Overview of Progress on Scattering Amplitudes

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ACCELERATE PARTICLES TO HIGH ENERGIES...

COLLIDE THEM....

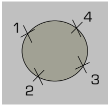
SEE WHAT COMES OUT!

smack!!

Gluon Amplitudes

Scattering n gluons at tree level

color-ordered



involves fast-growing number of Feynman diagrams

n	=	3	4	5	6	7	...
# Feynman diagrams	=	1	3	10	38	149	...

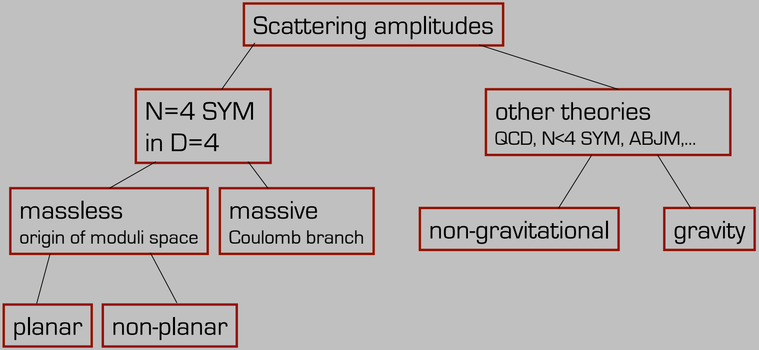
of increasing complexity, but sum of diagrams is extremely simple:

$$A_n(1^- 2^- 3^+ \dots n^+) = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}$$

Spinor helicity formalism: $p_{i\,\alpha\dot\alpha} = |i\rangle_\alpha [i|_{\dot\alpha}$.

- Why so simple?
- Better way to calculate?
- New 'dual' formulation?

Overview


Scattering amplitudes
N=4 SYM
in D=4
other theories
QCD, N<4 SYM, ABJM,...
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origin of moduli space
massive
Coulomb branch
non-gravitational
gravity
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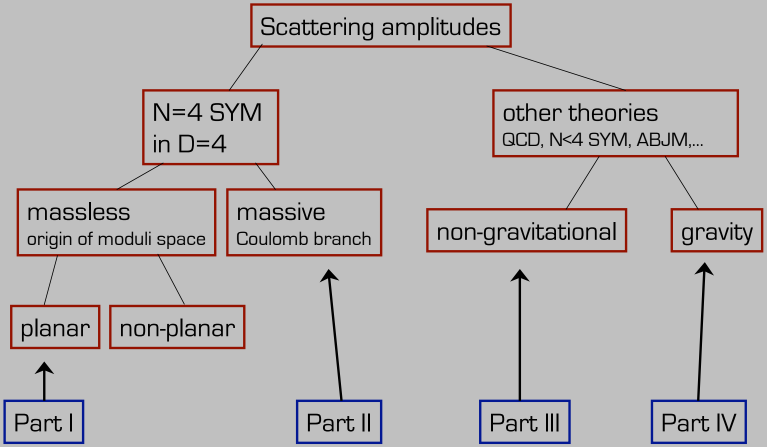
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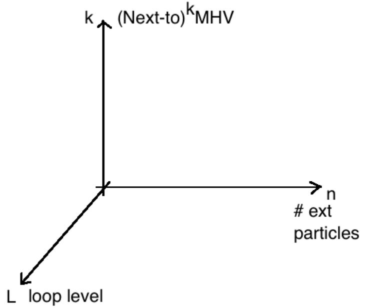
Part I
Part II
Part III
Part IV

Levels of difficulty



$$\mathrm{MHV} = \text{maximally helicity violating} = \text{sector of } A_n(--+\cdots+)$$
$$\mathrm{NMHV} = \text{sector of } A_n(---+\cdots+)$$
...
$$\mathrm{N}^k\mathrm{MHV} = \text{sector } A_n(\underbrace{-\cdots-}_{k+2}\underbrace{+\cdots+}_{n-k-2}).$$
...
$$\overline{\mathrm{MHV}} = \text{anti-MHV} = \mathrm{N}^{n-4}\mathrm{MHV} = \text{sector of } A_n(-\cdots-++)$$

Part I:

First tree-level amplitudes in $N = 4$ SYM.

Then loop-level amplitudes in planar $N = 4$ SYM.

Tree-level Gluon Amplitudes - recursively

Better ways to calculate!

Two powerful "recycling" techniques

following twistor-string [Witten'03]

- BCFW recursion relations [Britto,Cachazo,Feng'04] [BCF,Witten'05]
  Get A_n from $A_{n'<n}$
- CSW expansion [Cachazo,Svrcek,Witten'04]
  amplitude = sum of diagrams with on-shell MHV vertices.

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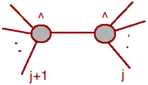
Both can be derived from complex deformation of momenta $p_i \longrightarrow \hat{p}_i = p_i + z\, q_i$ such that $\hat{p}_i^2 = 0$ and $\sum \hat{p}_i = 0$.

If $\quad (\star) \quad \hat{A}_n(z) \to 0$ as $z \to \infty \quad$ then

$$0 = \oint_{\mathcal{C}} \frac{\hat{A}_n(z)}{z} \quad \Longrightarrow \quad A_n = \hat{A}_n(0) = \sum_I \hat{A}_L(z_I) \frac{1}{P_I^2} \hat{A}_R(z_I) \; = \sum_j$$



When is a shift 'good'?

Tree-level Gluon Amplitudes - recursively

Better ways to calculate!

Two powerful "recycling" techniques

following twistor-string [Witten'03]

- BCFW recursion relations $\longleftarrow$ 2-line shift
  Get A_n from $A_{n'<n}$ $\qquad |1\rangle \to |1\rangle - z|2\rangle, \;\; |2] \to |2] + z|1]$
- CSW expansion $\longleftarrow$ "Risager shift" or all-line shift
  amplitude = sum of diagrams with on-shell MHV vertices.

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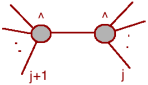


When is a shift 'good'?

$\mathcal{N} = 4$ SYM Superamplitudes $\mathcal{A}_n$

Promote democracy among the external states:

Collect amplitudes into Grassmann polynomials $\mathcal{A}_n(\Omega_1, \Omega_2, \ldots, \Omega_n)$ using superwavefunction [Ferber'78]

$$\Omega_i = G_i^{\,+} + \eta_{ia}\,\lambda_i^{\,a} - \frac{1}{2!}\eta_{ia}\eta_b\,S_i^{\,ab} - \frac{1}{3!}\eta_{ia}\eta_{ib}\eta_{ic}\,\lambda_i^{\,abc} + \eta_{i1}\eta_{i2}\eta_{i3}\eta_{i4}\,G_i^{\,-}\,.$$

Then

$$\mathcal{A}_n^{\mathrm{MHV}} = \frac{\delta^{(8)}(\sum_i |i\rangle\eta_i)}{\langle 12\rangle\langle 23\rangle\ldots\langle n1\rangle}$$ [Nair'88]

and

$$\mathcal{A}_n = \mathcal{A}_n^{\mathrm{MHV}}\big(1 + \mathcal{P}_n^{\mathrm{NMHV}} + \ldots\mathcal{P}_n^{\overline{\mathrm{MHV}}}\big)$$

where $\mathcal{P}_n^{\mathrm{N}^k\mathrm{MHV}}$ has Grassmann degree $4k$.

SUSY Ward identities $Q^a\mathcal{A}_n = \tilde{Q}_a\mathcal{A}_n = 0$.
In $\mathcal{N} = 4$ SYM & $\mathcal{N} = 8$ SG solved in terms of $SU(n-4)$ Young tableaux [H.E.,Freedman,Kiermaier'09]

Tree-level $\mathcal{N} = 4$ SYM

BCFW promoted to supershift:

[Arkani-Hamed,Cachazo,Kaplan'08] [Brandhuber,Heslop,Travaglini'08]

$$|1\rangle \to |1\rangle - z|2\rangle, \qquad |2] \to |2] + z|1], \qquad \eta_{2a} \to \eta_{2a} + z\,\eta_{1a}.$$

Now $\delta^{(8)}$-function invariant — all shifts "good".

Can SOLVE the super-BCFW recursion relations, all n, all N^{k}MHV:

$$\mathcal{P}_n^{\mathrm{N}^k\mathrm{MHV}} = \sum \text{"R-invariants"}$$

[Drummond & Henn'08]

So tree-level amplitudes of $\mathcal{N} = 4$ SYM solved!

Alternative tree-level solution:

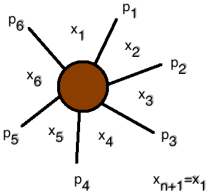
all N^{k}MHV tree superamplitudes from CSW.

[Georgio,Glover,Khoze'04] [HE,Freedman,Kiermaier'08]

A new symmetry

Introduce region variables x_i:

$p_i = x_i - x_{i+1}$



$\Longrightarrow$ mom. cons. automatic

Dual (super)conformal symmetry

acts on region variables as ordinary s.conf. sym.,

e.g. inversion $x_i^\mu \to x_i^\mu / x_i^2$ and $\langle i, i+1 \rangle \to \langle i, i+1 \rangle / x_i^2$

- Split-helicity gluon amplitudes $A_n(- \cdots - + \cdots +)$ transforms covariantly.
- Non-split-helicity amplitudes do NOT transform covariantly. $\frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \ldots \langle n1 \rangle}$
- Tree superamplitudes of $\mathcal{N} = 4$ SYM are dual superconf. covariant!
- The "R-invariants" are dual superconformal *invariant*.

[Drummond,Henn,Sokatchev,Korchemsky'08] [Brandhuber,Heslop,Travaglini'08] [Drummond,Henn'08]

More symmetry

Historically:

First hints of dual conformal sym. in loop calculations [Drummond,Henn,Smirnov,Sokatchev'06]

Then at strong coupling. [Alday,Maldacena'07]

Planar $\mathcal{N} = 4$ SYM has

- ordinary superconformal symmetry
- dual superconformal symmetry

These comprise to two lowest levels of a Yangian algebra

[Drummond,Henn,Plefka'09]

$\leftrightarrow$ Planar $\mathcal{N} = 4$ SYM integrable!

Note: symmetry generator must be extended to take into account collinear momenta.

[Bargheer,Beisert,Galleas,Loebbert,McLoughlin'09]

Good variables

1. spinor helicity $p_i = |i\rangle[i|$ makes null condition $p_i^2 = 0$ manifest.
2. region variables make mom. cons. manifest, but requires $(x_i - x_{i+1})^2 = p_i^2 = 0$.

Wouldn't it be nice to have unconstrained variables in which dual conformal symmetry is manifest!!?

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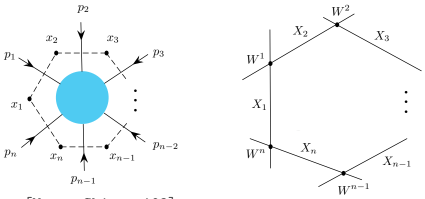
This is what momentum twistors do! [Hodges'09]



from [Mason,Skinner'09]

$x_i \;\rightarrow\; X_i$ line in $\mathbb{CP}^3$

$(x_i - x_{i+1})^2 = 0$
$\rightarrow X_i$ and X_{i+1} intersect

The intersection points W_i are the momentum twistors

Define $\langle ijkl\rangle = \epsilon_{ABCD} W_i^A W_j^B W_k^C W_l^D$

NMHV tree amplitudes

Now the "R-invariants" of the NMHV tree amplitudes can be expressed as $R_{nij} = [n, i, i+1, j, j+1]$, where $[ijklm] \equiv \frac{\delta^{(4)}(\chi_i \langle jklm \rangle + \text{cyclic})}{\langle jklm \rangle \langle klmi \rangle \langle lmij \rangle \langle mijk \rangle \langle ijkl \rangle}$.

In fact, the NMHV tree amplitudes can be written

$$\text{tree-level} \quad \mathcal{A}_n^{\text{NMHV}} = \mathcal{A}_n^{\text{MHV}} \sum_{i<j} [\star, i, i+1, j, j+1]$$

where

- $\star = W_n$ gives super-BCFW form of the superamplitude
- $\star = W_X$ for some reference momentum twistor W_X gives CSW form of the superamplitude.

[Bullimore,Mason,Skinner'10] [Bullimore'10] [Arkani-Hamed,Bourjaily,Cachazo,Trnka'10]

Amplitudes as geometry!?!

Proposal: [Hodges'09; Arkani-Hamed, Bourjaily, Cachazo, Hodges, Trnka'10]

Amplitudes are volumes of polytopes, and different triangulations correspond to different representations (BCFW/CSW/new).

Has been illustrated for tree-level NMHV and 1-loop MHV integrand.

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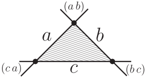
Here restrict to baby-version:

Let $[ijk]$ be the $\mathbb{CP}^2$-analogue of the R-invariants $[ijklm]$. The role of the "amplitude" is then played by $A_n = \sum_i [\star, i, i+1]$. For example, we have $A_4 = [234] + [241]$ for $\star = 2$.

$[abc]$ is area of dual space triangle: 

$= \quad [a\,b\,c]$

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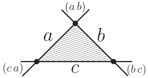
$[abc]$ is area of dual space triangle: 

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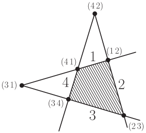
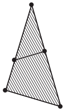
$A_4 =$ 

$=$  $-$ 

Area $\quad = [2\,3\,4] \quad + \quad [2\,4\,1]$

Tree-level amplitudes are fascinating!

Remarkable structures!

What about loop amplitudes?

Two aspects:
the **integrand** and
the **integrated** result.

Recursive approach to the loop level integrand

A key to tree-level rec'rel's was that A_n^{tree} is a rational function with only simple poles.

$A_n^{\mathrm{loops}} \ni$ Log's, Li_k's... much more complicated structure. And also need to regulate IR divergences.

Note: in the planar limit, the *integrand* of a loop amplitude is a well-defined quantity and a rational function.

Recurse it!

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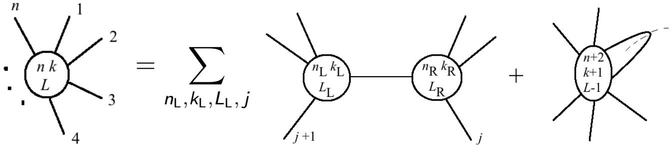


"usual" BCFW "forward limit"

with $n_L + n_R = n + 2$, $k_L + k_R = k - 1$ and $L_L + L_R = L$.

[Arkani-Hamed,Bourjaily,Cachazo,Caron-Huot,Trnka'10] [Caron-Huot'10] [Boels'10]

Integrands — planar $\mathcal{N} = 4$ SYM

Examples of integrand results:

- all 2-loop MHV integrand
- 7-point 2-loop NMHV integrand
- 5-point 3-loop MHV integrand

Yangian symmetry manifest.

[Arkani-Hamed,Bourjaily,Cachazo,Caron-Huot,Trnka'10]

How to integrate?

Integrating the integrand — planar $\mathcal{N} = 4$ SYM

IR regulator

Dimensional regularization breaks (dual) conf. sym.

Go on the Coulomb branch [Alday, Henn, Plefka, Schuster '09]

$\rightarrow$ dual conf. sym. restored when transformations include the masses!

Closely related: 6d and 10d maximally SYM planar amplitudes enjoy dual superconformal symmetry.

[Bern,Carrasco,Dennen,Huang,Ita'10] [Dennen,Huang'10] [Caron-Huot,O'Connell'10]

Interesting forms of integrand $\rightarrow$ easier integration?

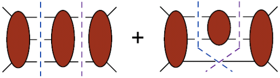
[Arkani-Hamed,Bourjaily,Cachazo,Trnka'10]

Loop-integrals satisfy differential equations $\rightarrow$ iterative structure

[Drummond,Henn,Trnka'10]

Systematics still needed.

Generalized unitarity

Generalized unitarity methods give powerful tool to reconstruct full loop amplitudes from trees:  .

see recent reviews [Britto'10] [Bern,Huang'11] [Carrasco,Johansson'11] and refs therein.

For example $N = 4$ SYM:

4-point $L = 1, 2, 3, 4$ planar amplitudes + non-planar integrands
$\rightarrow$ 4-point $L = 1, 2, 3, 4$ amplitudes in $N = 8$ supergravity

And methods implemented in numerical codes for QCD backgrounds.

$\rightarrow$ See also Dixon Strings'08 talk

ABDK/BDS ansatz [Anastasiou,Bern,Dixon,Kosower'03; Bern,Dixon,Smirnov'05]
captures infrared + collinear behavior of n-point MHV L-loop amplitude,
but fixes it only up to "remainder function" $R_n^{(L)}$.

(If) dual conformal symmetry

$$\Longrightarrow \quad R_n^{(L)} \text{ function of dual conformal cross-ratios,}$$

such as $u_{1346} = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} = \frac{s_{12} s_{45}}{s_{123} s_{345}}$ (using $x_{ij} = x_i - x_j = p_i + p_{i+1} + \cdots + p_{j-1}$)

What can we expect from loops? Planar Limit!

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No available cross-ratios for $n = 4, 5$. First non-trivial test $n = 6$, $L = 2$:

$R_6^{(2)} = 17$ pages of umph! Impressive calculation by [Del Duca, Duhr, Smirnov'10]

where "umph!" = polylogs Li_k and Goncharov logarithms.

What can we expect from loops? Good news!

With application of

- momentum twistors,
- "the Symbol", mathematical 'derivative'-type operation
- careful analysis of branch cuts, and
- Goncharov, the mathematician,

[Goncharov,Spradlin,Vergu,Volovich'10] reduced the result for $R_6^{(2)}$ to a ∼3-line expression of Li_k's and log's in combinations of uniform transcendentality 4. No Goncharov-logarithms left. Manifestly dual conf.inv.

→ see Strings'11 talk by Volovich.

The Lesson: There is hope for the future of higher-loop explorations!

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New super-Wilson-loop based proposal for calculating Symbol(Ampl).
[Caron-Huot'11]

Planar $\mathcal{N} = 4$ SYM

Why the focus on planar $\mathcal{N} = 4$ SYM?

1) It is the simplest!

So excellent "lab" for new methods.

2) It is related to QCD (and to quantum gravity)

'Solving' the scattering amplitudes is part of 'solving' the theory:

- spectrum of scaling dimensions
- correlation functions
- *scattering amplitudes*
- expectation values of Wilson-loops
- ...

What is the role of integrability for amplitudes?

The Grassmannian: a dual formulation?

Grassmannian $G(n, k)$ = space of k-planes in n-dimensions.

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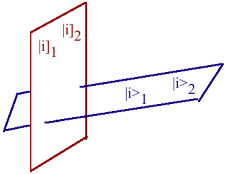
Outline:

Momentum conservation $\sum_{i=1}^{n} |i\rangle_\alpha [i|_{\dot\beta} = 0$.

Linearize momentum conservation condition by considering all k-planes containing the $|i\rangle$-plane.



Leeds to $G(n,k)$-integral

$$\mathcal{L}_{n,k} = \frac{1}{\mathrm{vol}(\mathrm{GL}(k))} \int \frac{d^{k\times n} C_{Aa} \quad \prod_{A=1}^{k} \delta^{4|4}(C_{Aa} W_a)}{(12\ldots k)(2\ldots k-1)\cdots(n\,1\ldots k-1)}$$

This generates **all Yangian invariants**.
It generates the $\mathcal{N}=4$ **tree amplitudes** and the "**Leading Singularities**".

[Drummond,Ferro'10] [Korchemsky,Sokatchev'10] [Mason,Skinner'10] [Spradlin,Volovich'10]

For more, see Arkani-Hamed Strings'10 talk

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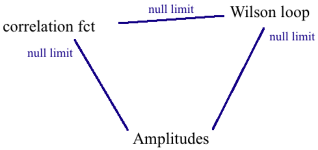
$\longrightarrow$ *Triality proposal*

Triality Prososal [Alday,Eden,Korchemsky,Maldacena,Sokatchev'10]

[Belitski,Korchemsky,Sokatchev'10] [Eden,Heslop,Korchemsky,Sokatchev'11]



For example "**square** of amplitude ≈ light-like limit of correlation function".

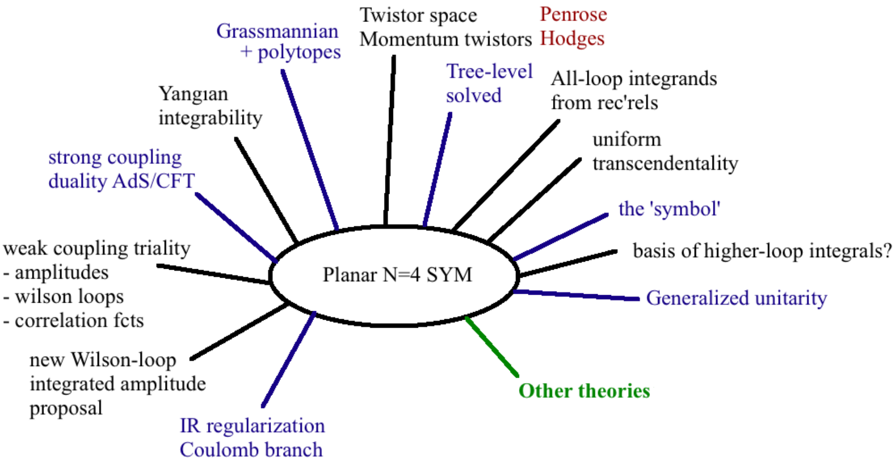
$$\big\langle \mathcal{O}(x_1)\dots\mathcal{O}(x_n)\big\rangle_{tree} = G_n = \sum_{k=0}^{n-4} a^k\, G_n^{(k)}$$

with 't Hooft couping $a = g^2 N_c/\pi^2$, $\mathcal{O}(x)$ is superfield operator version of "$\mathrm{Tr}\,\phi^2$", and k denotes Grassmann degree chiral superspace from truncated harmonic superspace

Proposal: $$\lim_{x^2_{i,i+1}\to 0} \frac{G_n}{G_n^{(k=0)}} = \left(\sum_{k=0}^{n-4} a^k\, \frac{\mathcal{A}_n^{\mathrm{N}^k\mathrm{MHV}}}{\mathcal{A}_n^{\mathrm{MHV}}}\right)^2 = \Big(1 + a\,\mathcal{P}_n^{\mathrm{NMHV}} + \dots\Big)^2$$

NMHV test passed. +loop-level generalization for *integrands*

Overview of planar $\mathcal{N} = 4$ SYM



Part II: Safari on the Coulomb branch

So far we discussed $\mathcal{N} = 4$ SYM at the origin of moduli space $\langle \phi^{ab} \rangle = 0$

Going on the Coulomb branch of $\mathcal{N} = 4$ is very useful for IR regularization [Alday,Henn,Plefka,Schuster'09] .

But to explore the Coulomb branch amplitudes with *massive* particles is also interesting in its own right.

That's Part II.

Most controlled theory with massive particles:

$\mathcal{N} = 4$ **SYM on the Coulomb branch.**

Specific scenario:

$U(N+M) \to U(N) \times U(M)$ by turning on VEVs

$$\langle (\phi^{12})_I{}^J \rangle = \langle (\phi^{34})_I{}^J \rangle = v\,\delta_I{}^J \quad \text{for } I, J \in U(M)$$

R-symmetry $SU(4) \to Sp(4) \supset \underbrace{SU(2)}_{12} \times \underbrace{SU(2)}_{34}$.

$$(A_\mu) \to \begin{pmatrix} (A_\mu)_{N\times N} & (W_\mu)_{N\times M} \\ (\overline{W}_\mu)_{M\times N} & (\widetilde{A}_\mu)_{M\times M} \end{pmatrix},$$

W's are massive "W-bosons", spin-1 of massive $\mathcal{N} = 4$ supermultiplet.

$W^L \propto (w^{12} + w^{34})$ is longitudinal polarization.

$\mathcal{N} = 4$ SYM on the Coulomb branch

Example:

Ultra-Helicity Violating (UHV) sector $A_n(W^- \bar{W}^+ + \cdots +)$ is now the simplest:

$$\mathcal{A}^{\text{UHV}}_{n,\text{tree}} = -\frac{m^2\,[3|\prod_{i=4}^{n-1}[m^2 - x_{i2}x_{2,i+1}]|n]}{\langle q1^\perp\rangle^2\langle q2^\perp\rangle^2\ \langle 34\rangle\langle 45\rangle\cdots\langle n-1,n\rangle\prod_{i=3}^{n-1}(P_{2i}^2+m^2)}\ \delta^{(4)}(\langle qi^\perp\rangle\eta_{ia})$$

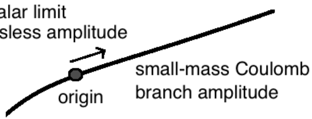
[Craig,H.E.,Kiermaier,Slatyer'11] [Boels,Schwinn'11] [Ferrario,Rodrigo,Talavera'06]

q is reference null vector: $p_i = p_i^\perp + \frac{m_i^2}{2q.p_i}\,q$.

Massive spinor helicity formalism [Dittmaier'98] [Cohen,H.E.,Kiermaier'10]

Massless $\longrightarrow$ massive

Soft-scalar limits probe nearby moduli-space



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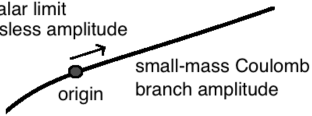

Example:

$$A_4(W^- \overline{W}^+ g^+ g^+) \;=\; -\frac{\langle 1^\perp 2^\perp\rangle^2 [34]}{\langle 34\rangle (P_{23}^2 + m^2)} \;\longrightarrow\; -m^2 \frac{\langle 1|q|2]}{\langle 2|q|1] P_{23}^2} \quad \text{for} \quad m^2 \ll P_{ij}^2$$

$$\langle\phi\rangle^2 \; A_6(g^- \phi^{12}_{\epsilon q_1} \phi^{34}_{\epsilon q_2} g^+ g^+ g^+) + (q_1 \leftrightarrow q_2) \;\longrightarrow\; -m^2 \frac{\langle 1|q|2]}{\langle 2|q|1] P_{23}^2} \quad \text{as} \quad \epsilon \longrightarrow 0$$

Several other examples of leading-order match [Craig,H.E.,Kiermaier,Slatyer'11]

Can the entire massive propagator be re-summed through multiple soft-scalar limits!

Yes! Works beautifully! [Kiermaier'11]

Motivates new proposal for CSW-type expansion for Coulomb branch amplitudes.

Part III: Other non-gravitational theories

One goal is practical applications of amplitude techniques to QCD backgrounds.

Another goal is to further advance the formal developments.

In part III, we leave the safe comfort of $\mathcal{N} = 4$ SYM.

General theories

Tree-level on-shell recursion relations (BCFW, CSW) have been a major input for the developments in $\mathcal{N} = 4$ SYM.

But when are they valid? (for which amplitudes in which theories?)

Link between factorization needed for BCFW and various classic S-matrix result (spin ≤ 2, interacting spin-1 is Yang-Mills theory w/ anti-sym structure constants etc). [Cachazo,Benincasa'07]

Other types of recursion?

All-line shift recursion relations

Validity of CSW in $\mathcal{N}=4$ SYM can be derived using "all-line shift" recursion relations

$$|\hat{i}] = |i] + z\, c_i |X]\,, \qquad |\hat{i}\rangle = |i\rangle\,, \qquad \underbrace{\sum_{i=1}^{n} c_i |i\rangle = 0}_{mom.cons.}$$

[H.E.,Freedman,Kiermaier'08] following minus-line shift [Risager'05]; SUSY-shift version [Kiermaier,Naculich'09]

Validity (i.e. large-z falloff of amplitudes) at tree-level in *general* 4d Lorentz invariant local theories

- with or without supersymmetry,
- with massless and/or massive particles,
- with or without non-renormalizable effective interactions

can be explored by simple dim'l analysis and little group scaling.

Result: For helicity$^{*)}$ amplitude $A_n(1^{h_1} 2^{h_2} \ldots n^{h_n})$, under above all-line shift

$$\hat{A}_n(z) \to z^s \quad \text{as} \quad z \to \infty\,, \quad \text{with} \quad 2s = 4 - n - c + \sum_{i=1}^{n} h_i$$

with c = mass-dimension of product of contributing couplings. [Cohen,H.E.,Kiermaier'10]

*) "frame" fixed by massive spinor helicity formalism.

All-line shift recursion relations

What is the physics of the sufficient condition $2s = 4 - n - c + \sum_{i=1}^{n} h_i < 0$?

— when can we even expect tree-level "on-shell constructibility"?

Example: scalar-QED $|D\phi|^2 \ni A_\mu \, \phi \, \partial^\mu \bar{\phi}$ and $A_\mu A^\mu \phi \, \bar{\phi}$

- $A_4(-_1 \, -_2 \, \phi_3 \, \phi_4)$ has $2s = 4 - 4 - 0 + (-2) = -2 < 0$ (c=0)

  so on-shell constructible. (actually, sum of two CSW-diagram = 0)

  $\rightarrow$ No info needed about 4-pt contact term, which is in $\mathcal{L}$ to ensure *off-shell* gauge-inv.

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- $A_4(\phi_1 \, \phi_2 \, \phi_3 \, \phi_4)$ has $2s = 4 - 4 - 0 + 0 = 0$ so NOT on-shell constructible.

  Why? Well, how are rec'rel's supposed to know if theory has $\lambda |\phi|^4$?

  That is indep. gauge invariant input.

So must supply the individual needed gauge indep. input.

Rec'rel's take care of the rest.

Add more structure to rec'rel's, e.g. SUSY, and they take care of more.

From $\mathcal{N} = 4$ SYM to $\mathcal{N} < 4$ SYM and beyond

- Tree-level direct map from $\mathcal{N} = 4$ SYM to QCD w/ massless quarks.
  [Dixon,Henn,Plefka,Schuster'10]

- Tree-level truncation at superamplitude level from $\mathcal{N} = 4$ SYM to pure $\mathcal{N} = 0, 1, 2, 3$ SYM. ($\mathcal{N} = 3$ is equiv. to $\mathcal{N} = 4$)
  [Bern,Carrasco,Ita,Johannson,Roiban'09] [H.E.,Huang,Peng'11]

- 1-loop β-function coeff. b_0 from $\sum c_{\text{bubbles}} = -b_0\, A_n^{\text{tree}}$
  [Dixon] [Arkani-Hamed,Cachazo,Kaplan'08] [H.E.,Huang,Peng'11]

- The Next-to-Simplest QFT's: no bubbles & triangles [Lal,Raju'09]
  following $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ SG no-triangle, no-bubbles, no-rationals results of
  [Bern/Kosower/Dunbar'91-92] [Bjerrum-Bohr,Dunbar,Ita,Perkins,Risager'06]
  [Bjerrum-Bohr,Vanhove'08] [Arkani-Hamed,Cachazo,Kaplan'08]

- Dual superconformal sym: explicitly verified in 6d and 10d maxSYM. Some tools for higher-D and lower-D amplitudes developed. [Cheung,O'Connell'09]
  [Bern,Carrasco,Dennen,Huang,Ita '10] [Dennen,Huang'10] [Caron-Huot,O'Connell'10]
  [Boels'09] [Dennen,Huang,Siegel'09]

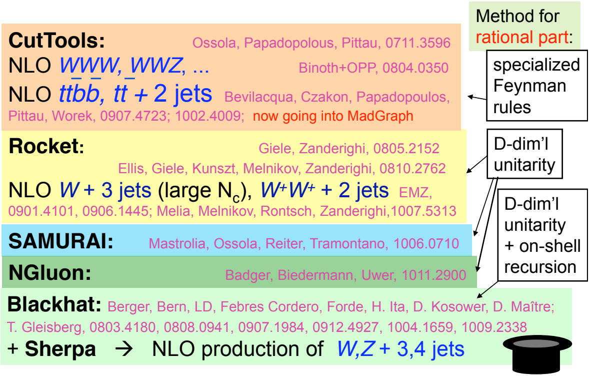
- 3d: ABJM amplitudes [Huang,Lipstein'10$_{\times 2}$] [Agarwal,Beisert,McLoughlin'08]
  [Bargheer,Loebbert,Meneghelli'10] [Lee'10] [Gang,Huang,Koh,Lee,Lipstein'10]

QCD

And of course the on-shell recursion relations + generalized unitarity methods have been applied in countless applications to calculate QCD-backgrounds relevant for collider experiments.

Implemented and automated in numerical codes.

From Dixon's "QFT 2011" talk:



Part IV: Gravity Amplitudes

From "maximally practical" to ...

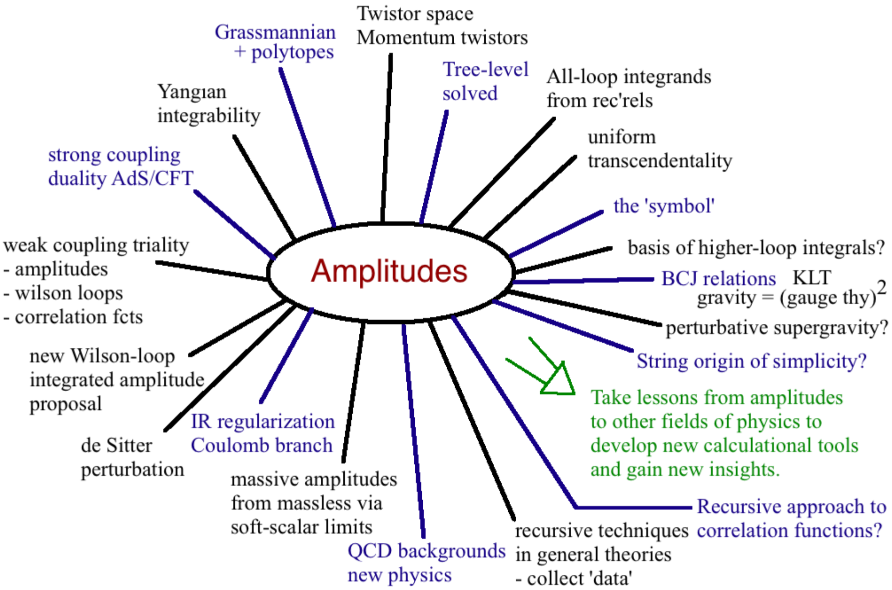
Part IV: Gravity Amplitudes.

Compare SYM with supergravity Amplitudes

Some similarities/differences:

$\mathcal{N} = 4$ SYM	$\mathcal{N} = 8$ SG	
color-ordered	no color-ordering	
Super-BCFW valid $\hat{A}_n(z) \sim \frac{1}{z}$ (adj.) $\frac{1}{z^2}$ (non-adj)	Super-BCFW valid $\hat{A}_n(z) \sim \frac{1}{z^2}$	[Arkani-Hamed,Cachazo,Kaplan'08]
CSW valid	generally no CSW-exp.	[Bianchi,H.E.,Freedman'08]
BCJ relations	$\Longrightarrow$ BCJ squaring relations	[Bern,Carrasco,Johansson'08]
UV finite	finite? where 1st divergence?	
planar limit $\rightarrow$ x_i-variables $\rightarrow$ integrand well-defined	how to define good variables??	
Dual + ord. superconf. sym $\Longrightarrow$ Yangian	Anything??? Bonus structure???	

Overview


Amplitudes
Twistor space
Momentum twistors
Grassmannian
+ polytopes
Tree-level
solved
All-loop integrands
from rec'rels
Yangian
integrability
uniform
transcendentality
strong coupling
duality AdS/CFT
the 'symbol'
basis of higher-loop integrals?
weak coupling triality
- amplitudes
- wilson loops
- correlation fcts
BCJ relations
KLT
gravity = (gauge thy)^2
perturbative supergravity?
String origin of simplicity?
new Wilson-loop
integrated amplitude
proposal
Take lessons from amplitudes
to other fields of physics to
develop new calculational tools
and gain new insights.
IR regularization
Coulomb branch
de Sitter
perturbation
massive amplitudes
from massless via
soft-scalar limits
Recursive approach to
correlation functions?
recursive techniques
in general theories
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QCD backgrounds
new physics