strings 2011

UPPSALA JUNE 27-JULY 2

A-polynomial, B-model, and S-duality

Sergei Gukov

Organizers:
Lars Brink
Ulf Danielsson
Ulf Lindström
Joseph Minahan
Antti Niemi
Maxim Zabzine

Conclusions

In this talk, I will present evidence for the following new results about Chern-Simons theory:

- Hidden symmetry

$$G \to {}^L G \qquad \hbar \to {}^L\hbar = -\frac{4\pi^2}{\hbar}$$

- A-polynomial = space of SUSY vacua in 3d N=2 gauge theory
- simple formula that turns classical curves $A(x,y) = 0$ into quantum operators

Chern-Simons gauge theory

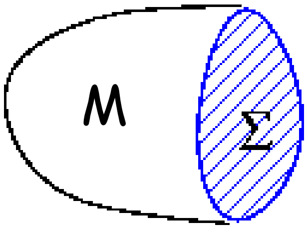
$$S = \int_M \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)$$

- non-abelian interacting gauge theory
- has a long history …
- has many applications …

Chern-Simons gauge theory

$$S = \int_M \mathrm{Tr}\left(A \wedge dA + \tfrac{2}{3} A \wedge A \wedge A\right)$$

M = 3-manifold (possibly with boundary)



$$Z(M) = \int e^{-S/\hbar}\, \mathcal{D}A$$

"quantum invariant" of M [E.Witten]

- depends on the choice of the gauge group
- depends on the "coupling constant" $\hbar$

$$\boxed{q = e^{\hbar}}$$

Modular Form?

Z() = $^{-2} - q^{-1} + 1 - q^{1} + q^{2}$

Jones polynomial

Z() =

[E.Witten]

q

Modular Form?

$Z($  $) = q^{-2} - q^{-1} + 1 - q^{1} + q^{2}$

Jones polynomial

- "experiments" with Don Zagier (circa 2007): analytically continue to complex q

leads to $\mathbf{G} \rightsquigarrow \mathbf{G}_{\mathbb{C}}$ [E.Witten]

Modular Form?

$Z($$)$ $= q^{-2} - q^{-1} + 1 - q^{1} + q^{2}$

Jones polynomial

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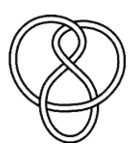
leads to $G \rightsquigarrow G_{\mathbb{C}}$

$Z($$)$ $= 1 - (q^{-1} - 2 - q^{1}) + \ldots$

$$= \sum_{m=0}^{\infty} \prod_{n=1}^{m} (1 - q^{n})(1 - q^{-n})$$

Modular Form?

$Z($$) = q^{-2} - q^{-1} + 1 - q^{1} + q^{2}$

Jones polynomial

- “experiments” … 2007): analytically

$$= 1 - (q^{-1} - 2 - q^{1}) + \ldots$$

$$= \sum_{m=0}^{\infty} \prod_{n=1}^{m} (1 - q^{n})(1 - q^{-n})$$

Realization in String Theory

6d five-brane theory
on $M \times S^3$

[J.Gauntlett, N.Kim]
:
[T.Dimofte, S.G, L.Hollands]
[S.Cecotti, A.Neitzke, C.Vafa]
[E.Witten]
[K.Hosomichi, S.Lee, J.Park]
[Y.Terashima, M.Yamazaki]
:

$SL(2)$ Chern-Simons
on 3-manifold M

3d $\mathcal{N} = 2$ "effective"
gauge theory on S^3

“Geometric Engineering”

6d five-brane theory
on $M \times S^3$

(partial topological
twist along M)

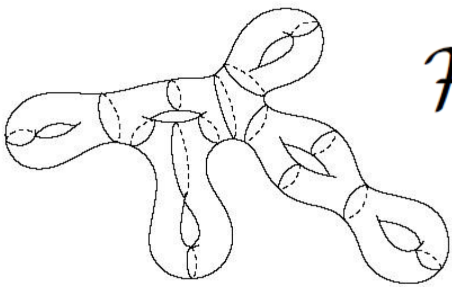
(Ω-deformation
along S^3)

$\mathbf{SL(2)}$ Chern-Simons
on 3-manifold M

3d $\mathcal{N} = 2$ “effective”
gauge theory on S^3

3d analog of AGT correspondence

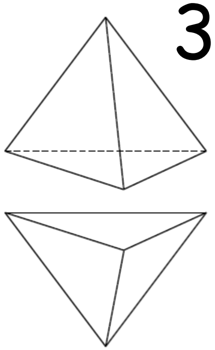
2-manifold C $\rightsquigarrow$ 4d $\mathcal{N} = 2$ SUSY gauge theory on $\mathbb{R} \times S^3$



$$\mathcal{H}_b^{Liouv}(C) \;=\; \mathcal{H}_{\epsilon_{1,2}}^{\mathcal{N}=2}(S^3)$$

[L.F.Alday, D.Gaiotto, Y.Tachikawa]

3-manifold M $\rightsquigarrow$ 3d $\mathcal{N} = 2$ "effective" gauge theory on S^3



$$Z^{CS}(M;\hbar) \;=\; Z^{3d}(\epsilon_1/\epsilon_2)$$

Reduction on M = $\mathbb{R}$ x C

6d five-brane theory
on M x S^3

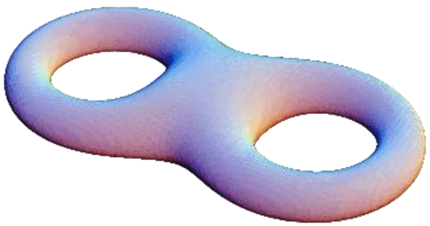
$$\mathcal{H}_b^{Liouv}(C) = \mathcal{H}_{\epsilon_{1,2}}^{\mathcal{N}=2}(S^3)$$

SL(2) Chern-Simons
on 3-manifold M

3d $\mathcal{N} = 2$ "effective"
gauge theory on S^3

Reduction on $M = \mathbb{R} \times C$

[L.F.Alday, D.Gaiotto, Y.Tachikawa]
[N.Nekrasov, E.Witten]



6d five-brane theory

on $\mathbb{R} \times C \times S^3$

$\mathbf{SL(2)}$ Chern-Simons

on 3-manifold $\mathbb{R} \times C$

4d $\mathcal{N} = 2$ SUSY gauge

theory on $\mathbb{R} \times S^3$

$$\mathcal{H}_{\hbar}^{CS}(C) \;=\; \mathcal{H}_{b}^{Liouv}(C) \;=\; \mathcal{H}_{\epsilon_{1,2}}^{\mathcal{N}=2}(S^3)$$

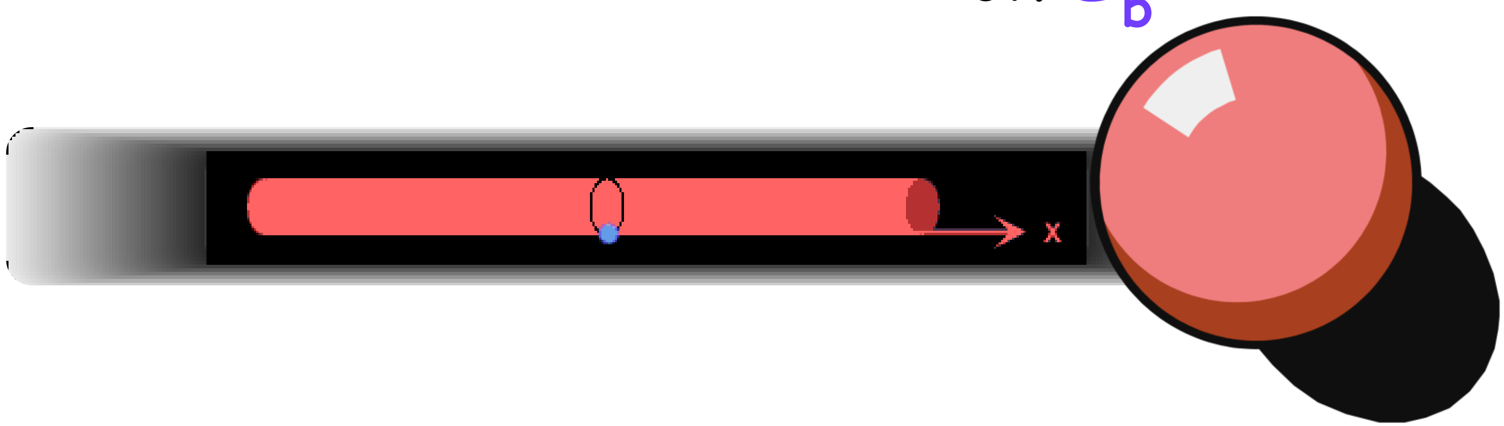
$$\hbar = 2\pi i b^2 = 2\pi i \frac{\epsilon_1}{\epsilon_2}$$

The semi-classical limit $\hbar \sim b^2 \to 0$

6d five-brane theory
on $M \times S^3$

$SL(2)$ Chern-Simons
on 3-manifold M

3d $\mathcal{N} = 2$ theory $T(M)$
on S^3_b



The semi-classical limit ‡ h w b² l 0

[T.Dimofte, S.G., L.Hollands]

5d $\mathcal{N}$=2 super-Yang-Mills

on **M** x $\mathbb{R}^2$

(partial topological twist along **M**)

(Ω-deformation along $\mathbb{R}^2$)

$\mathbf{SL(2)}$ Chern-Simons on 3-manifold **M**

3d $\mathcal{N} = 2$ theory **T(M)** on $\mathbb{R}^2$x S^1

under $SO(5) \ldots \times SO(5) \ldots \quad SO(3)' \ldots \times U(1) \ldots \times U(1)_R$

bosons : $\ldots (0, \pm 2)$

fermions : $(4,4) \to \mathbf{3}^{(\pm1,\pm1)} \oplus \mathbf{1}^{(\pm1,\pm1)}$

[M.Blau, G.Thompson]

Flat connections = SUSY Moduli

[T.Dimofte, S.G., L.Hollands]

5d $\mathcal{N}$=2 super-Yang-Mills

on **M** x $\mathbb{R}^2$

(partial topological twist along **M**)

(Ω-deformation along $\mathbb{R}^2$)

$\mathbf{SL(2)}$ Chern-Simons on 3-manifold **M**

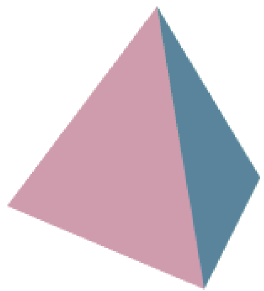
3d $\mathcal{N}=2$ theory **T(M)** on $\mathbb{R}^2$ x S^1

classical solutions:

$$d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = 0$$

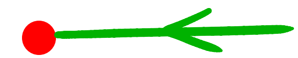
$\longleftrightarrow$

SUSY vacua



T1

charged chiral



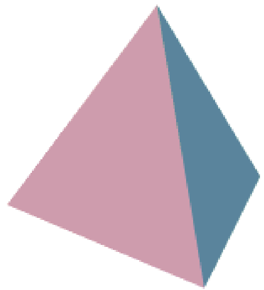
U(1) $k = -1/2$

3d $\mathcal{N} = 2$ theory T(M):

3-manifold M:

3-manifold **M**:

3d $\mathcal{N} = 2$ theory **T(M)**:



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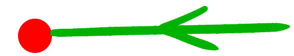

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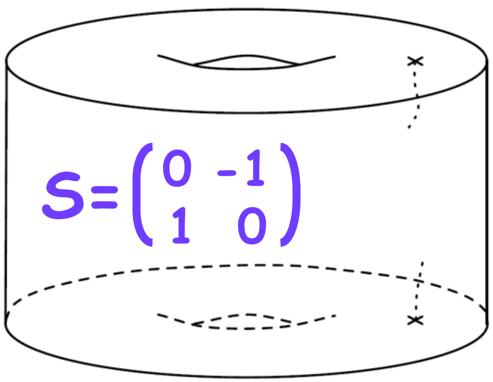
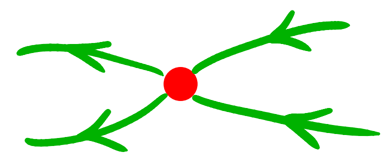
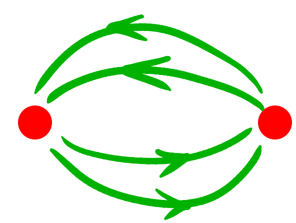


T[SU(2)]

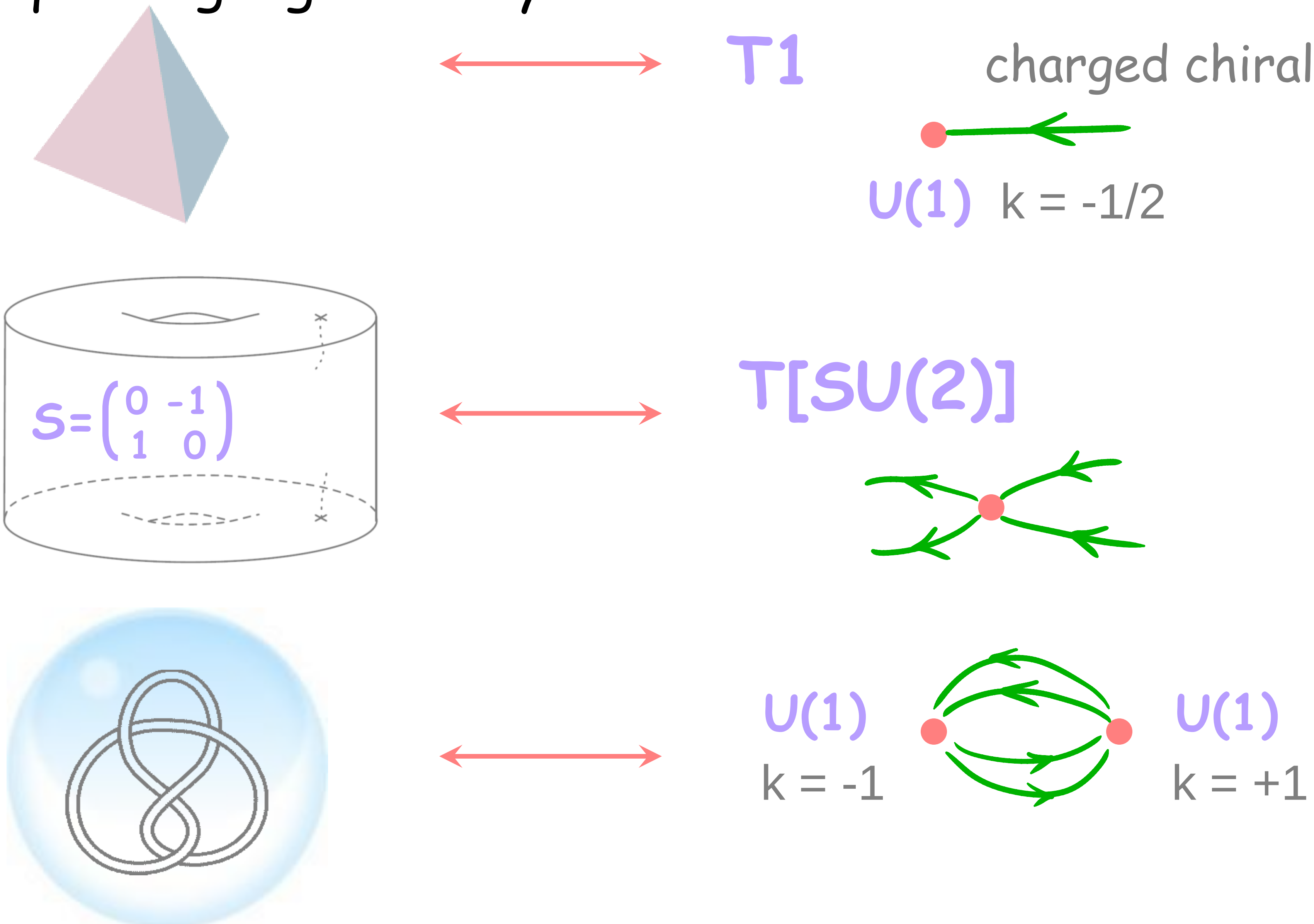


U(1)
k = -1



U(1)
k = +1

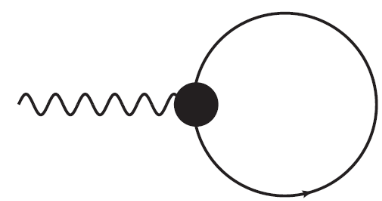
- Notice, in all these examples **T(M)** is an Abelian quiver gauge theory

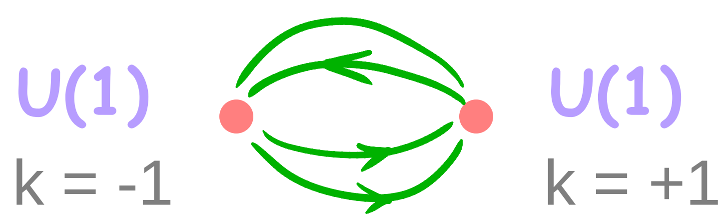


- Notice, in all these examples **T(M)** is an Abelian quiver gauge theory

- Twisted superpotential has the following general form, *cf.* 2d $\mathcal{N} = 2$ SQED:

$$\mathcal{W} = \sum (\sigma_i - m_k) \left(\log \frac{\sigma_i - m_k}{\mu} - 1 \right)$$





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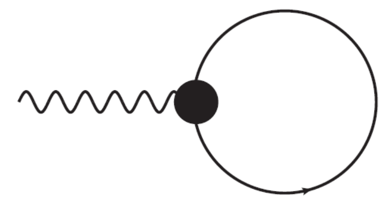
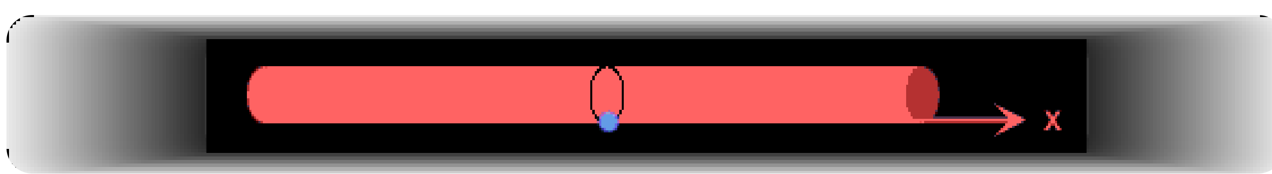


- Twisted superpotential in a theory on $\mathbb{R}^2 \times S^1_R$:

$$\mathcal{W} = \frac{i}{R}\sum \mathrm{Li}_2\left(e^{iR(\sigma_i - m_k)}\right)$$



Algebraic curves from 3d $\mathcal{N} = 2$ theories

[T.Dimofte, S.G.]

$$\mathcal{W}(\sigma_i, v_k)$$

1. Extremize w.r.t all dynamical fields σ_i

$$\frac{\partial \mathcal{W}}{\partial \sigma_i} = 0$$

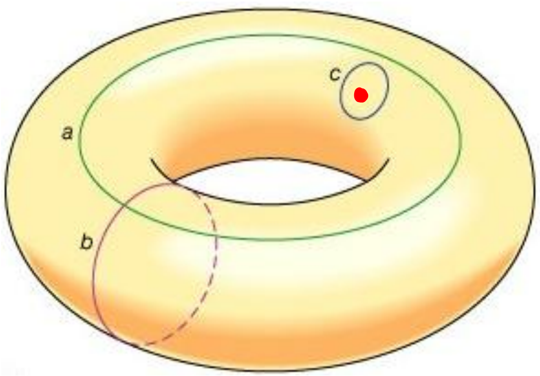
2. Introduce "duals" of all non-dynamical parameters v_k (twisted masses, FI terms, etc.)

$$u_k := \frac{\partial \mathcal{W}}{\partial v_k}$$

3. Expect something nice to happen ...

Algebraic curves from 3d $\mathcal{N}=2$ theories

$C = T^2 \backslash \{p\}$



4d $\mathcal{N}=2^*$ theory

duality group:

$\Gamma(C) = PSL(2,\mathbb{Z})$

[R.Donagi, E.Witten]

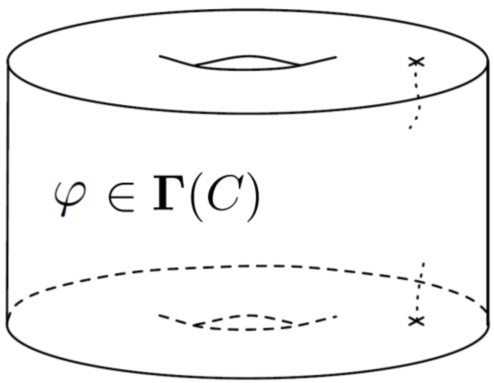
$\mathcal{V}$ = eigenvalue of **SL(2)** holonomy around the puncture

mass of the adjoint matter multiplet

m_{adj}

Algebraic curves from 3d $\mathcal{N}=2$ theories

mapping cylinder



Duality wall in
4d $\mathcal{N}=2^*$ theory

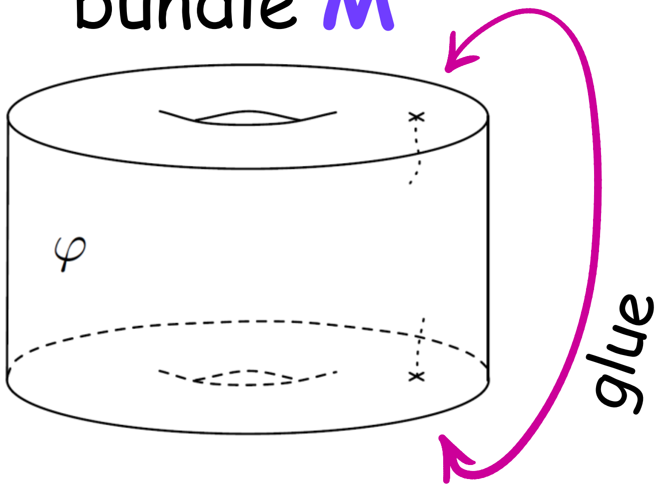
$$\mathbf{\Gamma}(C) = PSL(2,\mathbb{Z})$$

[N.Drukker, D.Gaiotto, J.Gomis]
[K.Hosomichi, S.Lee, J.Park]

$\mathcal{V}$ = eigenvalue of **SL(2)** holonomy around the puncture

mass of the adjoint matter multiplet m_{adj}

punctured torus bundle **M** $\longleftrightarrow$ 3d $\mathcal{N} = 2$ theory **T(M)**



$$u \; := \; \frac{\partial \mathcal{W}_{\text{eff}}(v)}{\partial v}$$

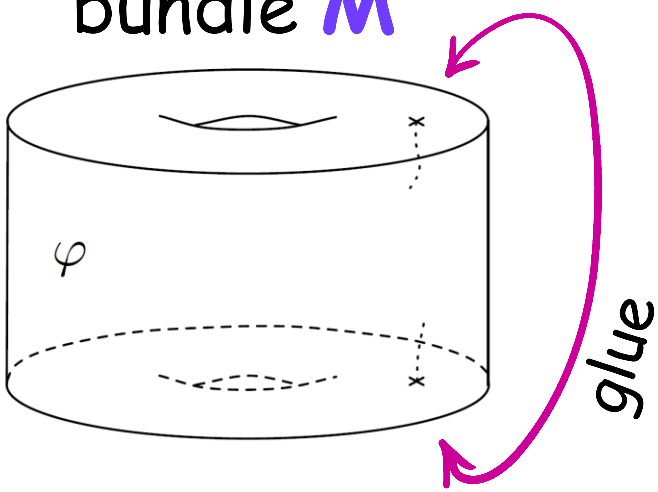
$x = e^u \qquad y = e^v$

$$\boxed{A(x,y) = 0}$$

In Chern-Simons theory with a Wilson loop, the polynomial $A(x,y)$ is a topological invariant called the A-polynomial and plays a role similar to that of the Seiberg-Witten curve in $\mathcal{N} = 2$ gauge theory.

[S.G.] *Strings'03*

punctured torus bundle **M** ⟷ 3d $\mathcal{N} = 2$ theory **T(M)**



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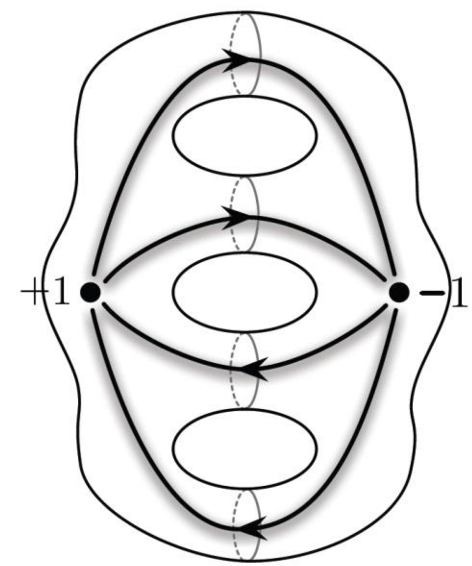
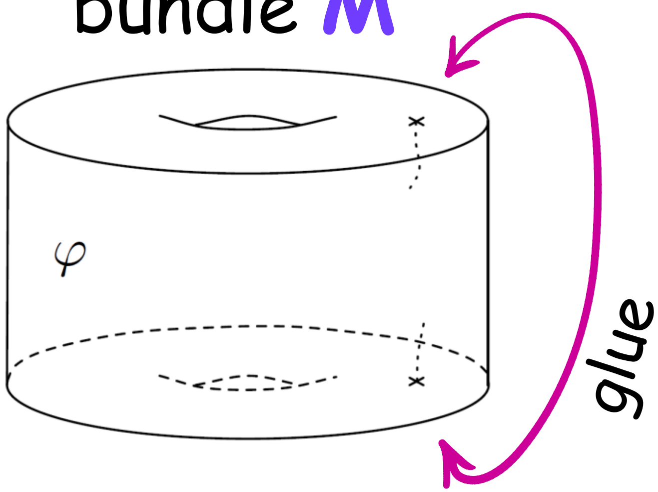
Example:

$\varphi = TST^{-1}S^{-1}$

⟷



punctured torus bundle **M** $\longleftrightarrow$ 3d $\mathcal{N}=2$ theory **T(M)**



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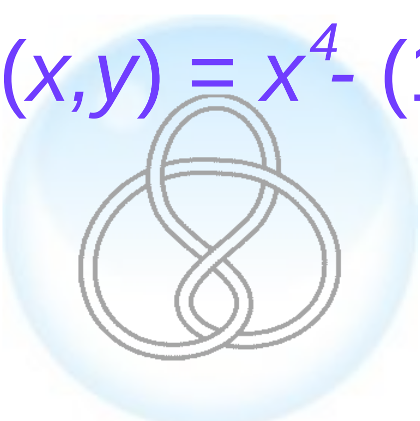
$$x = e^u \qquad y = e^v$$

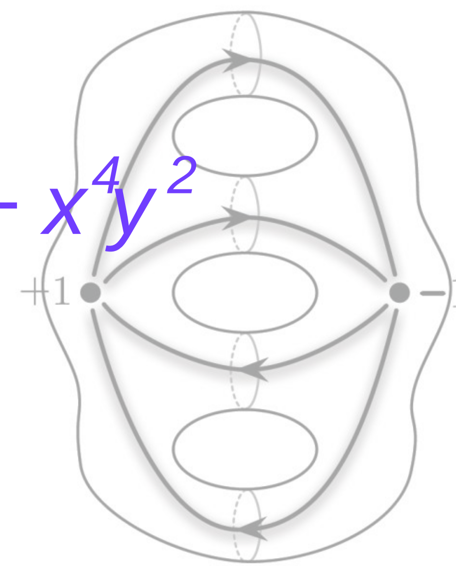
$$\boxed{A(x,y) = 0}$$

<u>Example:</u>

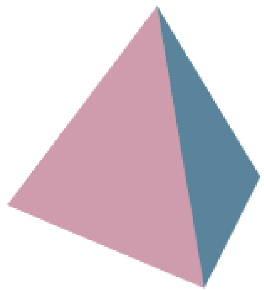
$$\varphi = TST^{-1}S^{-1}$$

$$A(x,y) = x^4 - (1 - x^2 - 2x^4 - x^6 + x^8)y + x^4 y^2$$



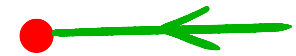


One more example

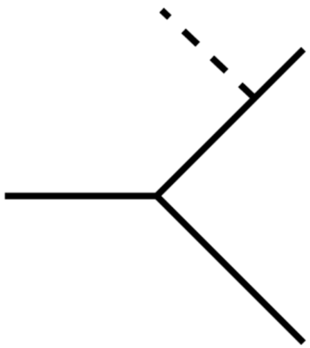


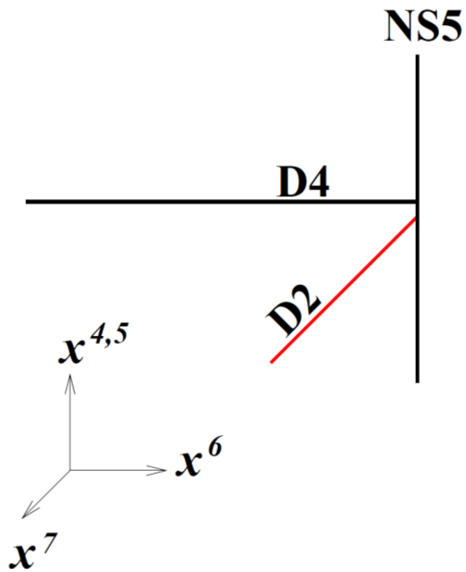
T1

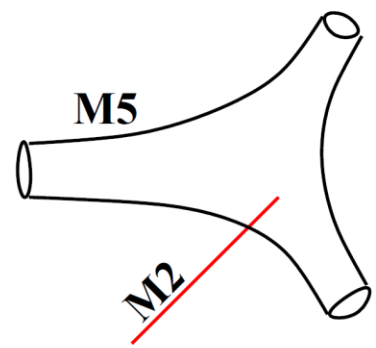
charged chiral



U(1) $k = -1/2$



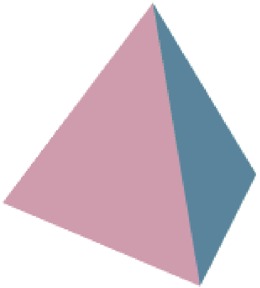




$$Z^{3d}(\epsilon_1/\epsilon_2) = s_b = \Phi_\hbar$$

"quantum dilogarithm"

Computing SL(2) partition functions



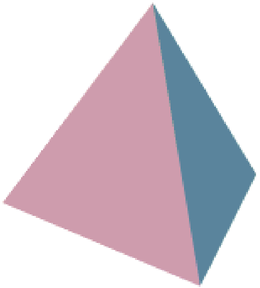
[T.Dimofte, S.G., J.Lenells, D.Zagier]
[T.Dimofte]

$$Z^{CS}(M;\hbar) \;=\; \int_{C_\rho} \prod_{j=1}^{N} \Phi_\hbar(\Delta_j)^{\pm 1} \prod_{i=1}^{N-b_0(\Sigma)} \frac{dp_i}{\sqrt{4\pi\hbar}}$$

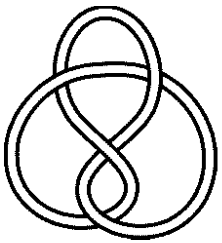
$$\overset{\hbar\to 0}{\sim} \; \exp\left(\frac{1}{\hbar}\mathcal{W} + \mathcal{O}(\log\hbar)\right)$$

Computing SL(2) partition functions

[T.Dimofte, S.G., J.Lenells, D.Zagier]
[T.Dimofte]

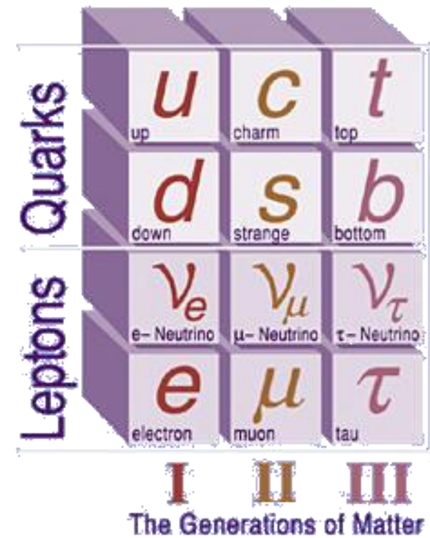


$$Z^{CS}(M;\hbar) \ = \int_{C_\alpha} dp \frac{\Phi_\hbar(p-u)}{\Phi_\hbar(-p-u)} e^{-\frac{2pu}{\hbar}}$$

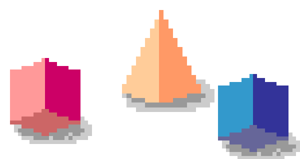


$$\stackrel{\hbar\to 0}{\sim} \ \exp\left(\frac{1}{\hbar}\mathcal{W} + \mathcal{O}(\log\hbar)\right)$$

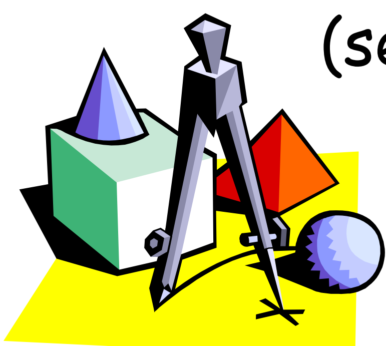
Standard Model of 3-manifolds ?



- We can build 3-manifolds from basic building blocks (tetrahedra, etc.)



- What are the corresponding building blocks of 3d $\mathcal{N} = 2$ SUSY theories? Is there a simple dictionary?
- YES! Work in progress with T.Dimofte and D.Gaiotto



(see Davide's talk)



The End