Holography and Hydrodynamics

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Outline

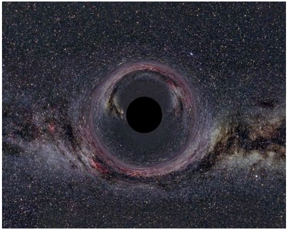
Collaborators and References

With B. Keren-Zur, C. Eling, G. Falkovich, I. Fouxon, I. Itkin, X. Liu, Y. Neiman, M. Rabinovich.

PRL **101** (2008) 261602, JHEP **0903**, 120 (2009), PLB **680**, 496 (2009), JHEP **1002**, 069 (2010), PLB **694**, 261 (2010), , JFM **644**, 465 (2010), JHEP **1006**, 006 (2010), CMP **52**, 43 (2011), JHEP **1103**, 006 (2011), JHEP **1012**, 086 (2010), JHEP **1103**, 023 (2011), JHEP **1102**, 070 (2011), JHEP |bf 1106, 007 (2011), , arXiv:1106.2683, arXiv:1106.3576 and to appear.

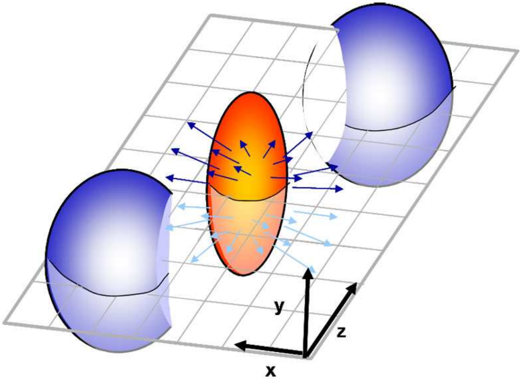
Fluid Dynamics and Gravity

- The AdS/CFT correspondence relates fluid dynamics to black hole dynamics: hydrodynamic regime of the correspondence.



Heavy-Ion Collision

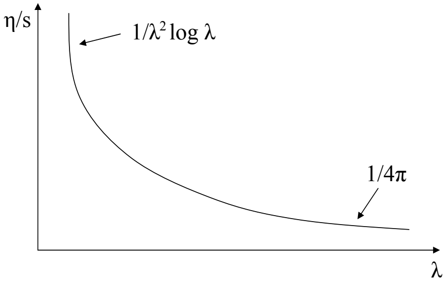
- The QCD plasma produced at RHIC and LHC seems to exhibit a strong coupling dynamics $\alpha_s(T_{RHIC}) \sim O(1)$.



The Shear Viscosity to Entropy Density Ratio $\frac{\eta}{s}$

$$T_{\mu\nu} = T_{\mu\nu}^{Ideal} + \eta T_{\mu\nu}^{Viscous}$$
$$\partial^{\mu} T_{\mu\nu} = 0$$

- In Einstein's gravity: $\frac{\eta}{s} = \frac{1}{4\pi}$ (Policastro, Son, Starinets:2001).



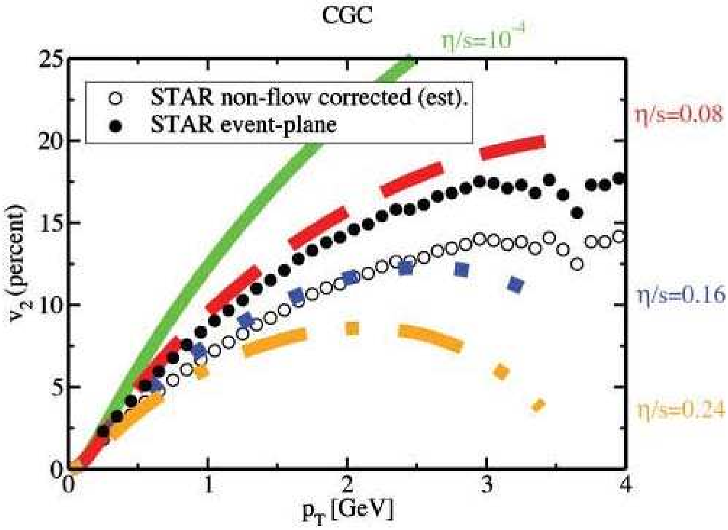
Elliptic Flow

- Hydrodynamic simulations at low shear viscosity to entropy ratio are consistent with RHIC Data.
- The elliptic flow parameter is the second Fourier coefficient $v_2 = \langle Cos(2\phi) \rangle$ of the azimuthal momentum distribution $dN/d\phi$

$$\frac{dN}{d\phi} \sim 1 + 2v_2 Cos(2\phi)$$

Elliptic Flow

- Luzum, Romatschke:2008



The Hydrodynamic Modes

- The effective degrees of freedom are charge densities $\rho(\vec{x},t)$, and the hydrodynamics equations are conservation laws

$$\partial_t \rho + \partial_i j^i = 0 \quad (1)$$

- The constitutive relations express j^i in terms of ρ and its derivatives. For instance $j^i = -D\partial^i \rho$, and we get

$$\partial_t \rho - D\partial_i \partial^i \rho = 0 \quad (2)$$

- Writing $\rho(\vec{k},t) = \int d^3 x e^{-i\vec{k}\cdot\vec{x}\rho(\vec{x},t)}$ we have

$$\rho(\vec{k},t) = e^{-Dk^2 t}\rho(\vec{k},t=0) \quad (3)$$

  This is the characteristic behaviour of hydrodynamic mode. It has a life $\tau(k) = \frac{1}{Dk^2}$ which is infinite in the limit $k \to 0$.

Relativistic Hydrodynamics

- Hydrodynamics applies under the condition that the correlation length of the fluid l_{cor} is much smaller than the characteristic scale L of variations of the macroscopic fields

$$Kn \equiv l_{cor}/L \ll 1 \tag{4}$$

- Hydrodynamics equations are conservation laws

$$\partial_\mu T^{\mu\nu} = 0, \qquad \partial_\mu J^\mu_a = 0 \tag{5}$$

Relativistic Hydrodynamics

- The equations of relativistic hydrodynamics are determined by the constitutive relation expressing $T^{\mu\nu}$ and J^{μ}_{a} in terms of the energy density $\epsilon(x)$, the pressure $p(x)$, the charge densities $\rho_a(x)$ and the four-velocity field $u^{\mu}(x)$ satisfying $u_{\mu}u^{\mu} = -1$.
- The constitutive relation has the form of a series in the small parameter $Kn \ll 1$,

$$T^{\mu\nu}(x) = \sum_{l=0}^{\infty} T^{\mu\nu}_{(l)}(x), \quad T^{\mu\nu}_{(l)} \sim (Kn)^l \tag{6}$$

Ideal Hydrodynamics

- Keeping only the first term in the series gives ideal hydrodynamics and the stress-energy tensor reads

$$T^{\mu\nu}_{(0)} = \epsilon u^\mu u^\nu + p\left(\eta^{\mu\nu} + u^\mu u^\nu\right) \tag{7}$$

  The equation of state $\epsilon(p)$ is an additional input.
- CFT hydrodynamics: $T^\mu_\mu = 0$, $\epsilon = 3p \sim T^4$ and the stress-energy tensor reads

$$T^{\mu\nu}_{(0)} = T^4[\eta^{\mu\nu} + 4u^\mu u^\nu] \tag{8}$$

Viscous Hydrodynamics

- The dissipative hydrodynamics is obtained by keeping $l = 1$ term in the series. The stress-energy tensor reads (Landau Frame: $u_\mu T^{\mu\nu}_{(1)} = 0$)

$$T^{\mu\nu}_{(1)} = -\eta\sigma^{\mu\nu} - \xi(\partial_\alpha u^\alpha)\,(\eta^{\mu\nu} + u^\mu u^\nu) \tag{9}$$

where

$$\sigma^{\mu\nu} = (\partial^\mu u^\nu + \partial^\nu u^\mu + u^\nu u^\rho \partial_\rho u^\mu + u^\mu u^\rho \partial_\rho u^\nu - \frac{2}{3}\partial_\alpha u^\alpha [\eta^{\mu\nu} + u^\mu u^\nu] \tag{10}$$

- The dissipative hydrodynamics of a CFT is determined by only one kinetic coefficient - the shear viscosity η

$$T^{\mu\nu}_{(1)} = -\eta\sigma^{\mu\nu} \tag{11}$$

Gravitational Dual Description

- Consider the five-dimensional Einstein equations with negative cosmological constant

$$R_{mn} + 4g_{mn} = 0, \qquad R = -20 \tag{12}$$

- These equations have a particular "thermal equilibrium" solution - the boosted black brane

$$ds^2 = -2u_\mu dx^\mu dr - r^2 f[br] u_\mu u_\nu dx^\mu dx^\nu + r^2 P_{\mu\nu} dx^\mu dx^\nu \tag{13}$$

where

$$f(r) = 1 - \frac{1}{r^4}, \quad P^{\mu\nu} = u^\mu u^\nu + \eta^{\mu\nu}$$

and the constant $T = 1/\pi b$ is the temperature.

Gravitational Description

- One looks for a solution of the Einstein equation by the method of variation of constants (Bhattacharya et.al.: 2007)

$$g_{mn} = (g_0)_{mn} + \delta g_{mn} \tag{14}$$

$$(g_0)_{mn} dy^m dy^n = -2u_\mu(x^\alpha) dx^\mu dr - r^2 f[b(x^\alpha)r] u_\mu(x^\alpha) u_\nu(x^\alpha) dx^\mu dx^\nu + r^2 P_{\mu\nu}(x^\alpha) dx^\mu dx^\nu \tag{15}$$

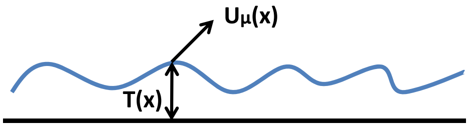
- $y = (x^\mu, r)$. As in the Boltzmann equation, the condition of constructibility of the series solution produces equations for $u^\mu(x^\alpha)$ and $T(x^\alpha) = 1/\pi b(x^\alpha)$. The series for g_{mn} is the series in the Knudsen number of the boundary CFT hydrodynamics.

Horizon Dynamics

- The way the black brane horizon geometry encodes the boundary fluid dynamics is reminiscent of the *Membrane Paradigm* (Damour:1979) in classical general relativity, according to which any black hole has a fictitious fluid living on its horizon.
- The dynamics of the event horizon of a black brane in asymptotically AdS space-time (Gauss-Codazzi equations) is described by the Navier-Stokes equations (Eling,Y.O.:2009).
- The two approaches are related by an RG flow (Bredberg et. al: 2010).

The Horizon Geometry

- Thermalization in field theory is the process of black hole creation in gravity.
- Hydrodynamics is the deformation of the black hole geometry.



The Bulk Viscosity ζ

- (Eling,Y.O.:2011) The bulk viscosity is captured by the horizon dynamics.
- Consider $(d+1)$-dimensional gravitational backgrounds holographically describing thermal states in strongly coupled d-dimensional field theories. The $(d+1)$-dimensional gravitational action reads

$$I = \frac{1}{16\pi}\int \sqrt{-g}d^{d+1}x\left(\mathcal{R} - \frac{1}{2}\sum_i (\partial\phi_i)^2 - V(\phi_i)\right) + I_{gauge} \ , \tag{16}$$

$V(\phi_i)$ represents the potential for the scalar fields and I_{gauge} represents the action of gauge fields (abelian or non-abelian) A^a_μ.

The Bulk Viscosity ζ

- The null horizon focusing equation obtained by projecting the field equations of (16) on the horizon is equivalent via the fluid/gravity correspondence to the entropy balance law of the fluid.
- Using this equation we derived

$$\frac{\zeta}{\eta} = \sum_i \left(s\frac{d\phi_i^H}{ds} + \rho^a \frac{d\phi_i^H}{d\rho^a} \right)^2$$

η is the shear viscosity, s is the entropy density, ρ^a are the charges associated with the gauge fields A_μ^a, and ϕ_i^H are the values of the scalar fields on the horizon.

Horizon Geometry and the Focusing Equation

- The formula seems to hold exactly in some cases (Buchel,Gursoy,Kiritsis:2011).
- In fact it is an exact formula (Eling,Y.O.: to appear).
- We consider in the framework of the fluid/gravity correspondence the dynamics of hypersurfaces located in the holographic radial direction at $r = r_c$.
- We prove that all these hypersurfaces evolve, to all orders in the derivative expansion and including all higher curvature corrections, according to the same hydrodynamics equations with identical transport coefficients.

Anomalies

- The hydrodynamics description exhibits an interesting effect when a global symmetry current of the microscopic theory is anomalous

$$D_\mu J^\mu_\alpha = \frac{1}{8} C_{\alpha\beta\gamma}\epsilon^{\mu\nu\rho\sigma} F^\beta_{\mu\nu} F^\gamma_{\rho\sigma}$$

- The form of an anomalous symmetry current is modified in the hydrodynamic description by a term proportional to the vorticity of the fluid

$$\omega^\mu \equiv \frac{1}{2}\epsilon^{\mu\nu\lambda\rho} u_\nu \partial_\lambda u_\rho$$

- This has been first discovered in the context of the the fluid/gravity correspondence (Erdmenger et.al, Banerjee et.al.:2008).

Anomalies

- The global symmetry current takes the form

$$j_a^\mu = \rho_a u^\mu + \sigma_a{}^b \left(E_b^\mu - T P^{\mu\nu} D_\nu \frac{\mu_b}{T} \right) + \xi_a \omega^\mu + \xi_{ab}^{(B)} B^{b\mu}$$

where ρ_a, T, μ_a and σ_a^b are the charge densities, temperature, chemical potentials and the conductivities of the medium.

Anomalies

- The anomaly coefficients are (Son,Surowka:2009; Neiman, Y.O.:2010)

$$
\begin{aligned}
\xi_a &= C_{abc}\mu^b\mu^c + 2\beta_a T^2 - \frac{2n_a}{\epsilon + p}\left(\frac{1}{3}C_{bcd}\mu^b\mu^c\mu^d + 2\beta_b\mu^b T^2\right) \\
\xi_{ab}^{(B)} &= C_{abc}\mu^c - \frac{n_a}{\epsilon + p}\left(\frac{1}{2}C_{bcd}\mu^c\mu^d + \beta_b T^2\right)
\end{aligned}
$$

  C_{abc} is the coefficient of the triangle anomaly of the currents j_a^μ,j_b^μ and j_c^μ.

- β_a seem to correspond to the $T^2 J$ gravitational anomaly (free fermions calculation by Landsteiner et.al: 2011).
- Generalization to arbitrary even dimension (Loganayagam:2011).

QCD Axial Anomaly in HIC

- In very energetic collisions the hot dense QCD matter can go through a phase transition to a deconfined phase described by a fluid-like collective motion of quarks and gluons. We consider a deconfined QCD fluid phase, with three light flavors and chiral symmetry restoration.
- We consider the experimental implications of the axial current triangle diagram anomaly in a hydrodynamic description of high density QCD.

Chiral Magnetic and Vortical Effects

- (Kharzeev, Son :2011) Chiral magnetic effect: charge separation. Chiral vortical effect: baryon number separation.

$$\vec{J} = \frac{N_c \mu_5}{2\pi^2} \left(tr(VAQ)\vec{B} + tr(VAB) 2\mu_B \vec{\omega} \right)$$

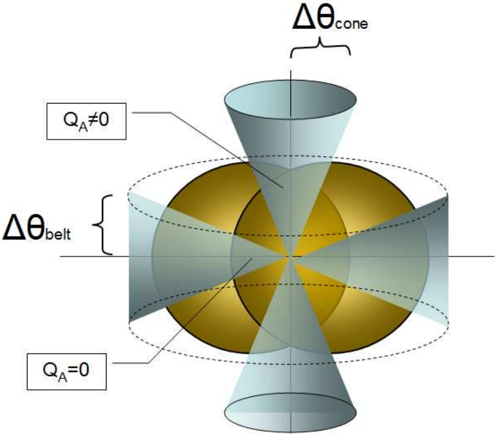
  The electromagnetic current corresponds to V = Q and the baryon current to V = B.
- The ratio between the baryon asymmetry and the charge asymmetry increases when the center of mass energy is lowered.

QCD Axial Anomaly in HIC

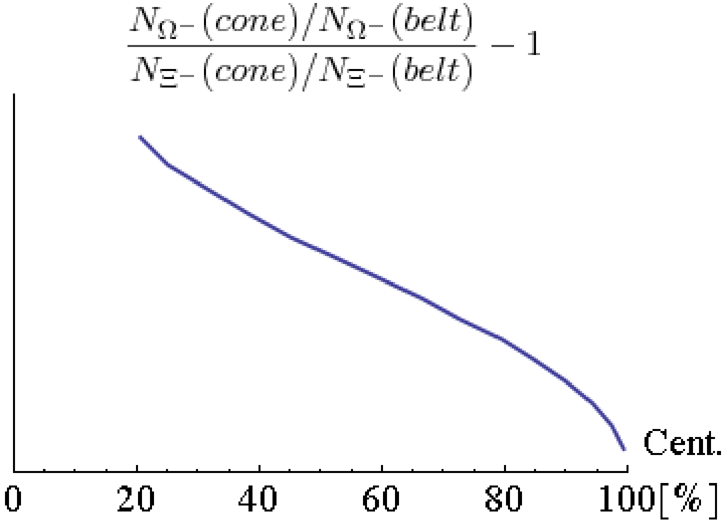
- (Keren-Zur,Y.O.:2010): The basic idea is that the the axial charge density, in a locally uniform flow of massless fermions, is a measure of the alignment between the fermion spins.
- When the QCD fluid freezes out and the quarks bind to form hadrons, aligned spins result in spin-excited hadrons. The ratio between spin-excited and low spin hadron production and its angular distribution may therefore be used as a measurement of the axial charge distribution.
- We predict the qualitative angular distribution and centrality dependence of the axial charge.

QCD Axial Anomaly in HIC

- Our main proposal is that for off-central collisions we expect enhancement of Ω^- production along the rotation axis of the collision



QCD Axial Anomaly in HIC

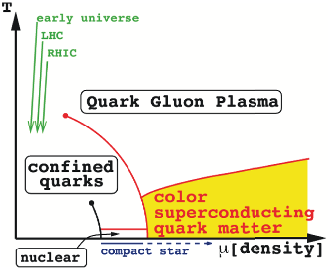


Superfluid Hydrodynamics

- The most remarkable property of liquid helium below the λ-point is superfluidity. It is the ability of the fluid to flow inside narrow capillaries without friction, discovered by Kapitza.
- The hydrodynamics of a superfluid consists of two motions: the motion of the normal part of the fluid, and the motion of the superfluid part which is an irrotational one, i.e. its velocity is curl free (Tisza,Landau).
- A superfluid can be described as a fluid with a spontaneously broken symmetry, where the superfluid component is the condensate, and its velocity is proportional to the Goldstone phase gradient.
- The hydrodynamics of relativistic superfluids is relevant to the study of neutron stars, and highly dense quark matter at the low temperature Color-Flavor locked phase (Alford, Rajagopal, Wilczek:1997).

Superfluid Hydrodynamics - CFL Phase

- Anomalous transport (Bhattacharya et. al.:2011).



- (Neiman,Y.O.:2011) A Chiral Electric Effect:

$$J^{a\mu} = C^{cgd}(\delta^a_c - \frac{n^a\mu_c}{\epsilon+p})(\delta^b_d - \frac{n^b\mu_d}{\epsilon+p})\epsilon^{\mu\nu\rho\sigma}u_\nu\xi_{g\rho}(E_{b\sigma} - T\partial_\sigma\frac{\mu_b}{T})$$

Relativistic Turbulence

- The Reynolds number is

$$\mathcal{R}_e \sim \frac{TL}{\eta/s} \tag{17}$$

  When $\mathcal{R}_e$ is large we expect turbulence.
- Is there a universal structure in relativistic turbulence?

Relativistic Turbulence

- In relativistic hydrodynamics we consider the hydrodynamics equation with a random force term

$$\partial^{\nu} T_{\mu\nu} = f_{\mu} \tag{18}$$

  and derive the exact scaling relation (Fouxon,Y.O.:2009)

$$\langle T_{0j}(0,t) T_{ij}(r,t) \rangle = \epsilon r_i$$

  where $d \langle T_{0j}(0,t) f_j(0,t) \rangle \equiv \epsilon$.
- In the non-relativistic limit we get the Kolmogorov 1941 exact scaling relation.

RHIC and LHC

- Does turbulence show up in RHIC and LHC?
- For gold collisions at RHIC, the characteristic scale L is the radius of a gold nucleus $L \sim 6$ Fermi, the temperature is the QCD scale $T \sim 200$ MeV, and $\frac{\eta}{s} \sim \frac{1}{4\pi}$ is a characteristic value of strongly coupled gauge theories.
- With these R_e is too small for an experimental realization of the steady state relativistic turbulence.

Charged Hydrodynamics

- In charged hydrodynamics with a conserved symmetry current J^μ, we derive in the inertial range the exact scaling relation
$$\langle J_0(0,t)J_i(r,t)\rangle = \epsilon r_i \tag{19}$$
- An experimental setup, where one may be able to study universal properties of relativistic turbulence is condensed matter physics.
- It was noticed (Sheehy,Schmalian:2007) that there is an emergent relativistic symmetry of electrons in graphene near its quantum critical point, for which a relativistic nearly ideal fluid description may be appropriate.

Non-relativistic Limit: Navier-Stokes Equations

- The incompressible Navier-Stokes equations can be obtained in the nonrelativistic limit of relativistic hydrodynamics (Fouxon,Y.O.,Bhattacharyya, Wadia, Minwalla:2008).
-

$$
\begin{aligned}
\partial_\mu u^\mu + (d-1) D ln T &= \frac{1}{2\pi T}\sigma_{\mu\nu}\sigma^{\mu\nu} \ , \\
a_\sigma + P^\mu_\sigma \partial_\mu ln T &= \frac{1}{2\pi T} P^\mu_\sigma(\partial_\alpha \sigma^\alpha_\mu - (d-1)\sigma^\alpha_\mu a_\alpha) \ .
\end{aligned} \qquad (20)
$$

- Expand $u^\mu = (1 - v^2/2 + ..., v^i), T = T_0(1 + P + ...)$ where we scale $v^i \sim \varepsilon, \partial_i \sim \varepsilon, \partial_t \sim \varepsilon^2, P \sim \varepsilon^2$, where $\varepsilon \sim 1/c$.
- The first equation gives the incompressibility condition $\partial_i v^i = 0$. The second equation gives

$$
\partial_t v^i + v^j \partial_j v^i = -\partial^i P + \nu \Delta v^i \qquad (21)
$$

where $\nu = \frac{1}{4\pi T_0}$.

Turbulence

- The Reynolds number is

$$\mathcal{R}_e = \frac{LV}{\nu} \tag{22}$$

  where L and V are, respectively, a characteristic scale and velocity of the flow.
- Experimental and numerical analysis data show that for $\mathcal{R}_e \gg 100$ the flow is highly irregular (turbulent) with a complex spatio-temporal pattern formed by the turbulent velocity field.

Turbulence in Nature

- Most flows in nature are turbulent. This is simple to see by noting, for instance, that the viscosity of water is $\nu \simeq 10^{-6}\frac{m^2}{sec}$. Thus, a medium size river has a Reynolds number $\mathcal{R}_e \sim 10^7$.

The Turbulence Problem: Anomalous Scaling

- There is experimental and numerical evidence that in the range of distance scales $l \ll r \ll L$ (the inertial range) the flows exhibit a universal behavior

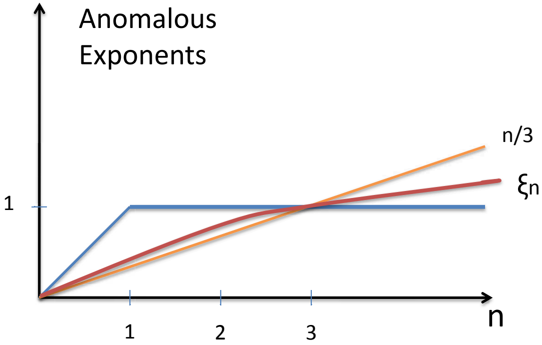
$$S_n(r) \equiv \left\langle \left((\mathbf{v}(\mathbf{x}) - \mathbf{v}(\mathbf{y})) \cdot \frac{\mathbf{r}}{r} \right)^n \right\rangle \sim r^{\xi_n} \tag{23}$$

  $\mathbf{r} \equiv \mathbf{x} - \mathbf{y}$.

- The 1941 exact scaling result of Kolmogorov $\xi_3 = 1$ agrees well with the experimental data, while the other exponents are measurable real numbers.

Anomalous Exponents

The major open problem of turbulence is to calculate the anomalous exponents ξ_n



Anomalous Exponents: New Results

- New scaling relations for incompressible turbulence (Falkovich,Fouxon,Y.O.:2009):

$$\langle v_i(r)p(r)v^2(0)\rangle = \epsilon r_i \tag{24}$$

- For compressible turbulence:

$$\left\langle \rho(0)v_j(0)\left(\rho(r)v_j(r)v_i(r) + p(r)\delta_{ij}\right)\right\rangle = \epsilon r_i \tag{25}$$

- When $\rho = constant$, it reduces to Kolmogorov relation:

$$\langle v_j(0,t)v_i(r,t)v_j(r,t)\rangle = \epsilon r_i \tag{26}$$

Rindler Horizon

- The Membrane Paradigm holds for a general non-singular null hypersurface, provided a large scale hydrodynamic limit exists. Thus, for instance, the dynamics of the Rindler acceleration horizon is also described by the incompressible Navier- Stokes equations (Eling,Fouxon,Y.O:2009).
- By expanding around the Rindler horizon, one shows that for every solution of the incompressible Navier-Stokes equation in d dimensions, there is a uniquely associated solution of the vacuum Einstein equations in $d+1$ dimensions (Bredberg et.al.:2011).
- May be used for holographic duality involving a flat space-time (Compere et.al.:2011).

Singularities in Hydrodynamics

- (Y.O.,M.Rabinovich:2010) The basic question concerning singularities in the hydrodynamic description, is whether starting with appropriate initial conditions, where the velocity vector field and its derivatives are bounded, can the system evolve such that it will exhibit within a finite time a blowup of the derivatives of the vector field.
- Physically, such singularities if present, indicate a breakdown of the effective hydrodynamic description at long distances and imply that some new degrees of freedom are required.

Singularities in Gravity

- The issue of hydrodynamic singularities has an analogue in gravity. Given an appropriate Cauchy data, will the evolving space-time geometry exhibit a naked singularity, i.e. a blowup of curvature invariants and the energy density of matter fields at a point not covered by a horizon.

Singularities in Gravity

- The Penrose inequality (Penrose:1973) is a conjecture relating the mass and the horizon area of any Cauchy initial data that if violated leads to a space-time naked singularity.

$$M_0 \geq M \geq \sqrt{A_H/16\pi} \geq \sqrt{A_{H0}/16\pi} \tag{27}$$

  and the initial Cauchy data should also satisfy the Penrose inequality $M_0 \geq \sqrt{A_{H0}/16\pi}$.
- The argument relies on the Hawking area theorem and the relaxation at late times to a Kerr solution, both assume the weak censorship hypothesis. Therefore, finding Cauchy data that violates the Penrose inequality implies finding a solution to Einstein equations in which a naked singularity is created.

Singularities in Gravity

- (I.Itkin,Y.O.:2011) The general form of the Penrose inequality for asymptotically locally AdS_{d+1} spaces, with electric charge q and a boundary topology characterized by $k = 0, \pm 1$:

$$M - M_0 \geq \frac{(d-1)\Omega_{d-1,k}}{16\pi}[q^2(\frac{\Omega_{d-1,k}}{A})^{\frac{d-2}{d-1}} + k(\frac{A}{\Omega_{d-1,k}})^{\frac{d-2}{d-1}} + \frac{1}{l^2}(\frac{A}{\Omega_{d-1,k}})^{\frac{d}{d-1}}] \tag{28}$$

where

$$M_0 = \frac{1}{8\pi}(-k)^{d/2}\frac{(d-1)!!^2}{d!}l^{d-2}\Omega_{d-1,k} \tag{29}$$

for d even and zero for d odd.

Outlook

- Turbulence and its universal structure.
- Singularities in hydrodynamics and cosmic censorship.
- Anomalous transport in normal fluids and superfluids.
- Applications to heavy ion collisions, QCD at high densities and Neutron stars.