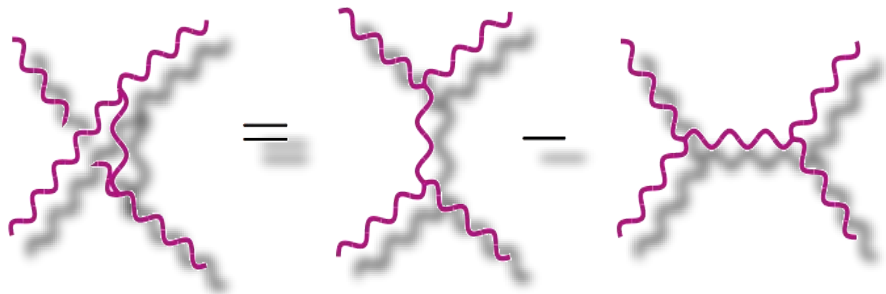
# *Lie Algebra Structures in Yang-Mills and Gravity Amplitudes*



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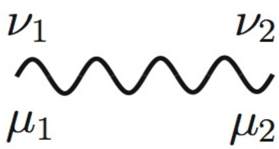
Based on work in collaboration with:
Z.Bern, J.J.Carrasco, L.Dixon, R.Roiban

# Outline

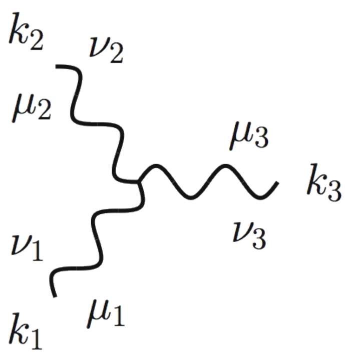
- Simple double-copy structure of gravity
- Duality between color and kinematics
  - Evidence at tree level
  - Explicit loop amplitudes with manifest duality
- Amplitude UV behavior from duality
- Kinematic Lie algebra and Lagrangian formulation
- Conclusion

# Einstein Gravity Feynman rules

de Donder gauge: $\mathcal{L} = \frac{2}{\kappa^2}\sqrt{g}R, \qquad g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$



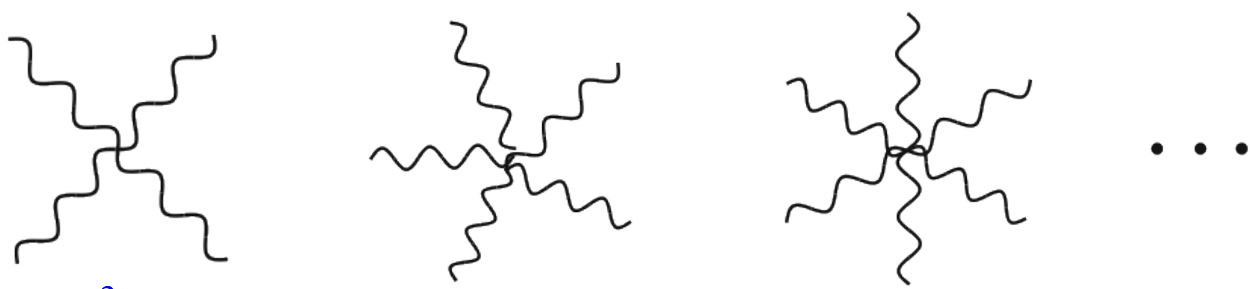

$$= \frac{1}{2}\left[\eta_{\mu_1\nu_1}\eta_{\mu_2\nu_2} + \eta_{\mu_1\nu_2}\eta_{\nu_1\mu_2} - \frac{2}{D-2}\eta_{\mu_1\mu_2}\eta_{\nu_1\nu_2}\right]\frac{i}{p^2 + i\epsilon}$$




$$\begin{aligned} = \mathrm{sym}[&-\frac{1}{2}P_3(k_1\cdot k_2\eta_{\mu_1\nu_1}\eta_{\mu_2\nu_2}\eta_{\mu_3\nu_3}) - \frac{1}{2}P_6(k_{1\mu_1}k_{1\nu_2}\eta_{\mu_1\nu_1}\eta_{\mu_3\nu_3}) + \frac{1}{2}P_3(k_1\cdot k_2\eta_{\mu_1\mu_2}\eta_{\nu_1\nu_2}\eta_{\mu_3\nu_3}) \\ &+P_6(k_1\cdot k_2\eta_{\mu_1\nu_1}\eta_{\mu_2\mu_3}\eta_{\nu_2\nu_3}) + 2P_3(k_{1\mu_2}k_{1\nu_3}\eta_{\mu_1\nu_1}\eta_{\nu_2\mu_3}) - P_3(k_{1\nu_2}k_{2\mu_1}\eta_{\nu_1\mu_1}\eta_{\mu_3\nu_3}) \\ &+P_3(k_{1\mu_3}k_{2\nu_3}\eta_{\mu_1\mu_2}\eta_{\nu_1\nu_2}) + P_6(k_{1\mu_3}k_{1\nu_3}\eta_{\mu_1\mu_2}\eta_{\nu_1\nu_2}) + 2P_6(k_{1\mu_2}k_{2\nu_3}\eta_{\nu_2\mu_1}\eta_{\nu_1\mu_3}) \\ &+2P_3(k_{1\mu_2}k_{2\mu_1}\eta_{\nu_2\mu_3}\eta_{\nu_3\nu_1}) - 2P_3(k_1\cdot k_2\eta_{\nu_1\mu_2}\eta_{\nu_2\mu_3}\eta_{\nu_3\mu_1})]\end{aligned}$$

After symmetrization
~ 100 terms !

higher order vertices...



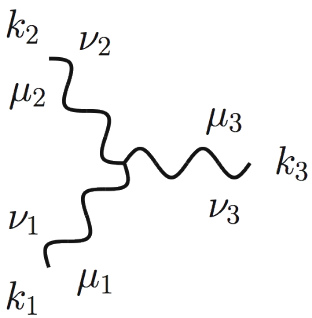
~$10^3$ terms

# On-shell simplifications

Graviton plane wave: $\varepsilon^{\mu}(p)\varepsilon^{\nu}(p)\, e^{ip\cdot x}$

Yang-Mills polarization

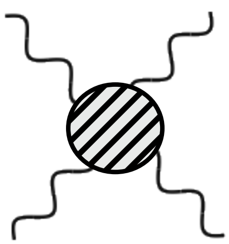
On-shell 3-graviton vertex:




$$= \quad i\kappa\Big(\eta_{\mu_1\mu_2}(k_1-k_2)_{\mu_3}+\text{cyclic}\Big)\Big(\eta_{\nu_1\nu_2}(k_1-k_2)_{\nu_3}+\text{cyclic}\Big)$$

Yang-Mills vertex

Gravity scattering amplitude:



$$M_4^{\text{tree}}(1,2,3,4) = -i\frac{st}{u}A_4^{\text{tree}}(1,2,3,4)\tilde{A}_4^{\text{tree}}(1,2,3,4)$$

Yang-Mills amplitude

On-shell gravity objects are "squares" of Yang-Mills objects !

- holds for the entire S-matrix  Bern, Carrasco, HJ [BCJ]

# Kawai-Lewellen-Tye Relations

String theory tree-level identity: closed string ~ (left open string) × (right open string)


$$A_n \sim \int \frac{dx_1 \cdots dx_n}{\mathcal{V}_{abc}} \prod_{1\le i<j\le n} |x_i - x_j|^{k_i\cdot k_j} \exp\Big[\sum_{i<j}\Big(\frac{\epsilon_i\cdot\epsilon_j}{(x_i-x_j)^2} + \frac{k_i\cdot\epsilon_j - k_j\cdot\epsilon_i}{(x_i-x_j)}\Big)\Big]\Big|_{\text{multi-linear}}$$

KLT relations emerge after nontrivial world-sheet integral identities

Field theory limit $\Rightarrow$ gravity theory ~ (gauge theory) × (gauge theory)

$$M_4^{\text{tree}}(1,2,3,4) = -is_{12} A_4^{\text{tree}}(1,2,3,4)\,\widetilde{A}_4^{\text{tree}}(1,2,4,3)$$

$$M_5^{\text{tree}}(1,2,3,4,5) = is_{12}s_{34} A_5^{\text{tree}}(1,2,3,4,5)\,\widetilde{A}_5^{\text{tree}}(2,1,4,3,5) + is_{13}s_{24} A_5^{\text{tree}}(1,3,2,4,5)\,\widetilde{A}_5^{\text{tree}}(3,1,4,2,5)$$

gravity states are products of gauge theory states:

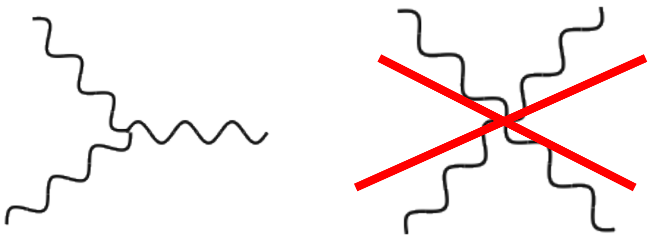
$$|1\rangle_{\text{grav}} = |1\rangle_{\text{gauge}} \otimes |1\rangle_{\text{gauge}}$$

# Gravity should be cubic

Yang-Mills → cubic

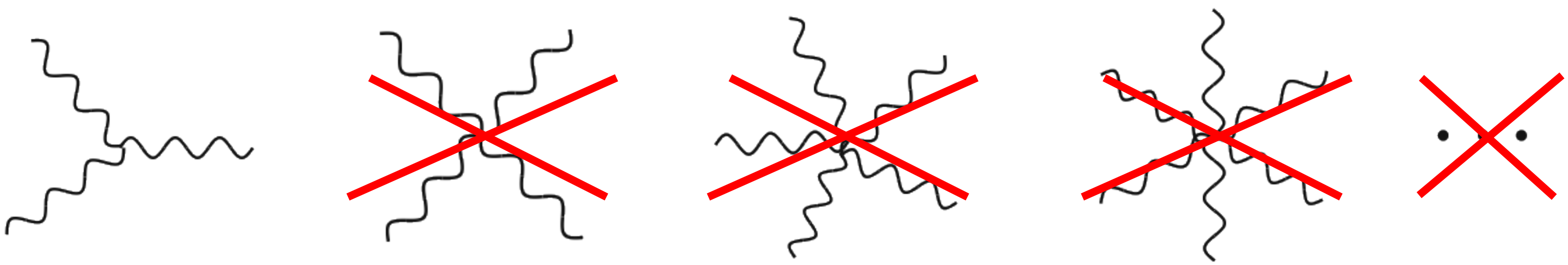
schematically: $\mathcal{L}_{\mathrm{YM}} \sim A\Box A + \partial A^3$

schematic derivative



Einstein gravity → cubic

schematically: $\mathcal{L}_{\mathrm{G}} \sim h\Box h + \partial^2 h^3$



And gravity should be a double copy of a YM theory:

$$h^{\mu\nu} \sim A^\mu A^\nu$$

$$V_{\mathrm{G}}(k_1,k_2,k_3) = V_{\mathrm{YM}}(k_1,k_2,k_3)V_{\mathrm{YM}}(k_1,k_2,k_3)$$

[BCJ]

# Gauge theory is the key

The simplicity of gravity stems from a novel structure in Yang-Mills

- represent amplitudes using cubic graphs only:

numerators

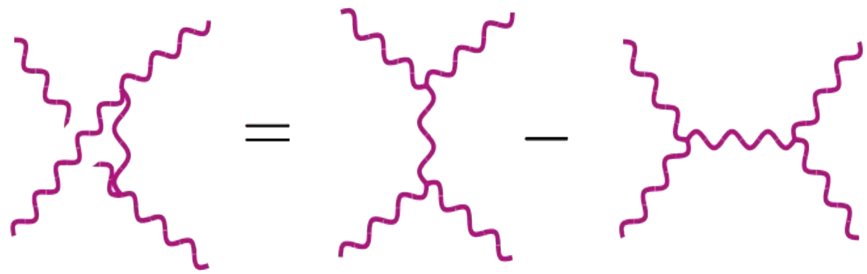
[BCJ]

color factors

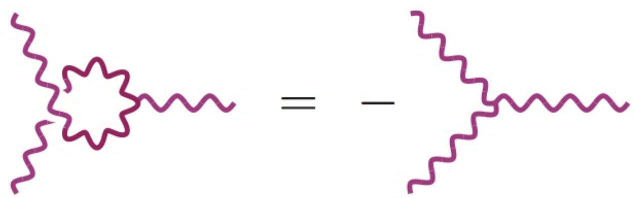
$$\mathcal{A}_m^{(L)} = \sum_{i\in\Gamma_3} \int \frac{d^{LD}\ell}{(2\pi)^{LD}} \frac{1}{S_i} \frac{n_i c_i}{p_{i_1}^2 p_{i_2}^2 p_{i_3}^2 \cdots p_{i_l}^2}$$

propagators

Diagram numerators satisfy the algebra:



Jacobi identity



antisymmetry

These are the same relations the color factors satisfy (Lie Algebra)

Duality: color $\leftrightarrow$ kinematics

# Gravity is a double copy

- Gravity amplitudes are obtained after replacing color by kinematics

$$\mathcal{A}_m^{(L)} = \sum_{i \in \Gamma_3} \int \frac{d^{LD}\ell}{(2\pi)^{LD}} \frac{1}{S_i} \frac{n_i c_i}{p_{i_1}^2 p_{i_2}^2 p_{i_3}^2 \cdots p_{i_l}^2}$$

[BCJ]

$$\mathcal{M}_m^{(L)} = \sum_{i \in \Gamma_3} \int \frac{d^{LD}\ell}{(2\pi)^{LD}} \frac{1}{S_i} \frac{n_i \tilde{n}_i}{p_{i_1}^2 p_{i_2}^2 p_{i_3}^2 \cdots p_{i_l}^2}$$

- The two numerators can belong to different theories:

$n_i$ $\tilde{n}_i$

$(\mathcal{N}=4) \times (\mathcal{N}=4) \rightarrow \mathcal{N}=8$ sugra

$(\mathcal{N}=4) \times (\mathcal{N}=2) \rightarrow \mathcal{N}=6$ sugra

$(\mathcal{N}=4) \times (\mathcal{N}=0) \rightarrow \mathcal{N}=4$ sugra

$(\mathcal{N}=0) \times (\mathcal{N}=0) \rightarrow$ Einstein gravity + axion+ dillaton

# Four-point example

- Usual tree-level decomposition

$$\mathcal{A}_n^{\text{tree}}(1,2,\ldots,n) = g^{n-2} \sum_{\mathcal{P}(2,\ldots,n)} \text{Tr}[T^{a_1}T^{a_2}\cdots T^{a_n}]\, A_n^{\text{tree}}(1,2,\ldots,n)$$

gauge invariant

- Alternative decomposition, 4pt example

$$\mathcal{A}_4^{\text{tree}}(1,2,3,4) = g^2\left(\frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u}\right)$$

color factors

$$c_u \equiv \tilde{f}^{a_4 a_2 b}\tilde{f}^{b a_3 a_1}$$

$$c_s \equiv \tilde{f}^{a_1 a_2 b}\tilde{f}^{b a_3 a_4}$$

$$c_t \equiv \tilde{f}^{a_2 a_3 b}\tilde{f}^{b a_4 a_1}$$

kinematic numerators

$n_s, n_t, n_u$

absorbs 4-pt contact terms

- Map

$$\tilde{f}^{abc} \equiv i\sqrt{2} f^{abc} = \text{Tr}([T^a, T^b]T^c)$$

color structures

$$A_4^{\text{tree}}(1,2,3,4) \equiv \frac{n_s}{s} + \frac{n_t}{t},$$

$$A_4^{\text{tree}}(1,3,4,2) \equiv -\frac{n_u}{u} - \frac{n_s}{s}$$

$$A_4^{\text{tree}}(1,4,2,3) \equiv -\frac{n_t}{t} + \frac{n_u}{u}$$

kinematic structures

# 4-pt kinematic Jacobi relation

$$\mathcal{A}_4^{\text{tree}}(1,2,3,4) = g^2\Big(\frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u}\Big)$$

**color factors**

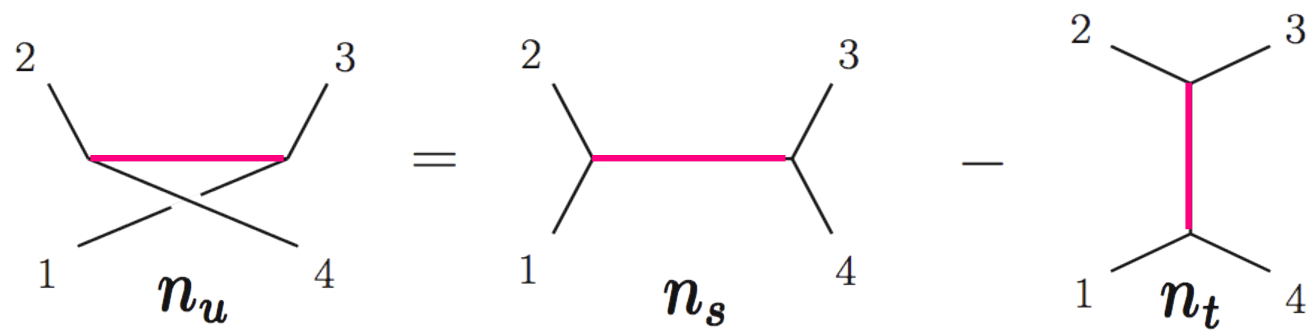
$$c_u \equiv \tilde{f}^{a_4 a_2 b}\tilde{f}^{b a_3 a_1}$$

$$c_s \equiv \tilde{f}^{a_1 a_2 b}\tilde{f}^{b a_3 a_4}$$

$$c_t \equiv \tilde{f}^{a_2 a_3 b}\tilde{f}^{b a_4 a_1}$$

- Jacobi identity for color and for kinematics

$$c_u = c_s - c_t \qquad \Leftrightarrow \qquad n_u = n_s - n_t$$




- Easy to check using Feynman rules

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- Kinematic numerators gauge dependent - but 4pt identity is gauge invariant

$$-n'_s + n'_t + n'_u = -n_s + n_t + n_u + \Delta(k_j, \varepsilon_j)(s+t+u) = 0$$

~ gauge parameter

# Generalized gauge transformation…

…explains why this kinematic structure has remained hidden.

$$\mathcal{A}_n^{\text{tree}} = \sum_i \frac{c_i n_i}{\prod_\alpha p_\alpha^2}$$

$(2n\text{-}5)!!$ cubic diagrams [BCJ]

Define "generalized gauge transformation" on amplitude as

$$n_i \to n_i + \Delta_i \qquad \text{such that} \qquad \sum_i \frac{c_i \Delta_i}{\prod_\alpha p_\alpha^2} = 0$$

Amplitudes invariant under this transformation, but not duality

$$n_i + n_j + n_k \neq 0 \qquad \not\Leftrightarrow \qquad c_i + c_j + c_k = 0$$

*To see the duality one must find the transformation that makes the numerators obey the algebra – in general a nontrivial task*

# Tree-Level Evidence

# Duality gives new amplitude relations

In color ordered tree amplitudes 3 legs can be fixed: $(n\text{-}3)!$ basis

4 points:

$$A_4^{\rm tree}(1,2,\{4\},3) = \frac{A_4^{\rm tree}(1,2,3,4)s_{14}}{s_{24}}$$

$s_{ij..} = (k_i+k_j+\ldots)^2$

5 points:

$$A_5^{\rm tree}(1,2,\{4\},3,\{5\}) = \frac{A_5^{\rm tree}(1,2,3,4,5)(s_{14}+s_{45}) + A_5^{\rm tree}(1,2,3,5,4)s_{14}}{s_{24}},$$

$$A_5^{\rm tree}(1,2,\{4,5\},3) = \frac{-A_5^{\rm tree}(1,2,3,4,5)s_{34}s_{15} - A_5^{\rm tree}(1,2,3,5,4)s_{14}(s_{245}+s_{35})}{s_{24}s_{245}}$$

…relations obtained for any multiplicity

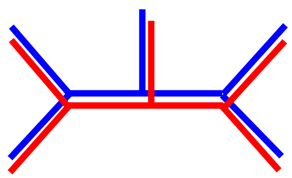
These were later found to be equivalent to monodromy relations on the open string worldsheet Bjerrum-Bohr, Damgaard, Vanhove; Stieberger

Also field theory proofs through BCFW: Feng, Huang, Jia; Chen, Du, Feng

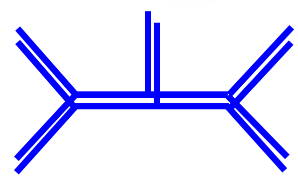
# Tree-level gravity checks

- Original conjecture checked through 8 points    Bern, Carrasco, HJ

$$\mathcal{A}_n^{\text{tree}} = \sum_i \frac{n_i c_i}{\prod_\alpha p_\alpha^2} \quad \Leftrightarrow \quad \mathcal{M}_n^{\text{tree}} = \sum_i \frac{n_i \tilde{n}_i}{\prod_\alpha p_\alpha^2}$$



$\Leftrightarrow$



double copy of YM

- All-multiplicity proof assuming gauge theory duality: Bern, Dennen, Huang, Kiermaier

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Work by Tye and Zhang connects to heterotic string

$$\mathcal{A}^{\text{het}}\Big|_{\alpha' \to 0} = \sum_i \frac{n_{\mathrm{L},i}\, \tilde{n}_{\mathrm{R},i}}{\prod_\beta p_\beta^2}$$

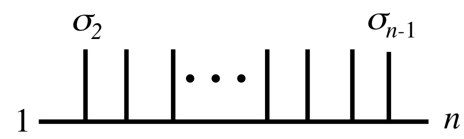
Left sector $\quad n_{\mathrm{L},i} \quad \Leftrightarrow \quad$ modes in spacetime $\quad R^{(1,D-1)}$

Right sector $\quad \tilde{n}_{\mathrm{R},i} \quad \Leftrightarrow \quad$ modes in spacetime $\quad R^{(1,D-1)} \times T^{N_c}$

# Some tree-level solutions

- All-multiplicity solution for non-local tree numerators using KLT
  Kiermaier; Bjerrum-Bohr, Damgaard, Sondergaard Vanhove

$$\mathcal{M}_n = i \sum_{\sigma \in S_{n-2}} n_{1,\sigma_2,\ldots,\sigma_{n-1},n} \times \tilde{A}_n(1,\sigma_2,\ldots,\sigma_{n-1},n)$$




- Explicit local numerators using pure-spinor methods in $D$=10
  Mafra, Schlotterer, Stieberger

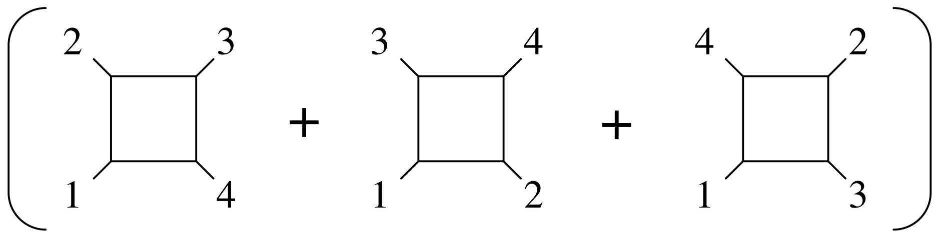
- Cubic Feynman rules obeying the duality for MHV sector of YM
  Monteiro and O'Connell

# Loop-Level Evidence

# Manifest duality in $\mathcal{N} = 4$ SYM 4-pt ampl.

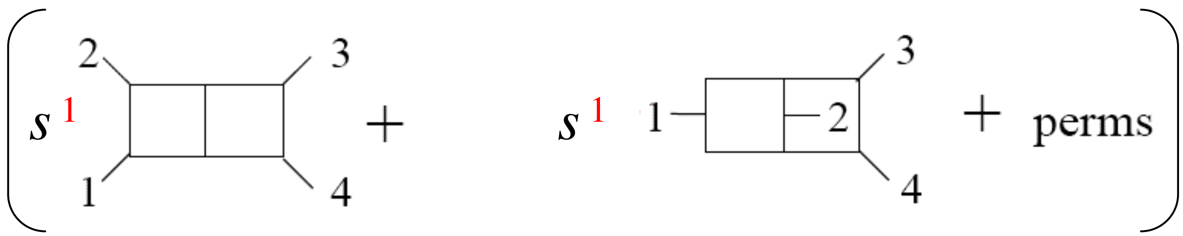
## Known cases of duality-satisfying loop amplitudes:

1-loop: $K^{1}$




Green, Schwarz, Brink (1982)

2-loop: $K^{1}$




Bern, Dixon, Dunbar, Perelstein and Rozowsky (1998)

prefactor contains helicity structure:

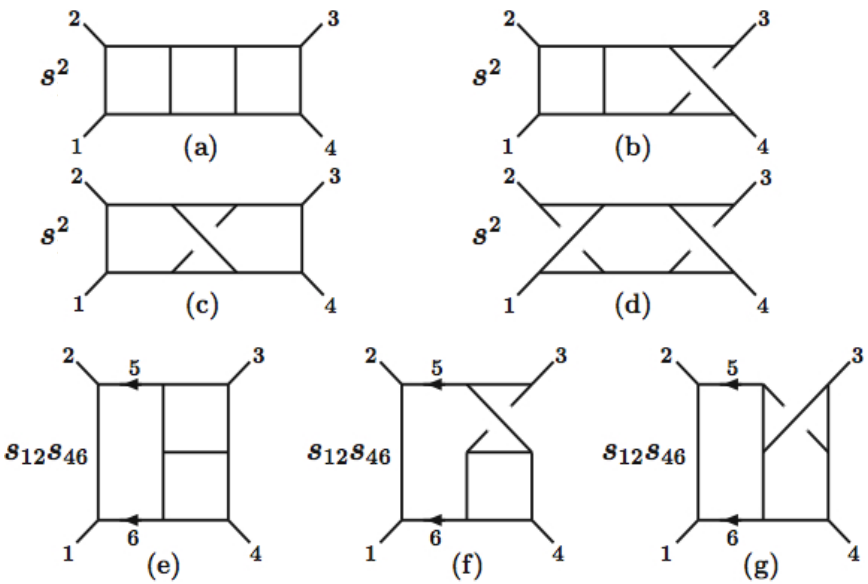
$$K = st A_4^{\text{tree}}$$

**Duality: $\mathcal{N} = 8$ SG is obtained if $1 \rightarrow 2$ (numerator squaring)**

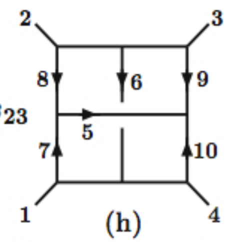
# Old form of 3-loop amplitude

Problem: no double copy in 0808.4112 [hep-th] (Bern, Carrasco, Dixon, HJ, Roiban)

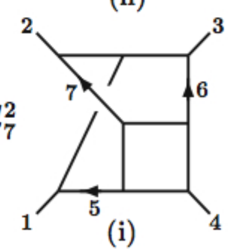
## N=4 SYM



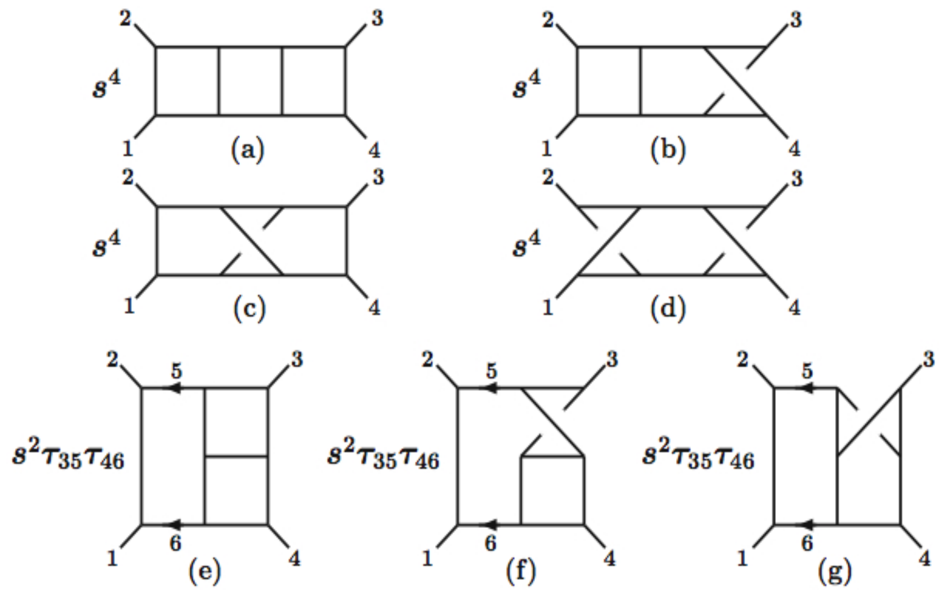

$$s_{12}(\tau_{26}+\tau_{36}) + s_{23}(\tau_{15}+\tau_{25}) + s_{12}s_{23}$$



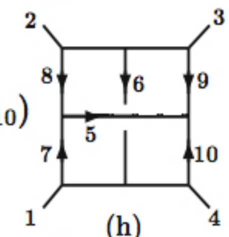

$$s_{12}s_{45} - s_{23}s_{46} - \tfrac{1}{3}(s_{12}-s_{23})l_7^2$$



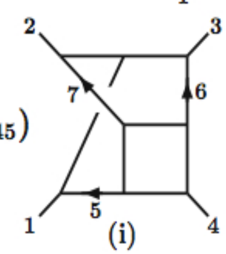

## N=8 SG




$$\begin{aligned}&(s(\tau_{26}+\tau_{36}) + t(\tau_{15}+\tau_{25}) + st)^2\\ &+(s^2(\tau_{26}+\tau_{36}) - t^2(\tau_{15}+\tau_{25}))(\tau_{17}+\tau_{28}+\tau_{39}+\tau_{4,10})\\ &+s^2(\tau_{17}\tau_{28}+\tau_{39}\tau_{4,10}) + t^2(\tau_{28}\tau_{39}+\tau_{17}\tau_{4,10})\\ &+u^2(\tau_{17}\tau_{39}+\tau_{28}\tau_{4,10})\end{aligned}$$




$$\begin{aligned}&(s\tau_{45} - t\tau_{46})^2 - \tau_{27}(s^2\tau_{45}+t^2\tau_{46})\\ &-\tau_{15}(s^2\tau_{47}+u^2\tau_{46}) - \tau_{36}(t^2\tau_{47}+u^2\tau_{45})\\ &+l_5^2 s^2 t + l_6^2 s t^2 - \tfrac{1}{3}l_7^2 stu\end{aligned}$$

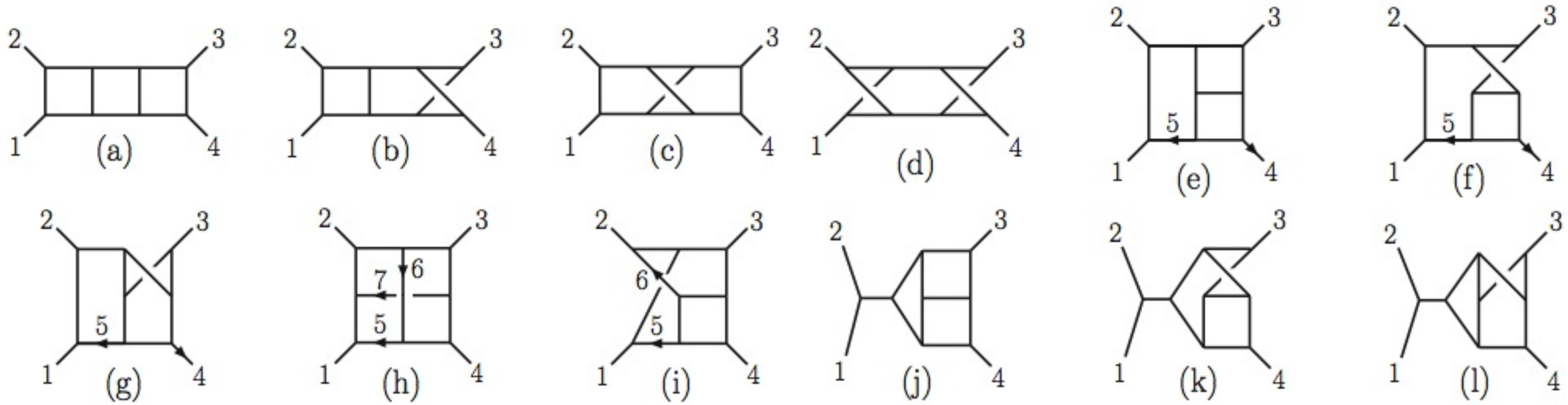



$$\tau_{ij} = 2k_i \cdot l_j$$

# After nontrivial reshuffling

3-loop $\mathcal{N}$=4 SYM admits manifest realization of duality – and $\mathcal{N}$=8 SG is simply the square

1004.0476 [hep-th]
Bern, Carrasco, HJ



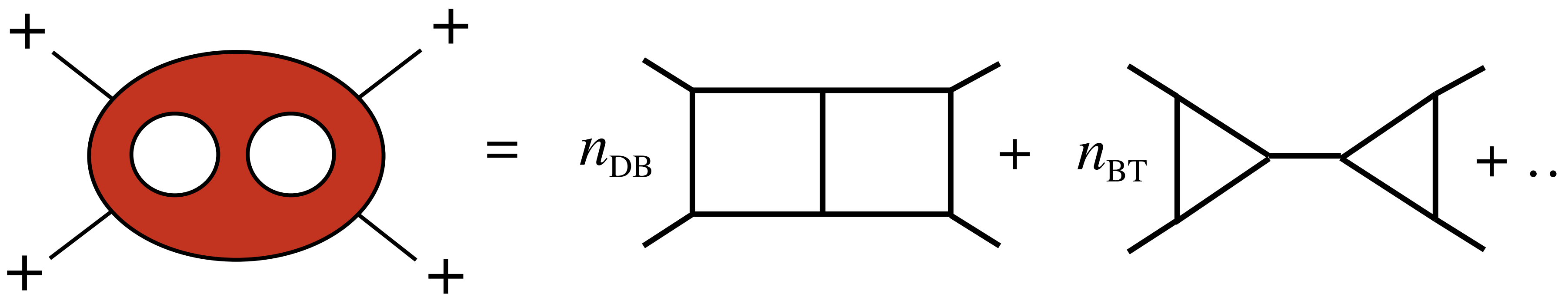

| Integral $I^{(x)}$ | $\mathcal{N}=4$ Super-Yang-Mills ($\sqrt{\mathcal{N}=8}$ supergravity) numerator |
|---|---|
| (a)–(d) | $s^2$ |
| (e)–(g) | $\big(s\,(-\tau_{35}+\tau_{45}+t)-t\,(\tau_{25}+\tau_{45})+u\,(\tau_{25}+\tau_{35})-s^2\big)/3$ |
| (h) | $\big(s\,(2\tau_{15}-\tau_{16}+2\tau_{26}-\tau_{27}+2\tau_{35}+\tau_{36}+\tau_{37}-u)$ $+t\,(\tau_{16}+\tau_{26}-\tau_{37}+2\tau_{36}-2\tau_{15}-2\tau_{27}-2\tau_{35}-3\tau_{17})+s^2\big)/3$ |
| (i) | $\big(s\,(-\tau_{25}-\tau_{26}-\tau_{35}+\tau_{36}+\tau_{45}+2t)$ $+t\,(\tau_{26}+\tau_{35}+2\tau_{36}+2\tau_{45}+3\tau_{46})+u\,\tau_{25}+s^2\big)/3$ |
| (j)-(l) | $s(t-u)/3$ |

$$\tau_{ij}=2k_i\cdot l_j$$

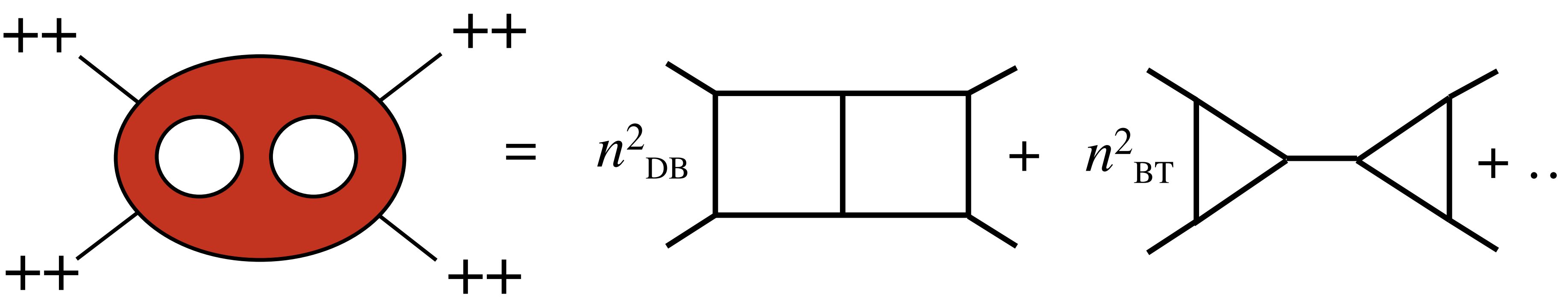
# Works for non-susy theories

1004.0476 [hep-th]
Bern, Carrasco, HJ

All-plus-helicity QCD amplitude:



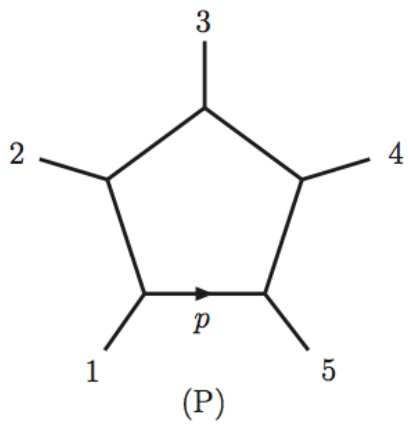

All-plus-helicity Einstein gravity amplitude:

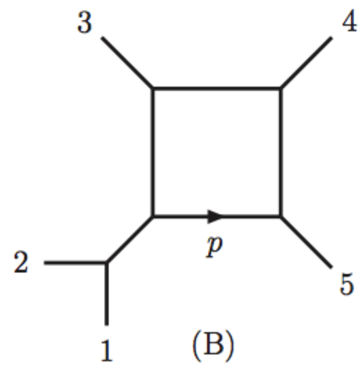



(with dilation and axion in loops)

# 1-loop 5-pts $\mathcal{N}=4$ SYM and $\mathcal{N}=8$ SG

Carrasco, HJ 1106.4711 [hep-th]



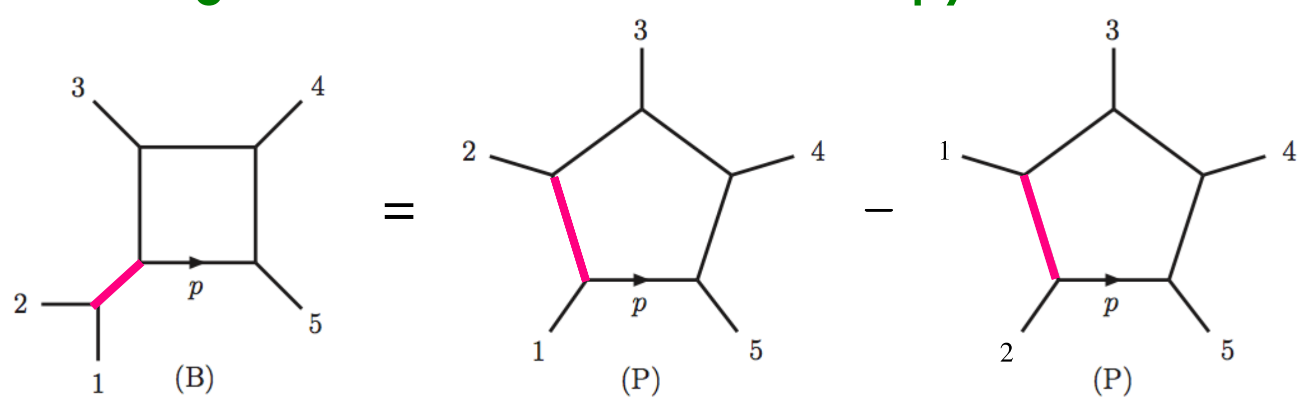




$$N^{(\mathrm{P})} = \beta_{12345} \equiv \delta^{(8)}(Q)\frac{[1\,2]\,[2\,3]\,[3\,4]\,[4\,5]\,[5\,1]}{4\,\varepsilon(1,2,3,4)}$$

$$N^{(\mathrm{B})} = \gamma_{12345} \equiv \delta^{(8)}(Q)\frac{[1\,2]^2\,[3\,4]\,[4\,5]\,[3\,5]}{4\,\varepsilon(1,2,3,4)}$$

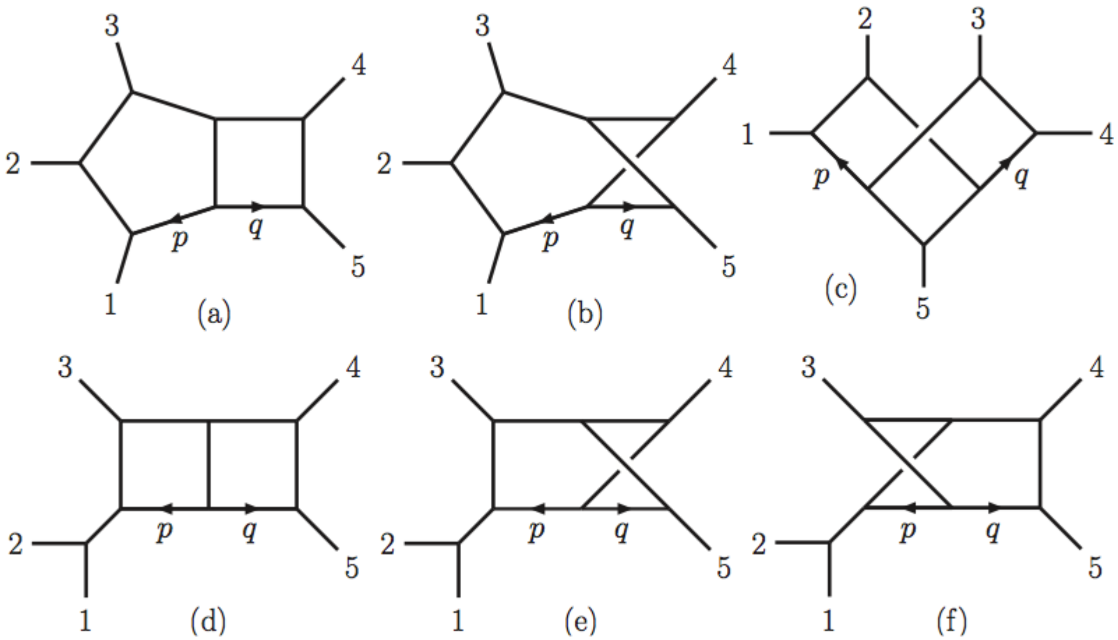
- The five-point amplitude makes the duality manifest !
- $\mathcal{N}=8$ SG is obtained through the numerator double copy

e.g. Jacobi relation:




$$N^{(\mathrm{B})}(1,2,3,4,p) = N^{(\mathrm{P})}(1,2,3,4,p) - N^{(\mathrm{P})}(2,1,3,4,p)$$

# 2-loop 5-pts $\mathcal{N}$=4 SYM and $\mathcal{N}$=8 SG




Carrasco, HJ
1106.4711 [hep-th]

The 2-loop 5-point amplitude with duality exposed

| $\mathcal{I}^{(x)}$ | $\mathcal{N}=4$ Super-Yang-Mills ($\sqrt{\mathcal{N}=8 \text{ supergravity}}$) numerator |
|---|---|
| (a),(b) | $\frac{1}{4}\Big(\gamma_{12}(2s_{45}-s_{12}+\tau_{2p}-\tau_{1p})+\gamma_{23}(s_{45}+2s_{12}-\tau_{2p}+\tau_{3p}) + 2\gamma_{45}(\tau_{5p}-\tau_{4p})+\gamma_{13}(s_{12}+s_{45}-\tau_{1p}+\tau_{3p})\Big)$ |
| (c) | $\frac{1}{4}\Big(\gamma_{15}(\tau_{5p}-\tau_{1p})+\gamma_{25}(s_{12}-\tau_{2p}+\tau_{5p})+\gamma_{12}(s_{34}+\tau_{2p}-\tau_{1p}+2s_{15}+2\tau_{1q}-2\tau_{2q}) + \gamma_{45}(\tau_{4q}-\tau_{5q})-\gamma_{35}(s_{34}-\tau_{3q}+\tau_{5q})+\gamma_{34}(s_{12}+\tau_{3q}-\tau_{4q}+2s_{45}+2\tau_{4p}-2\tau_{3p})\Big)$ |
| (d)-(f) | $\gamma_{12}s_{45}-\frac{1}{4}\Big(2\gamma_{12}+\gamma_{13}-\gamma_{23}\Big)s_{12}$ |

$\mathcal{N}$= 8 SG obtained from numerator double copies

$$\tau_{ip} = 2k_i \cdot p$$

$$\gamma_{12} \equiv \gamma_{12345}$$

# Duality and UV behavior

# UV properties of $\mathcal{N}$=8 supergravity

Quick status:

- Conventional superspace power counting forbids $L$=1,2 divergences Green, Schwarz, Brink (1982), Howe and Stelle (1989), Marcus and Sagnotti (1985)

- Three-loop divergence ruled out by calculation: Bern, Carrasco, Dixon, HJ, Kosower, Roiban, (2007), Bern, Carrasco, Dixon, HJ, Roiban (2008)

- $L$<7 loop divergences ruled out by counterterm analysis, using $E_{7(7)}$ symmetry and other methods, but a $L$=7 divergence is still possible Beisert, Elvang, Freedman, Kiermaier, Morales, Stieberger; Björnsson, Green, Bossard, Howe, Stelle, Vanhove, Kallosh, Ramond

Comparing $\mathcal{N}$=8 SG and $\mathcal{N}$=4 SYM UV behavior for $D$ > 4

Through four loops the theories diverge in exactly the same dimension:

$$D_c = 4 + \frac{6}{L} \qquad (L > 1)$$

Bern, Carrasco, Dixon, HJ, Kosower, Roiban

Confirmed using duality-satisfying amplitudes: UV behavior is manifest

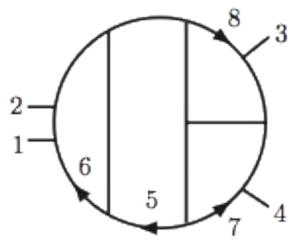
Bern, Carrasco, Dixon, HJ, Roiban

# 4-loops $\mathcal{N}$=4 SYM and $\mathcal{N}$=8 SG

Bern, Carrasco, Dixon, HJ, Roiban (to be published)

$$N_6^{\rm SYM} = \tfrac{1}{2} s_{12}^2 (\tau_{45} - \tau_{35} - s_{12})$$

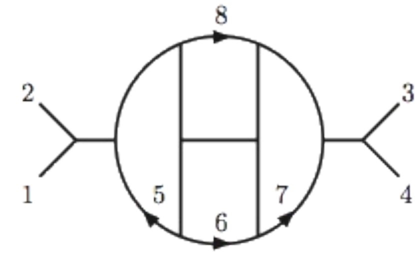
$$N_6^{\rm SG} = \left[\tfrac{1}{2} s_{12}^2 (\tau_{45} - \tau_{35} - s_{12})\right]^2$$




(6)

$$N_{80}^{\rm SYM} = 16 s_{12}^2 (s_{13} - s_{23})$$

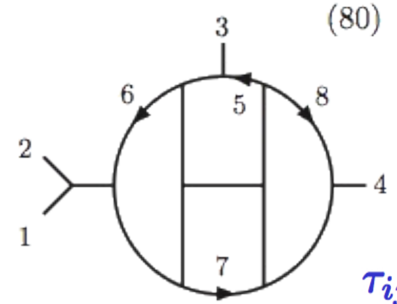
$$N_{80}^{\rm SG} = \left[16 s_{12}^2 (s_{13} - s_{23})\right]^2$$




(80)

$$N_{58}^{\rm SYM} = s_{12}(2 s_{13}(\tau_{45} - 3\tau_{35}) - s_{12}(s_{13} - s_{23} + 4\tau_{25} + 5\tau_{35} + \tau_{45}))$$

$$N_{58}^{\rm SG} = [s_{12}(2 s_{13}(\tau_{45} - 3\tau_{35}) - s_{12}(s_{13} - s_{23} + 4\tau_{25} + 5\tau_{35} + \tau_{45}))]^2$$




$\tau_{ij} = 2 k_i \cdot l_j$

(58)

- 85 diagrams in total
- Duality manifest
- Power counting manifest both $\mathcal{N}$=4 and $\mathcal{N}$=8
- Both diverge in $D$=11/2

$$\mathcal{M}_4^{(4)}\Big|_{\rm pole} = -\frac{23}{8}\left(\frac{\kappa}{2}\right)^{10} s_{12} s_{13} s_{23} (s_{12}^2 + s_{13}^2 + s_{23}^2)^2 \, M_4^{\rm tree} \, (V_1 + 2V_2 + V_8)$$

# Towards a kinematic Lie algebra

All available evidence suggest that exist kinematic numerators of gauge theory amplitudes that satisfy the same general algebra as the color structures of these theories.

Suggest the existence of a Lie algebra for the kinematics !

- In 1103.0312 [hep-th] Bern and Dennen investigate the trace structure of kinematical numerators.
- In 1105.2565 [hep-th] Monteiro and O'Connell identify a diffeomorphism Lie algebra in the self-dual Yang-Mills sector. From this they obtain the kinematic structure constants for MHV tree amplitudes.
- In 1004.0693 [hep-th] Bern, Dennen, Huang, Kiermaier work out the first terms of a duality-satisfying Lagrangian

# Lagrangian formulation

- First attempt at Lagrangian with manifest duality

1004.0693 [hep-th]
Bern, Dennen, Huang, Kiermaier

YM Lagrangian receives corrections at 5 points and higher

$$\mathcal{L}_{YM} = \mathcal{L} + \mathcal{L}'_5 + \mathcal{L}'_6 + \ldots$$

corrections proportional to the Jacobi identity (thus equal to zero)

$$\mathcal{L}'_5 \sim \mathrm{Tr}\,[A^\nu, A^\rho]\frac{1}{\Box}([[\partial_\mu A_\nu, A_\rho], A^\mu] + [[A_\rho, A^\mu], \partial_\mu A_\nu] + [[A^\mu, \partial_\mu A_\nu], A_\rho])$$

Introduction of auxiliary fields gives local cubic Lagrangian

$$\mathcal{L}_{YM} = \tfrac{1}{2}A^{a\mu}\Box A^a_\mu - B^{a\mu\nu\rho}\Box B^a_{\mu\nu\rho} - g f^{abc}(\partial_\mu A^a_\nu + \partial^\rho B^a_{\rho\mu\nu})A^{b\mu}A^{c\nu} \quad + \ldots$$

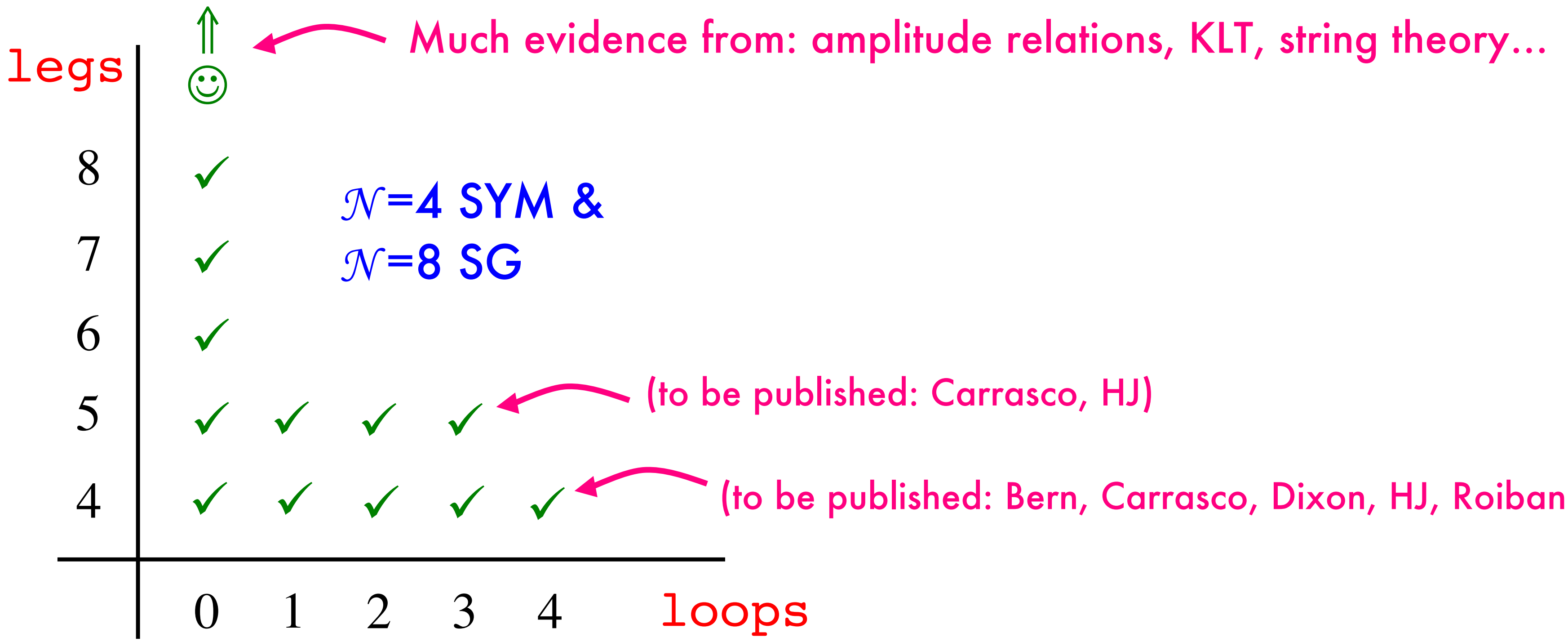
"squaring" gives gravity Lagrangian.

# Summary

- Gravity appears to be a double copy of Yang-Mills theory, order by order in the S-matrix.
- The double-copy structure in field theory becomes clear if kinematic numerators are treated on equal footing with color factors. Suggesting that a kinematic Lie algebra should exist.
- Nontrivial evidence at tree and loop level supports the duality.
- Lagrangian formulation, connection to string theory, give hints of future potential. Duality should be a key tool for nonplanar gauge theory and gravity calculations.
- What is the physical interpretation the duality ? What is the kinematic Lie algebra ? Further understanding of the connection to string theory may help answer these questions.

# Extra slides

# Summary of checks of duality




Less-SUSY theories:

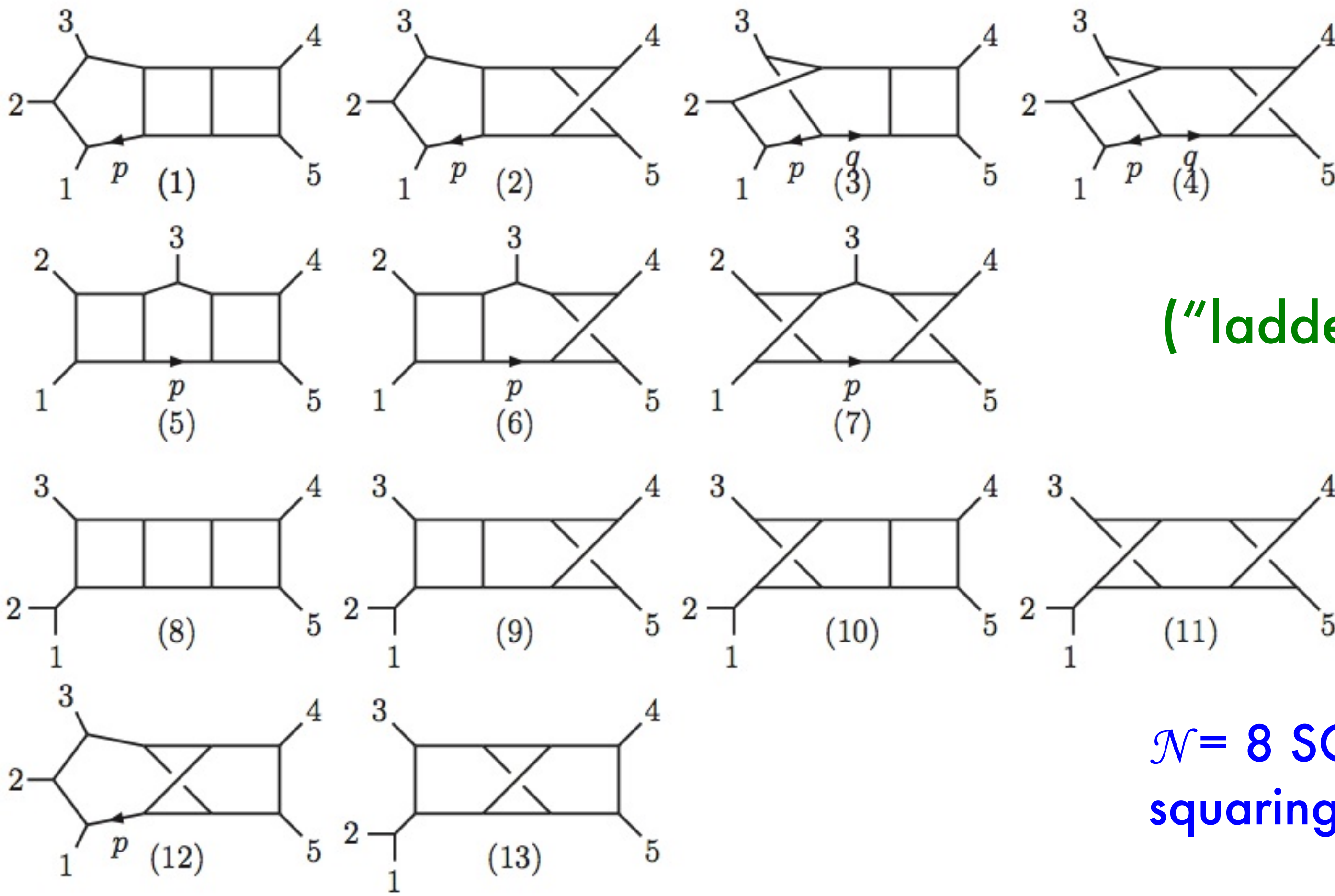
Tree level: all pure gauge theories have the same tree amplitudes as $\mathcal{N}$=4 SYM ✓

Two-loop $\mathcal{N}$=0 YM, 4p all-plus helicity ✓

# 3-loop 5-pts $\mathcal{N}=4$ SYM and $\mathcal{N}=8$ SG

Again the D-dimensional amplitude admits a representation with manifest duality

Carrasco, HJ
(to be published)




("ladder-like" diagrams)

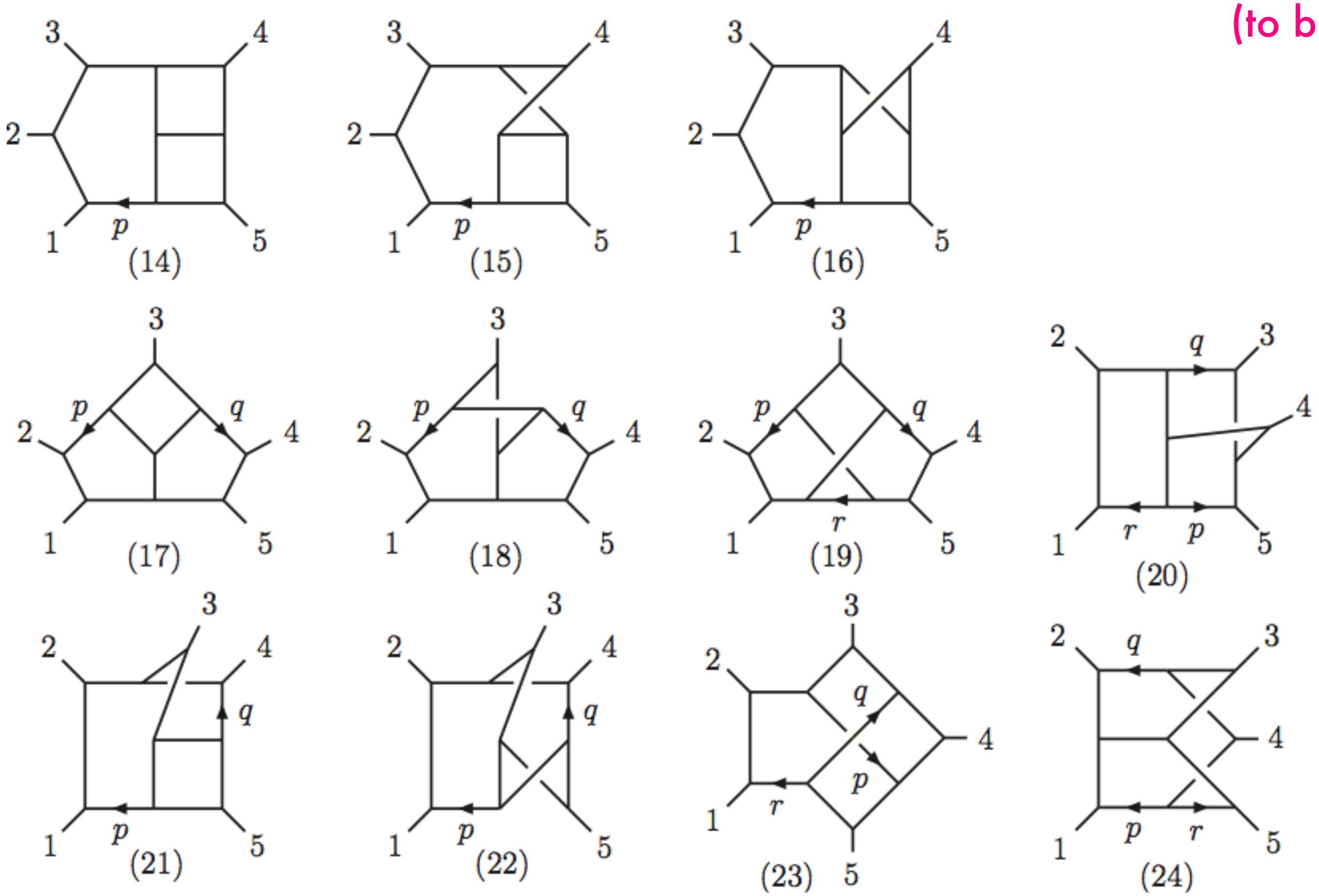
$\mathcal{N}$= 8 SG obtained from squaring the numerators

$$N_8 = -2\gamma_{[12]} s_{45}^2 + \frac{1}{6} s_{12}\Big(\gamma_{[13]}(2s_{13}+12s_{23}-s_{12}) - \gamma_{[23]}(2s_{23}+12s_{13}-s_{12}) - \gamma_{[12]}(7s_{12}-11s_{45})\Big)$$

# 3-loop 5-pts $\mathcal{N}$=4 SYM and $\mathcal{N}$=8 SG

some "Mercedes-like" diagrams…

Carrasco, HJ
(to be published)




$$N_{14} = \gamma_{[54]}(\tau_{1p}^2 + \tau_{2p}^2 + \tau_{3p}^2 + \tau_{4p}^2 + \tau_{5p}^2) + \text{ subleading in } p$$

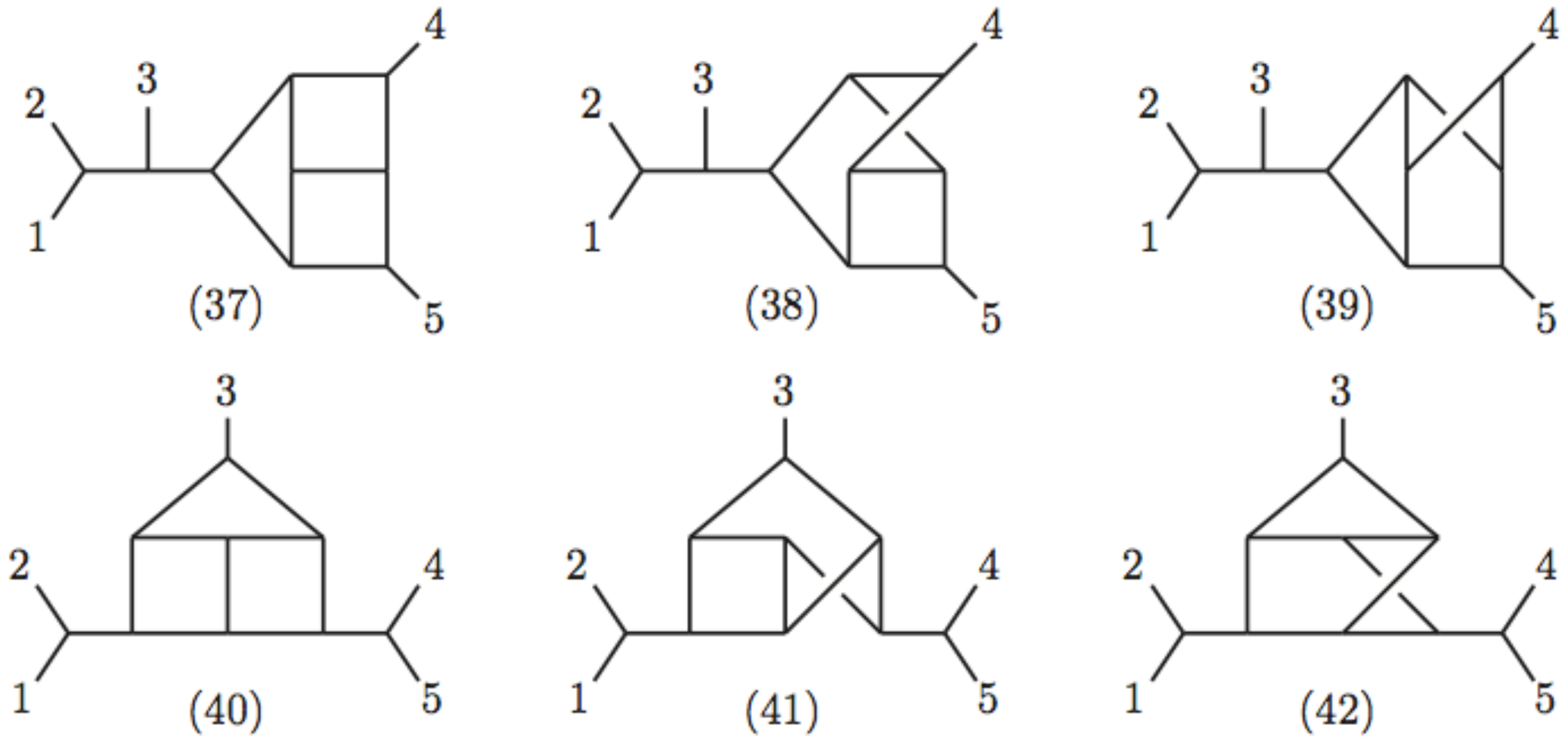
$$\tau_{ip}^2 = 2k_i \cdot p$$

# 3-loop 5-pts $\mathcal{N}$=4 SYM and $\mathcal{N}$=8 SG

...in total 42 diagrams.

Carrasco, HJ
(to be published)

Conveniently the UV divergent diagrams (in D=6) are very simple:



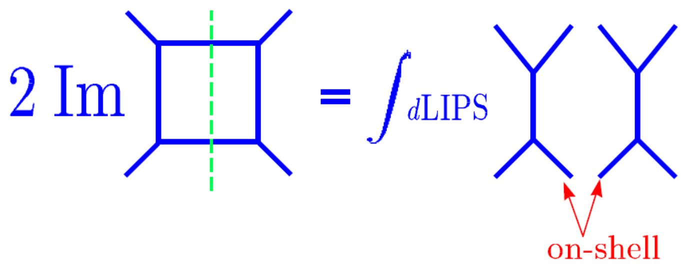

(for SG the UV div. comes from the other diagrams as well)

# Unitarity

**Optical theorem:**

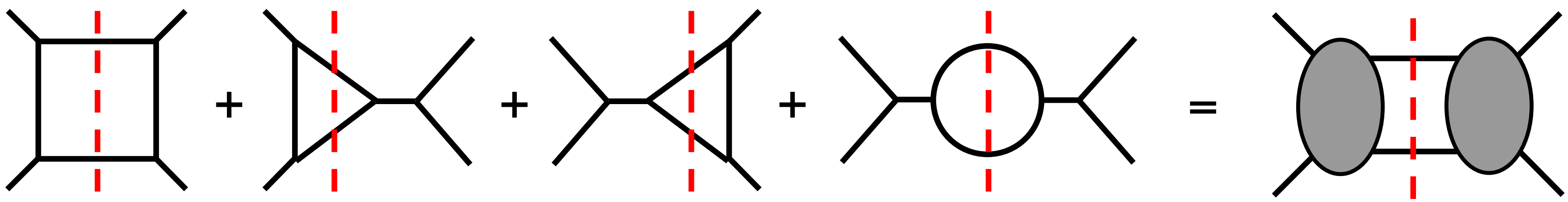
$$1 = S^\dagger S = (1 - iT^\dagger)(1 + iT)$$

$$2\,\mathrm{Im}\,T = T^\dagger T$$




The unitarity method reconstructs the amplitudes avoiding dispersion relations

Bern, Dixon, Dunbar, Kosower (1994)



Compute a cut: put loop legs on-shell in amplitude = sew trees amplitudes

checking every cut channel will fix the loop integrals

# Amplitude relations for any number of legs

Bern, Carrasco, HJ

- General relations for gauge theory partial amplitudes

$$A_n^{\rm tree}(1,2,\{\alpha\},3,\{\beta\}) = \sum_{\{\sigma\}_j\in {\rm POP}(\{\alpha\},\{\beta\})} A_n^{\rm tree}(1,2,3,\{\sigma\}_j) \prod_{k=4}^{m} \frac{\mathcal{F}(3,\{\sigma\}_j,1|k)}{s_{2,4,\ldots,k}}$$

where

$$\{\alpha\} \equiv \{4,5,\ldots,m-1,m\}, \qquad \{\beta\} \equiv \{m+1,m+2,\ldots,n-1,n\}$$

and

$$\mathcal{F}(3,\sigma_1,\sigma_2,\ldots,\sigma_{n-3},1|k) \equiv \mathcal{F}(\{\rho\}|k) = \begin{Bmatrix} \sum_{l=t_k}^{n-1} \mathcal{G}(k,\rho_l) & \text{if } t_{k-1} < t_k \\ -\sum_{l=1}^{t_k} \mathcal{G}(k,\rho_l) & \text{if } t_{k-1} > t_k \end{Bmatrix} + \begin{Bmatrix} s_{2,4,\ldots,k} & \text{if } t_{k-1} < t_k < t_{k+1} \\ -s_{2,4,\ldots,k} & \text{if } t_{k-1} > t_k > t_{k+1} \\ 0 & \text{else} \end{Bmatrix}$$

and

$$\mathcal{G}(i,j) = \begin{Bmatrix} s_{i,j} & \text{if } i < j \text{ or } j = 1,3 \\ 0 & \text{else} \end{Bmatrix}$$

and $t_k$ is the position of leg $k$ in the set $\{\rho\}$

$$A_n(\sigma_1,\sigma_2,..,\sigma_n) = \alpha_1 A_n(1,2,..,n) + \alpha_2 A_n(2,1,..,n) + \ldots + \alpha_{(n-3)!} A_n(3,2,..,n)$$

Basis size: $(n-3)!$ Compare to Kleiss-Kuijf relations $(n-2)!$

Recent proofs: Bjerrum-Bohr, Damgaard, Vanhove; Feng, Huang, Jia

# Gauge theory amplitude properties

- Tree level, adjoint representation

$$\mathcal{A}_n^{\text{tree}}(1,2,\ldots,n) = g^{n-2} \sum_{\mathcal{P}(2,\ldots,n)} \text{Tr}[T^{a_1}T^{a_2}\cdots T^{a_n}]\, A_n^{\text{tree}}(1,2,\ldots,n)$$

gauge invariant

- Well-known partial amplitude properties

$$A_n^{\text{tree}}(1,2,\ldots,n) = A_n^{\text{tree}}(2,\ldots,n,1)$$ cyclic symmetry

$$A_n^{\text{tree}}(1,2,\ldots,n) = (-1)^n A_n^{\text{tree}}(n,\ldots,2,1)$$ reflection symmetry

$(n$ -1)!/2

$$\sum_{\sigma\in\text{cyclic}} A_n^{\text{tree}}(1,\sigma(2,3,\ldots,n)) = 0$$ "photon"-decoupling identity

$$A_n^{\text{tree}}(1,\{\alpha\},n,\{\beta\}) = (-1)^{n_\beta} \sum_{\{\sigma\}_i\in\text{OP}(\{\alpha\},\{\beta^T\})} A_n^{\text{tree}}(1,\{\sigma\}_i,n)$$ Kleiss-Kuijf relations

$(n$ - 2)!

- *New relations reduce independent basis to* **$(n$ - 3)!** Bern, Carrasco, HJ