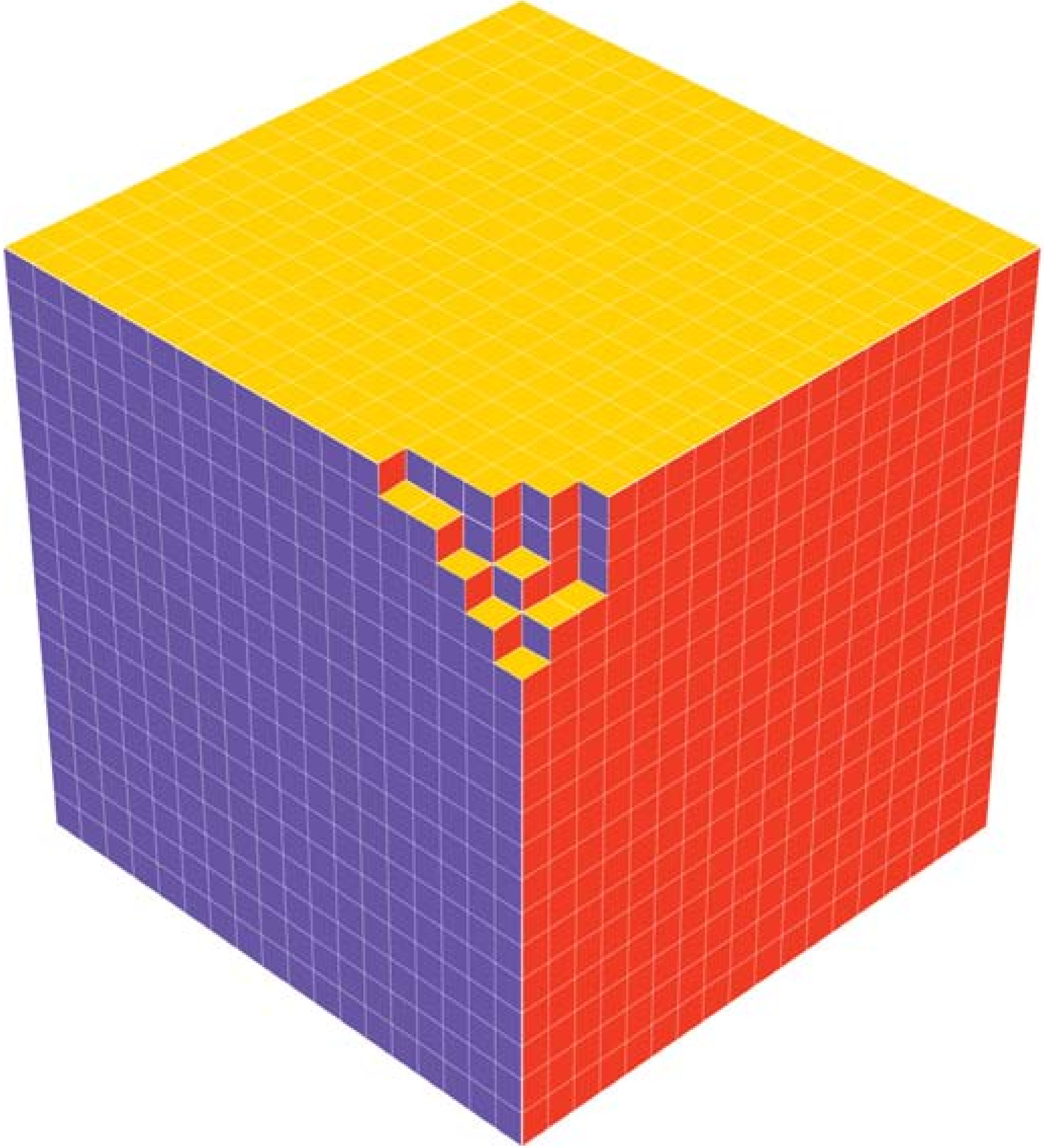
$$Z_{\text{crystal}} = \sum \Omega(n_0, n_a)\, e^{-g_s n_0 - t^a n_a}$$

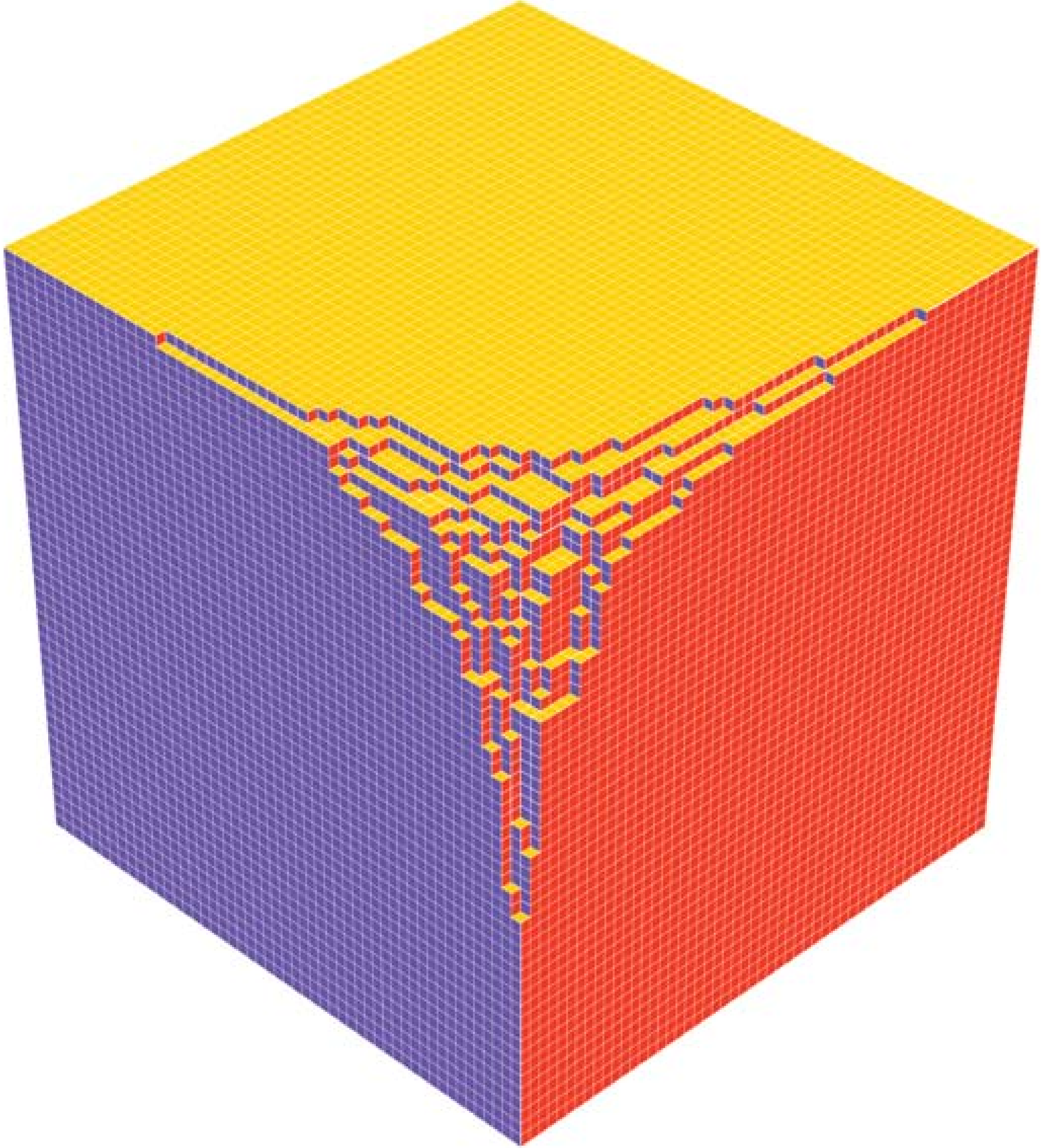
n_0 = # D0 branes

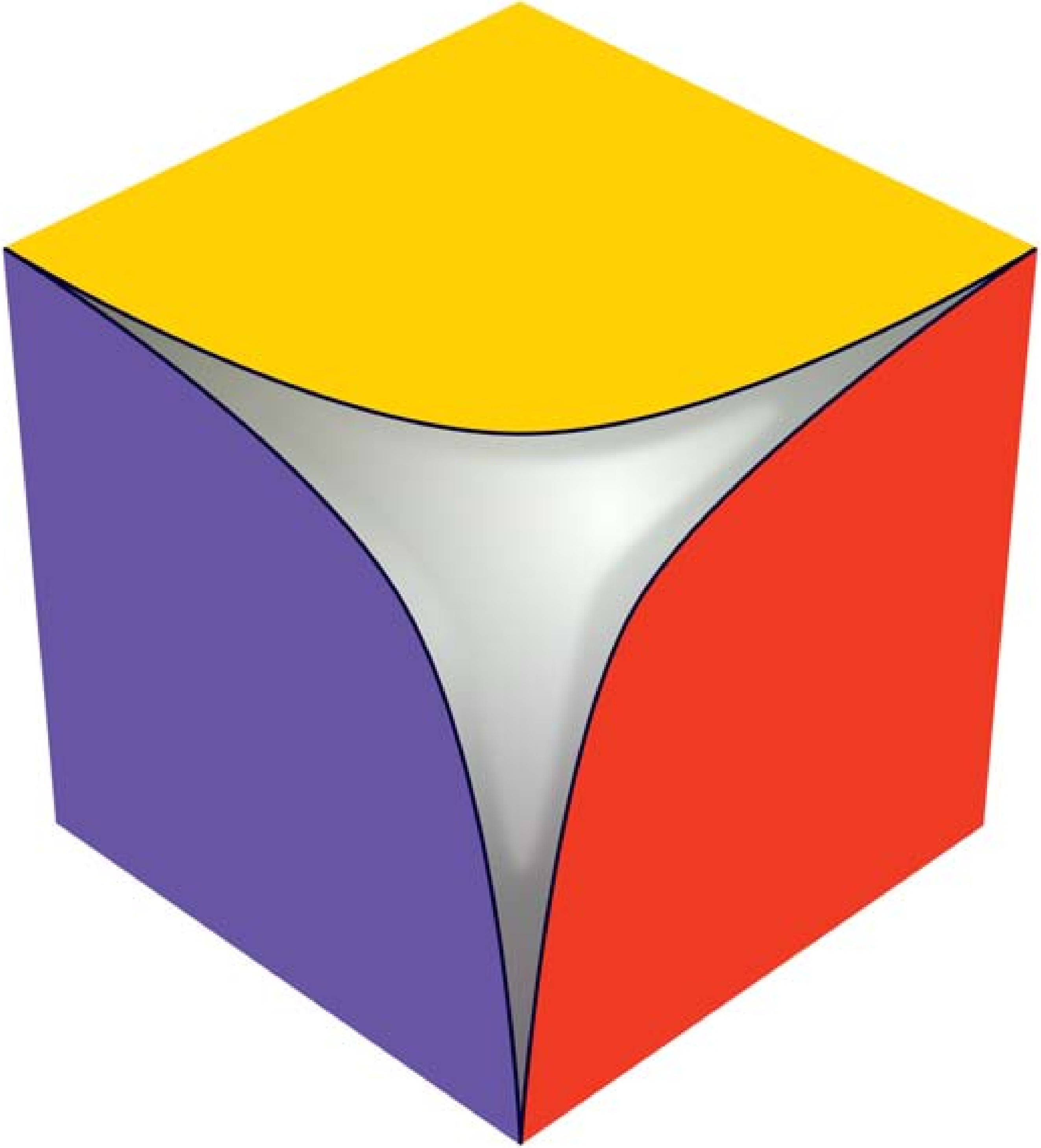
n_a = # D2 branes , $a = 1, \cdots, \dim H_2$

Kenyon, Okounkov and Sheffield evaluated Z_{crystal} and related its thermodynamic limit $g_s \to 0$ to the Amoeba of $F(x, y)$.



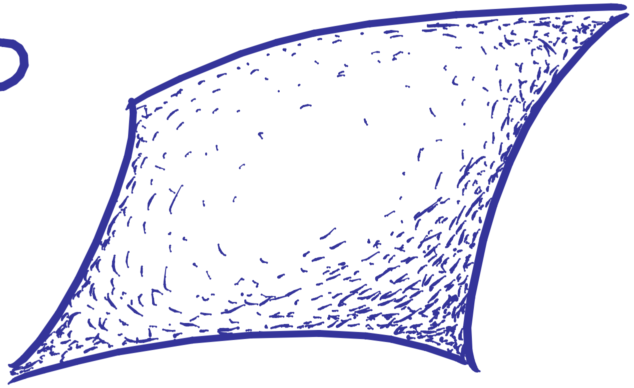


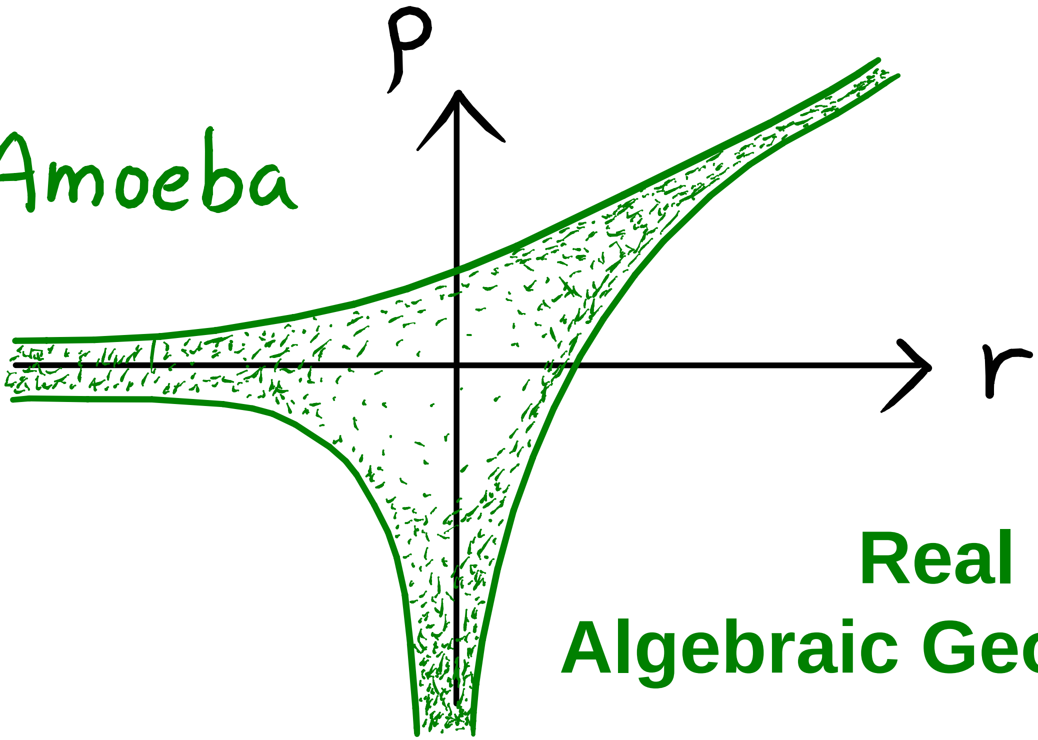




$F(x, y) = 0$

in $\mathbb{C}^2$

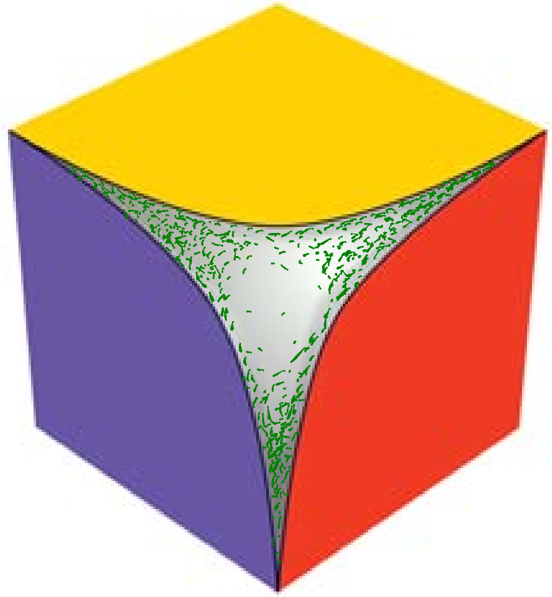




Amoeba

Real

Algebraic Geometry



From the work of Kenyon, Okounkov and Sheffield, one can deduce :

$$Z_{\text{crystal}} \sim \exp\left(-\frac{1}{g_s^2}\int_{-\infty}^{\infty} dx\,dy\, R(x,y)\right),$$

where

$$R(x,y) = \int_0^{2\pi} \frac{d\theta\, d\phi}{(2\pi)^2} \log F(e^{x+i\theta}, e^{y+i\phi})$$

Ronkin function

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$$\Uparrow$$

Ronkin function

$uv + F(x,y) = 0$

mirror of toric CY_3

We have shown that this is equal to the genus-0 topological string partition function.

$$\mathcal{F}_0 = \int_\gamma \omega$$

ω : holomorphic 3-form

γ : mirror of 6-cycle of toric CY_3

$$= \int dx\, dy\, R(x, y)$$

Yamazaki + H.O. [0902.3996]

Does this mean

$$Z_{crystal} = Z_{top} \; ?$$

Wall Crossing

Wall Crossing:

The number of BPS states depends on

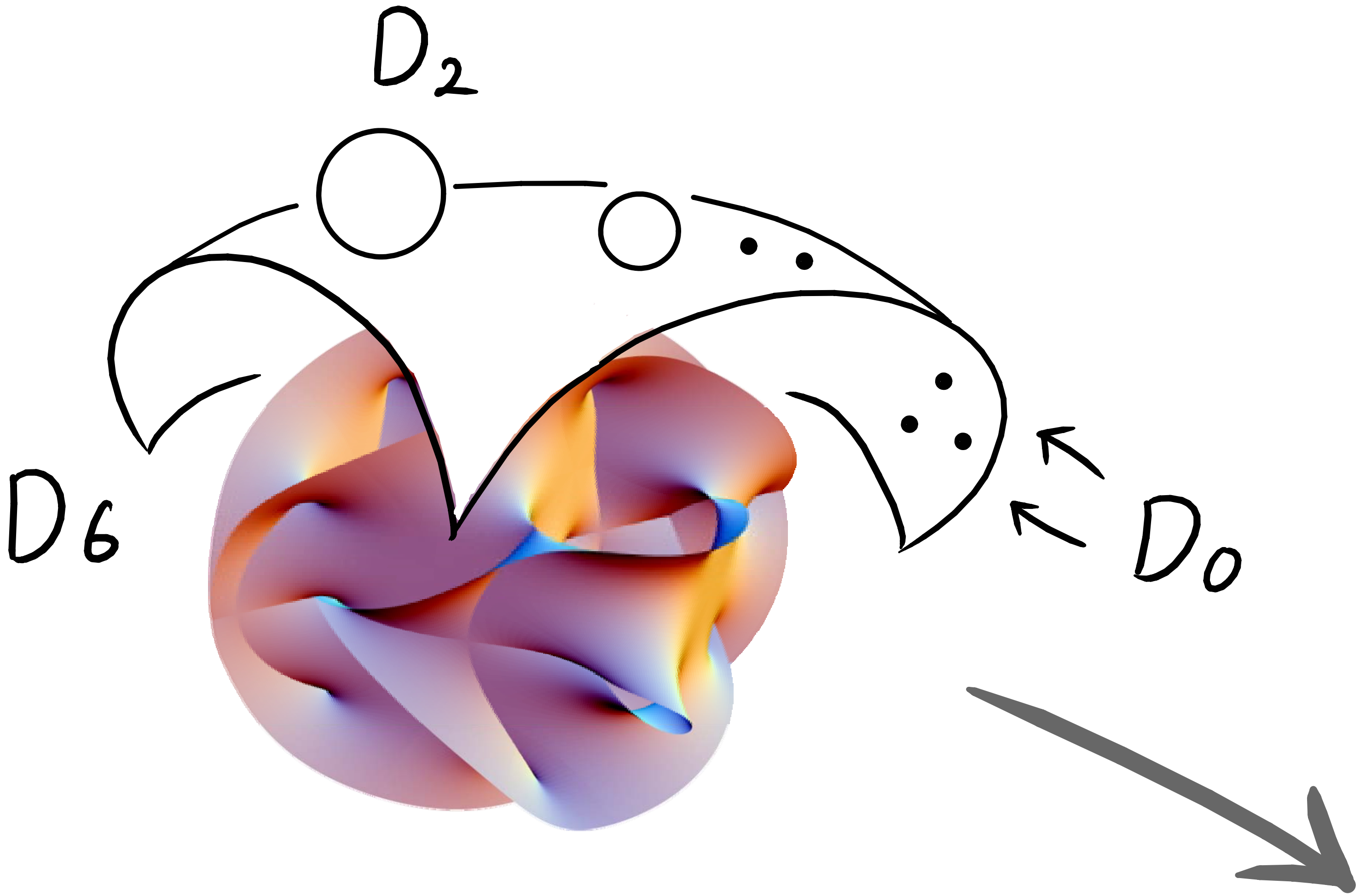
... the asymptotic values of the CY moduli.

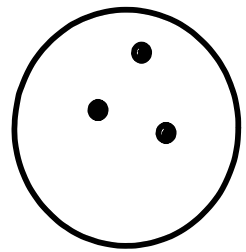
... the stability conditions on D brane bound states.

... how to treat the 1 D6 brane.

... the choice of the crystal ground state.

They are all related.





When can

D_2 / D_0 bound states

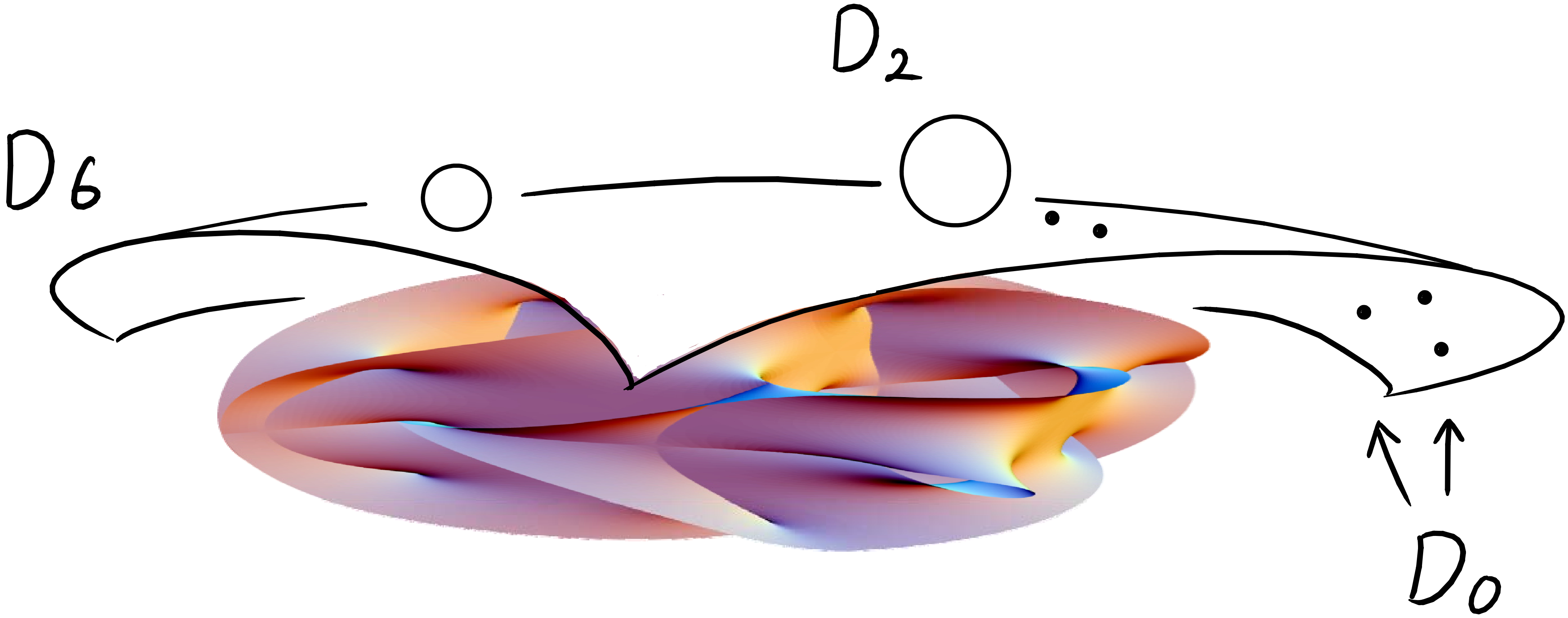
be emitted ?

A BPS particle can decay when the central charges of the fragments align.

For $1\,[D_6] + n_0\,[D_0] + n_i\,[D_2]_i$,

$$\text{Central Charge} = \infty\, e^{i\varphi} + n_0 + n_i t^i / g_s$$

$$\simeq \infty\, e^{i\varphi}$$



$$\text{volume}(CY_3) = \infty$$

A BPS particle can decay when the central charges of the fragments align.

For $1\,[D_6] + n_0\,[D_0] + n_i\,[D_2]_i$,

$$\text{Central Charge} = \infty\, e^{i\varphi} + n_0 + n_i t^i/g_s$$
$$\simeq \infty\, e^{i\varphi}$$

It can decay and emit $m_0\,[D_0] + m_i\,[D_2]_i$

if $\arg(\infty\, e^{i\varphi}) \sim \arg(m_0 + m_i t^i/g_s)$

i.e. $\mathrm{Im}\left[e^{-i\varphi}(m_0 + m_i t^i/g_s)\right] = 0$

Jafferis, Moore ('08)

When t/g_s : real,

$$\mathrm{Im}\left[e^{-i\varphi}\left(m_0+m_i\frac{t^i}{g_s}\right)\right]=-\sin\varphi\left(m_0+m_i\frac{t^i}{g_s}\right)$$

$\Rightarrow$ The walls are at $m_0 g_s + m_i t^i = 0$.

When t/g_s : real,

$$\mathrm{Im}\left[e^{-i\varphi}\left(m_0 + m_i \frac{t^i}{g_s}\right)\right] = -\sin\varphi \left(m_0 + m_i \frac{t^i}{g_s}\right)$$

$\Rightarrow$ The walls are at $m_0 g_s + m_i t^i = 0$.

$$Z_{DT} = \prod_{n_0, n_2} \left(1 \pm e^{-n_0 g_s - n_i t^i}\right)^{c_{n_0, n_2}}$$

Every time we cross a wall, we gain/lose a factor of $\left(1 \pm e^{-n_0 g_s - n_i t^i}\right)^{c_{n_0, n_2}}$.

e.g. Conifold

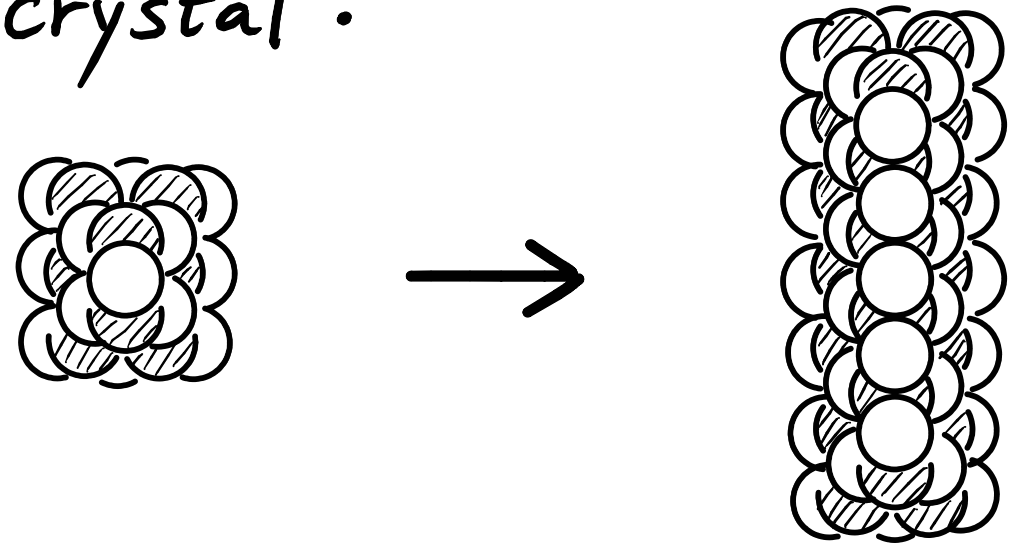
$$Z_{DT} = \frac{\prod_n (1-q^n Q)^n}{\prod_n (1-q^n)^{2n}} \qquad q = e^{-g_s}, \quad Q = e^{-t}$$

$$Z_{crystal} = \frac{\prod_n (1-q^n Q)^n (1-q^n Q^{-1})^n}{\prod_n (1-q^n)^{2n}}$$

The walls are at $q^n Q^{\pm 1} = 1$, $q^n = 1$.

Szendroi ('07); Nakajima, Nagao ('08), Jafferis, Moore ('08)

For the conifold, the wall crossing can also be interpreted as changing of the ground state of the crystal:



Chuang, Jafferis ('08)

Nagao, Nakajima ('08, v2)

This generalizes to an arbitrary toric CY_3 without compact 4 cycles.

Nagao ('08, '09)

Aganagic, Vafa, Yamazaki + H.O. ('09).

Unified Description of Chambers

For the conifold

$$Z_{crystal} = \frac{\prod_n (1-q^n Q)^n (1-q^n Q^{-1})^n}{\prod_n (1-q^n)^{2n}}$$

$$Z_{DT} = \frac{\prod_n (1-q^n Q)^n}{\prod_n (1-q^n)^{2n}}$$

$$Z_{top} = \frac{\prod_n (1-q^n Q)^n}{\prod_n (1-q^n)^{n}}$$

Unified Description of Chambers

For the conifold

$$Z_{\text{crystal}} = \frac{\prod_n (1-q^n Q)^n (1-q^n Q^{-1})^n}{\prod_n (1-q^n)^{2n}}$$

$$= Z_{\text{top}}(q, Q) \cdot Z_{\text{top}}(q, Q^{-1})$$

This generalizes to an arbitrary toric CY_3 without compact 4 cycles.

Aganagic, Vafa, Yamazaki + H.O. ('09).

This explains why the $g_s \to 0$ limit of the crystal melting model reproduced $\mathcal{F}_0$.

$$Z_{\text{crystal}} \sim \exp\left(-\frac{1}{g_s^2} \underbrace{\int_{-\infty}^{\infty} dx\,dy\, R(x,y)}_{\displaystyle \parallel} \right)$$

$\oint \omega$ of the mirror.

Yamazaki + H.O. ('09)

More generally, in any chamber between the DT and the Crystal Chambers, $^{\exists}Q_*$ such that

$$Z_{BPS} = Z_{top}(q, Q) \cdot Z_{top}(q, Q_* Q^{-1})$$

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free gas of mutually BPS particles

Aganagic, Vafa, Yamazaki + H.O. ('09).

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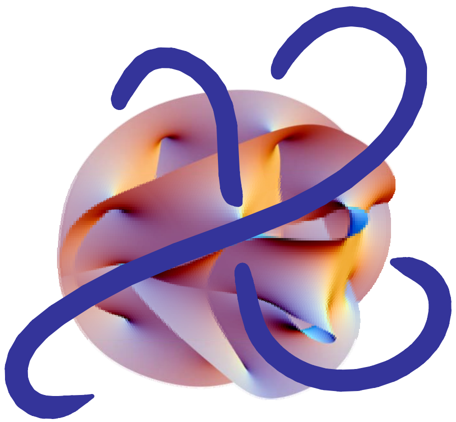
$$Z_{BPS} = Z_{top}(q, Q) \cdot Z_{top}(q, Q_* Q^{-1})$$

Note : This is not OSV.

We need

$$(g_s, t) \to (1/g_s, t/g_s)$$

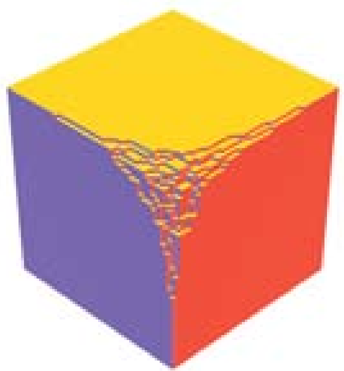
When t/g_s are real, the walls separating the Donaldson-Thomas theory and the crystal melting model are at $n_0 g_s + n_i t^i = 0$.



Donaldson-Thomas chamber :

$|g_s| \ll |t^i|$

perturbative in g_s.



Crystal Chamber :

$|g_s| \sim |t^i|$

non-perturbative in g_s.

Questions :

With compact 4-cycles ?

For compact CY_3 ?