Exact results for loop operators in 4d gauge theories

Strings 2011

Uppsala, Sweden, June 29

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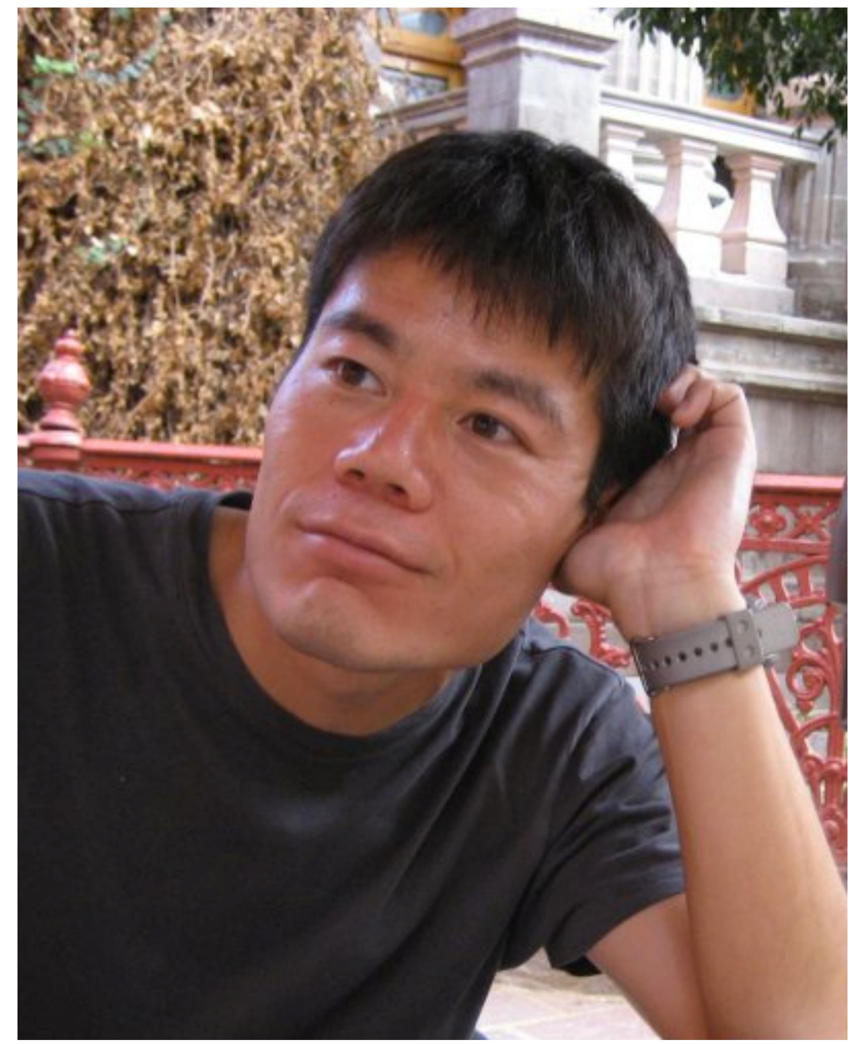
Based on
arXiv:1105.2568 with
J. Gomis and T. Okuda



Loop operators in 4d gauge theories

Wilson loop
electric

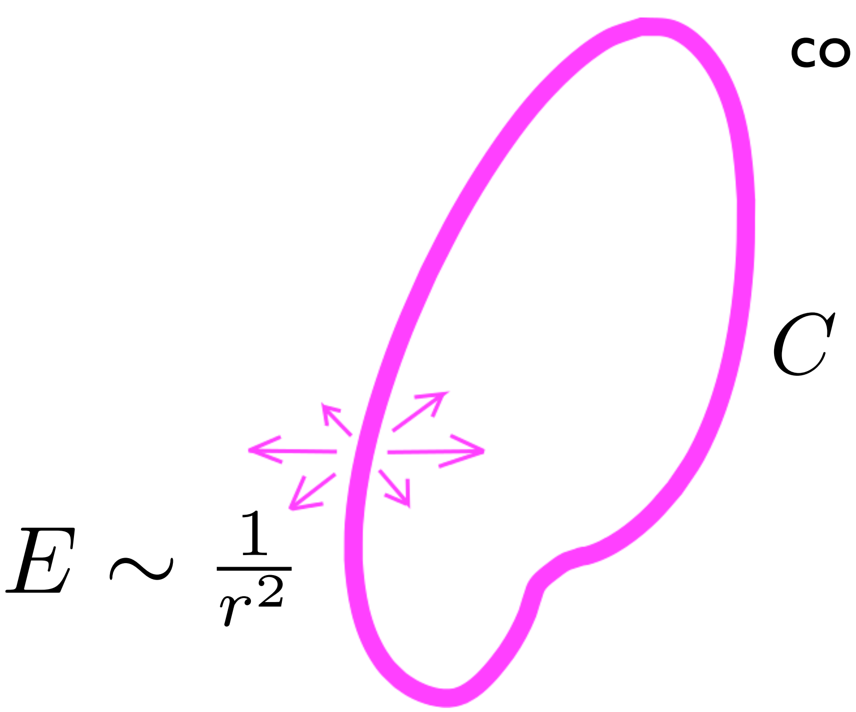
$$\exp \oint_C inA$$

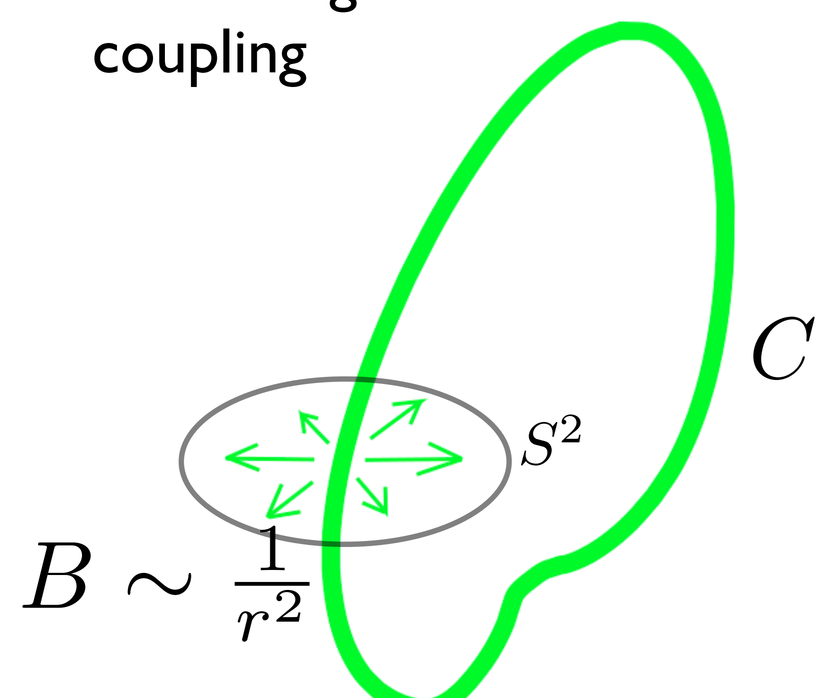
t' Hooft loop
magnetic

$$\int_{S^2} F = 2\pi m$$

EM duality

strong/weak coupling $\longleftrightarrow$ weak/strong coupling





S-duality for $\mathcal{N}=4$ SYM

Montone-Olive '77

$$\mathcal{L} = \frac{1}{2g_{YM}^2}\mathrm{Tr}F\wedge\star F + \frac{i\vartheta}{8\pi^2}\mathrm{Tr}F\wedge F + \dots$$

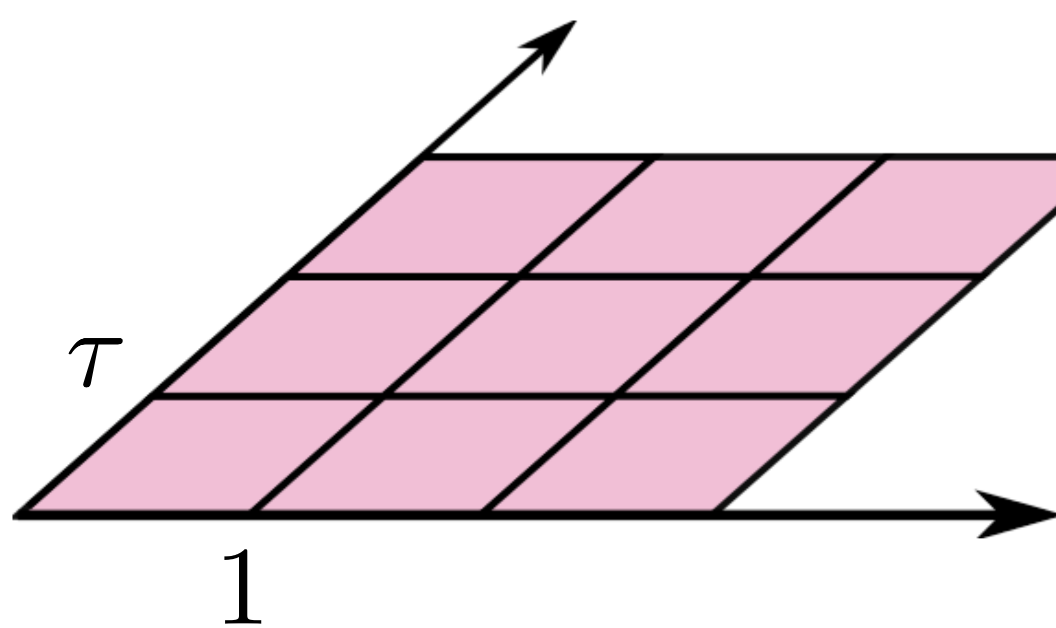
$$\tau = \frac{4\pi i}{g_{\mathrm{YM}}^2} + \frac{\vartheta}{2\pi}$$

EM duality $\quad \tau \leftrightarrow -\frac{1}{\tau}$

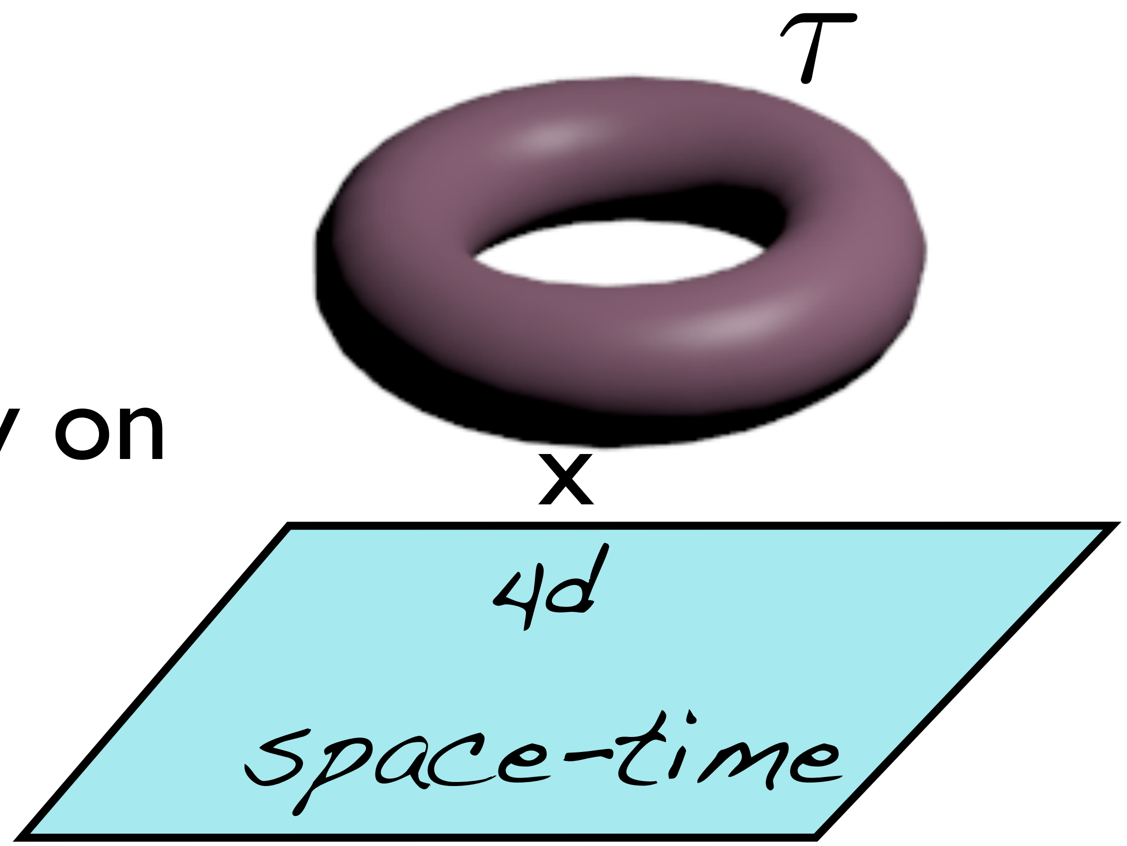
ϑ periodicity $\quad \tau \to \tau + 1$

$\Longrightarrow$ $SL(2,\mathbb{Z})$ symmetry (ADE theories)

Why modular group?



ADE (2,0) theory on



=

= ADE 4d $\mathcal{N} = 4$ SYM with coupling τ

geometric Langlands Kapustin, Witten

arbitrary curve w/ punctures $\Rightarrow$ $\mathcal{N} = 2$ SYM in 4d

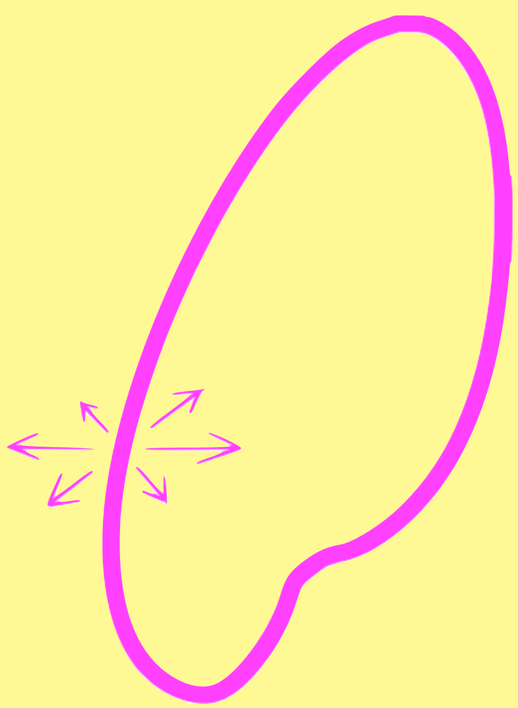
Moore's review talk

Can we make a computation?

(G, τ)

Wilson loop
rep R of G

$\langle W_R \rangle [\tau]$



C

?

$^L\tau = -1/\tau$

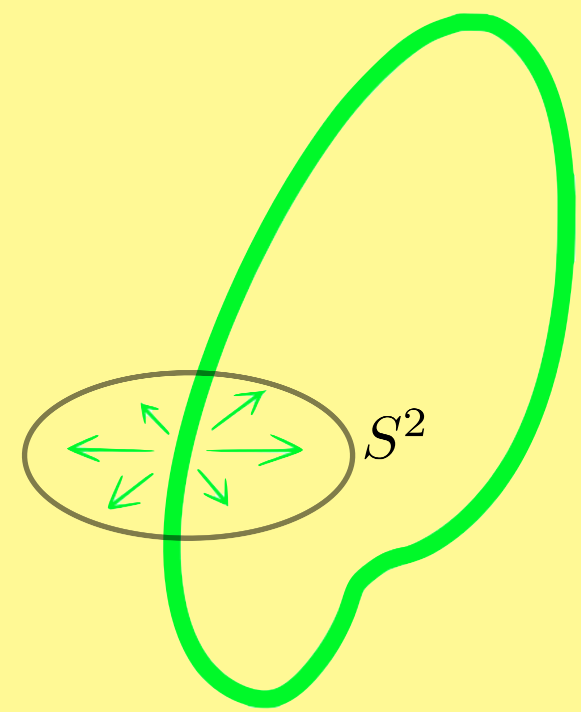
weak - strong

strong - weak

$({}^L G, {}^L \tau)$

't Hooft loop
rep R of G

$\langle T_R \rangle [{}^L \tau]$

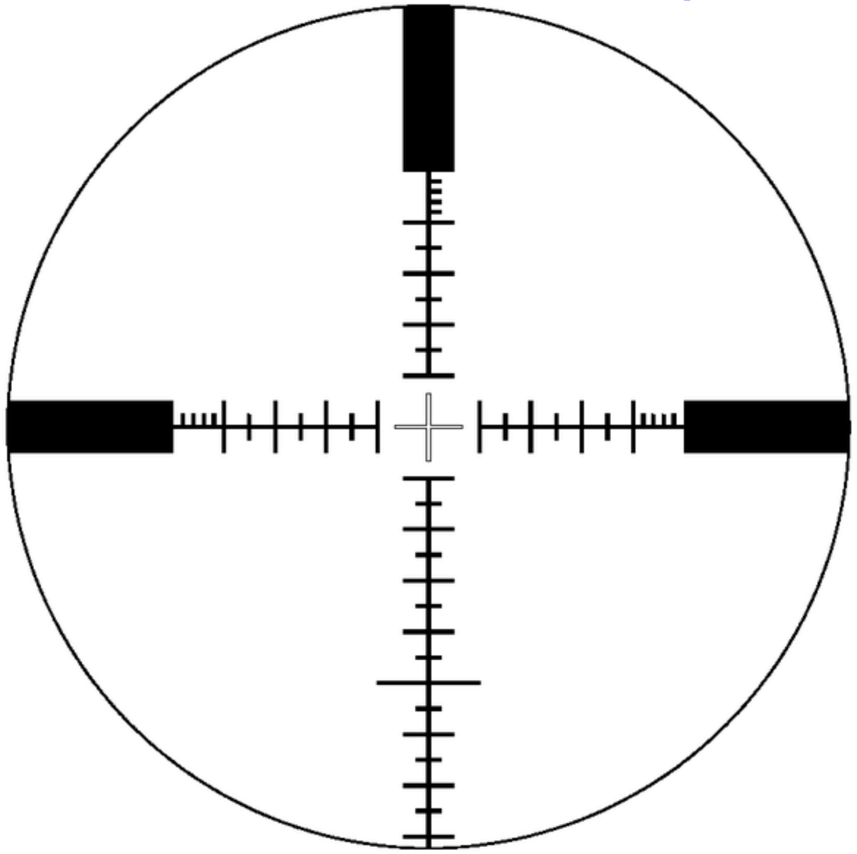


C

WANTED

exact
computation

Atiyah-Bott,Singer,Duistermaat-Heckman,Berline-Vergne,Witten



Localization

compute the integral

$$Z_0 = \int [D\phi]\, e^{-S_0[\phi]}$$

given

$$QS_0 = 0$$

where Q is fermionic symmetry,

and $Q^2 = R$ is bosonic symmetry

Witten

Deform the action

$$S_t[\phi] = S_0[\phi] + t\{Q, V\}$$

assuming

$$\{Q^2, V\} = 0$$

Consider

$$Z_t = \int [D\phi]\, e^{-S_0[\phi] - t\{Q, V[\phi]\}}$$

Integrate by parts

$$\partial_t Z_t = 0$$

Hence

integral over zeroes of QV in 1-loop approximation(easy)

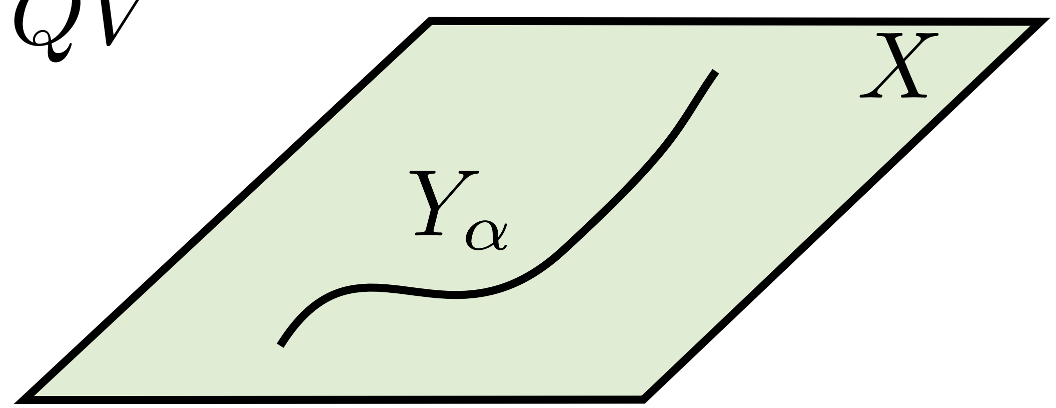
$$Z_0 = Z_\infty$$

original integral (hard)

$$Z = \int_X e^{-S} = \sum_\alpha \int_{Y_\alpha \subset X} e^{-S|_{Y_\alpha}} Z_{1-loop}[N_{Y_\alpha}]$$

X - the space of all fields

Y_α - the zeroes of QV



Take

$$V = (\overline{Q\Psi}, \Psi)$$

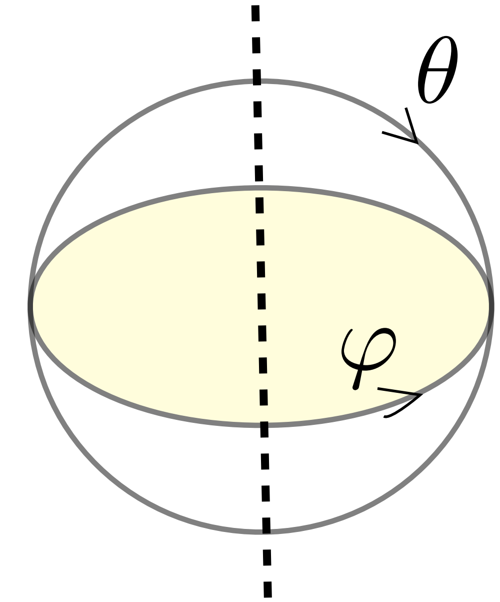
then zeroes of QV

are precisely zeroes of $Q\Psi$

Example

Compute

$$Z(\beta) = \int_{S^2} \exp(\omega + \beta \cos\theta)$$



where

$$\omega = \sin\theta d\theta \wedge d\varphi$$

is the volume form

Straight integration gives

$$Z(\beta) = 2\pi \frac{\exp(\beta) - \exp(-\beta)}{\beta}$$

Localization exercise

Consider the operator

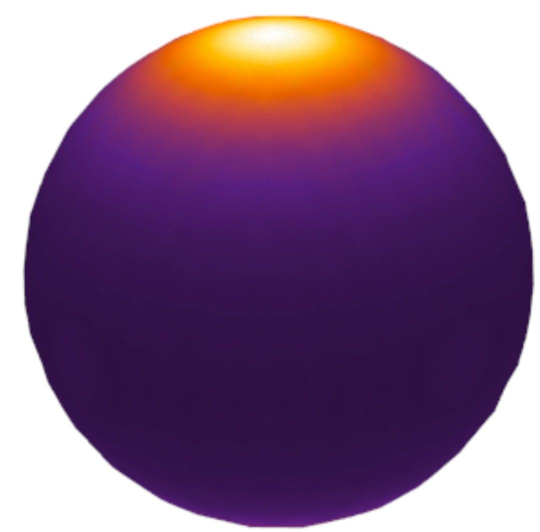
$$Q = d - \beta i_v \qquad (v = \partial_\phi)$$

$$Q^2 = -\beta \mathcal{L}_v$$

Notice that

$$Z(\beta) = \int \exp S \qquad (S = \omega + \beta \cos\theta)$$

with $QS = 0$



Deform the action using

$$V = \sin^2\theta \, d\varphi$$

Then

$$Z(t) = \int \omega(1 + 2t\cos\theta)\exp(\beta\cos\theta - t\sin^2\theta)$$

In the limit $t \to \infty$ the integrand localizes

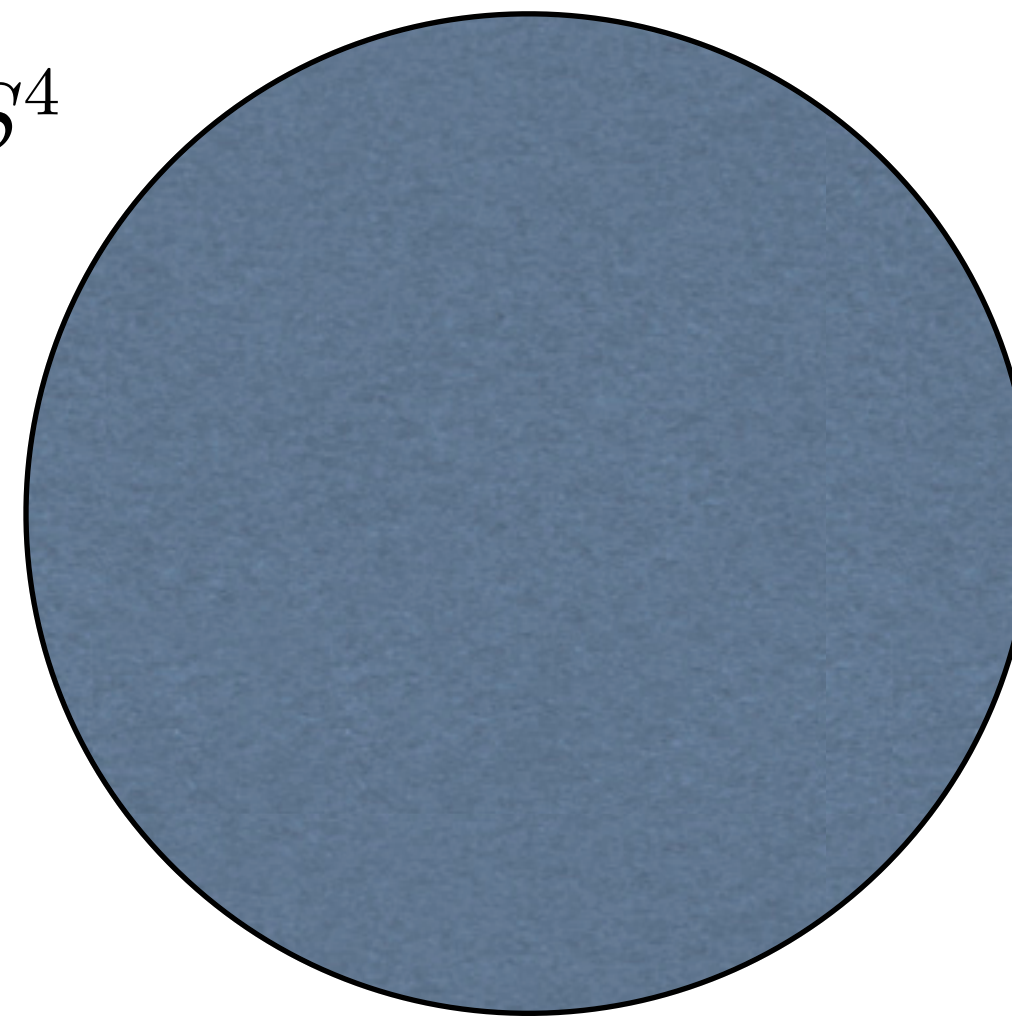
to $\theta = 0$ and $\theta = \pi$

$\mathcal{N} = 2$ supersymmetry on S^4

$$OSp(2|4) \subset SL(1|2, \mathbb{H})$$

- 8 fermionic generators
- $Sp(4) \simeq SO(5)$: isometry of S^4
- $SO(2)$: R-symmetry

Seiberg's talk



$\mathcal{N}=2$ vector multiplet

$$(A_\mu, \Phi_0, \Phi_9, \Psi, K_i)$$

$$S_{\text{vect}} = \frac{1}{g_{\text{YM}}^2}\int \sqrt{g}d^4x\left(\frac{1}{2}F^2 + (D\Phi)^2 + \frac{R}{6}\Phi^2 + \frac{1}{2}[\Phi_a, \Phi_b]^2 + K^2 + \Psi D\Psi\right)$$

$\mathcal{N}=2$ hypermultiplet

To introduce hypermultiplet masses in $OSp(2|4)$ theory on S^4:

1. gauge the flavor symmetry
2. give expectation value to Φ_0 in the flavor vector multiplet

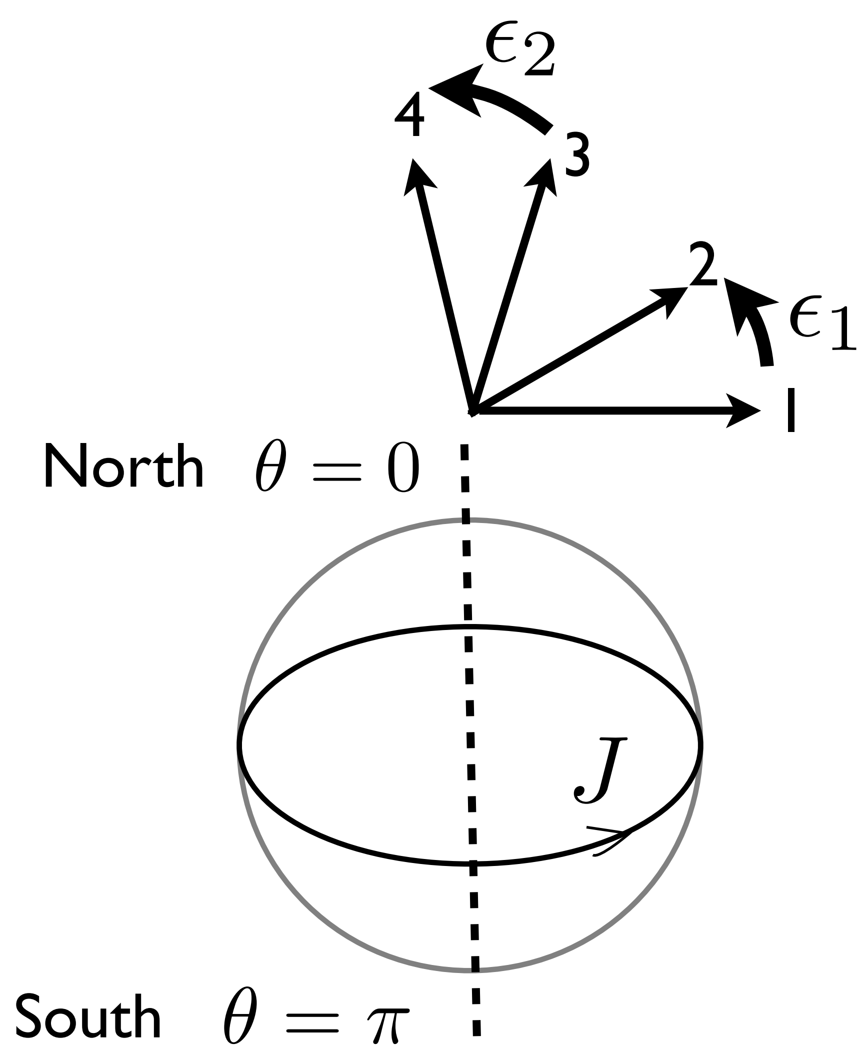
Choice of supercharge Q

$$Q^2 = J + R + G_\phi$$

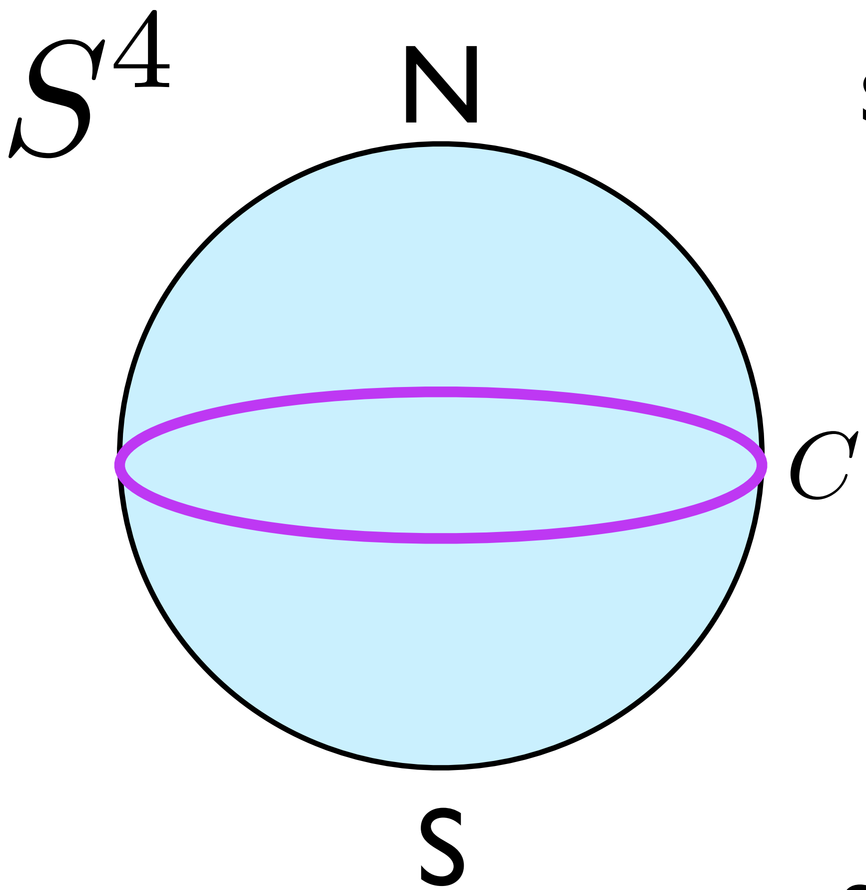
J – space-time rotation

R – R-symmetry

G_ϕ – gauge transformation by $\phi = i\Phi_0 - \cos\theta\Phi_9$



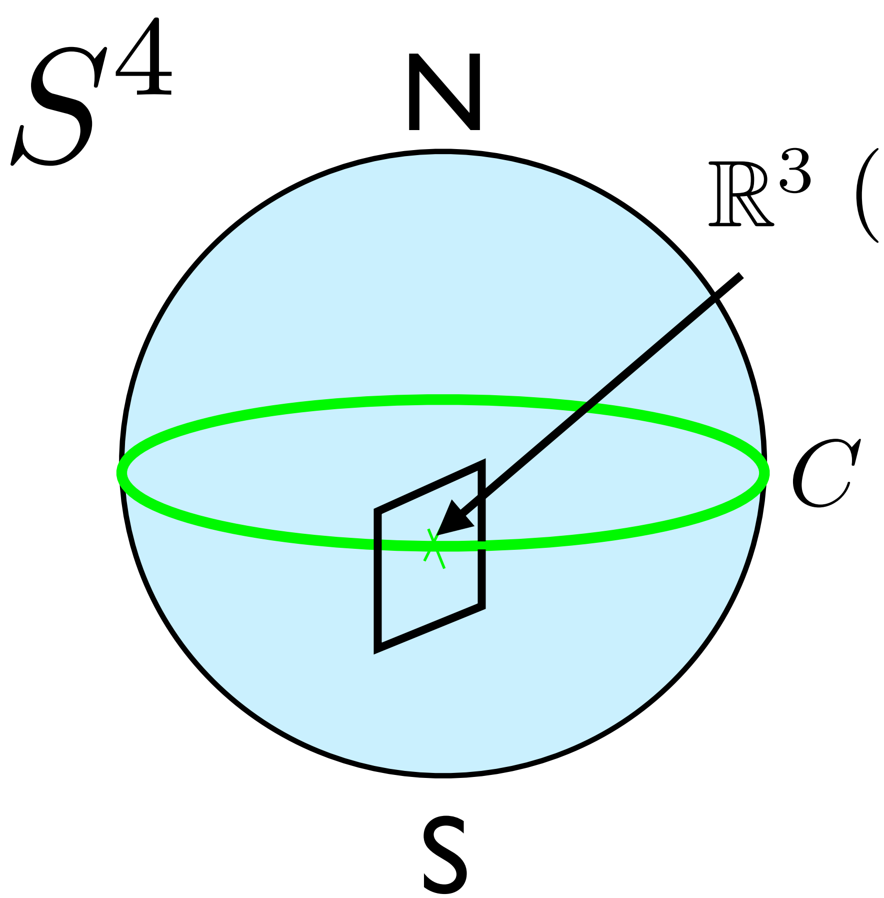
$$\epsilon_1 = \epsilon_2 = 1/r$$



susy Wilson loop

$$W_R(C) = \mathrm{tr}_R \,\mathrm{Pexp} \oint_C (A dx + i\Phi_0 ds)$$

R – rep of G

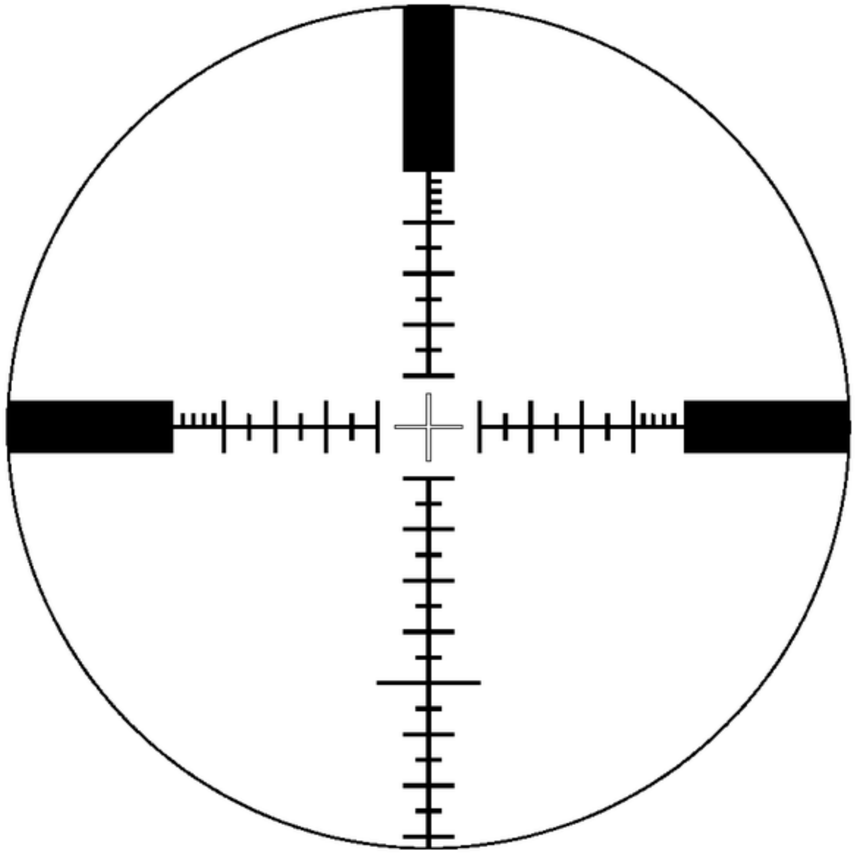


susy 't Hooft loop

$$\mathbb{R}^3\ (x_i) : F_A \to -\star \frac{B}{2}\frac{dx}{x^2} \qquad \Phi_9 \to \frac{B}{2}\frac{1}{x}$$

$B \in \mathfrak{g}$ – weight of $\mathfrak{g}^\vee$

$$D_A \Phi_9 = \star F_A$$



NOW LET'S LOCALIZE

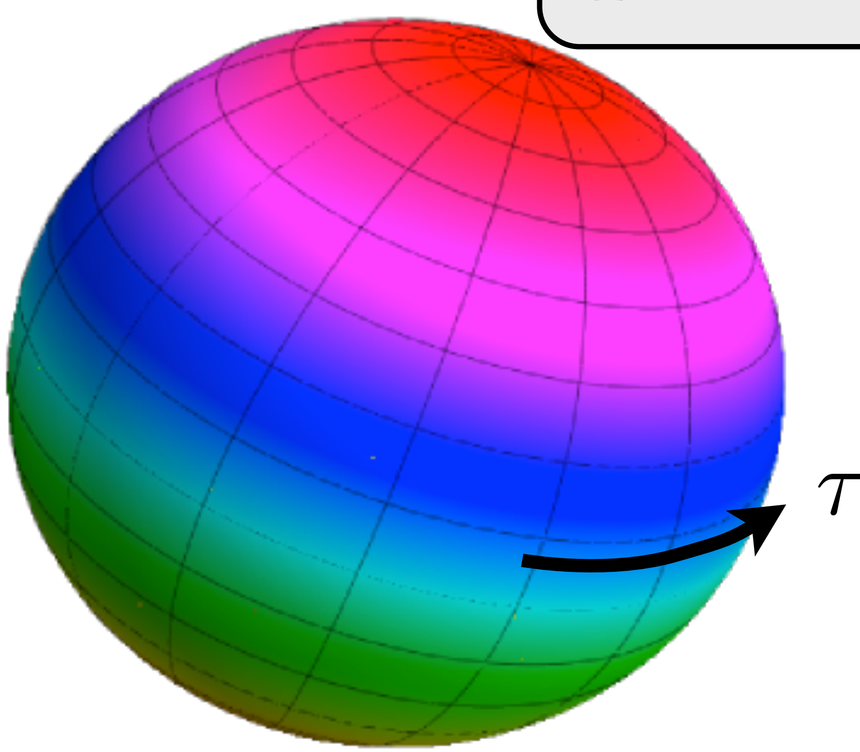
1. Find the localization loci Y_α (solve $Q\Psi = 0$).
2. Compute $\exp(-S|_{Y_\alpha})$
3. Compute the determinant $Z_{1-loop}[N_{Y_\alpha}]$
4. Integrate over Y_α and sum over α

Step 1. Find the localization loci Y_α (solve $Q\Psi = 0$)

$$Q\Psi = \frac{1}{2} F_{mn}\Gamma^{mn}\varepsilon - \frac{1}{2}\phi_a \Gamma^{a\mu}\nabla_\mu \varepsilon + iK_i \Gamma_{8i+4}\varepsilon$$

ε – spinor on S^4 defining Q

$$\boxed{F^+ = 0, \quad D\Phi_9 = 0}$$



$$\boxed{D\Phi_9 = \star_3 F, \quad [D_\tau, \cdot] = 0}$$

$$\boxed{F^- = 0, \quad D\Phi_9 = 0}$$

Vanishing theorem

the only possible solutions to the susy equations smooth everywhere except at the 't Hooft loop singularity are given by

$$Y_0 = \boxed{\begin{aligned} & A_\mu = A_\mu^{\mathrm{bg}} \\ & \Phi_9 = \Phi_9^{\mathrm{bg}} \\ & K_i = 0, \quad i = 1, 2 \\ & \Phi_0 = \Phi_0^{\mathrm{bg}} + a \\ & K_3 = -a \\ & \qquad\qquad\qquad a \in \mathfrak{h} \end{aligned}}$$

The original infinite dimensional path integral localizes to finite (rank G) dimensional integral

$$Z = \int [DA\, D\Phi \ldots] e^{-S} = \int_{\mathfrak{h}} [da] e^{-S(a)} Z_{1-loop}(a)$$

+ *non-perturbative corrections from other loci* Y_α

Step 2. Compute $e^{-S|_{Y_0}}$

't Hooft loop:

$$S(a) = -\frac{8\pi^2}{g^2}\mathrm{Tr}a^2 + \left(\frac{2\pi^2}{g^2} + \frac{g^2\vartheta^2}{32\pi^2}\right)\mathrm{Tr}B^2$$

rewrite:

$$S(a) = -\pi i\tau \mathrm{Tr}\hat{a}_N^2 + \pi i\bar{\tau}\mathrm{Tr}\hat{a}_S^2$$

$$\hat{a}_N = ia - \frac{\epsilon}{2}B$$

$$\hat{a}_S = ia + \frac{\epsilon}{2}B$$

$$Q^2 = J + R + G_\phi$$

$$\phi = i\Phi_0 - \cos\theta\Phi_9$$

$$\Phi_9^{\mathrm{bg}}(N) = \Phi_9^{\mathrm{bg}}(S) = \frac{\epsilon}{2}B$$

hence

$$e^{-S(a)} = e^{\pi i\tau \mathrm{Tr}\hat{a}_N^2}\cdot e^{-\pi i\bar{\tau}\mathrm{Tr}\hat{a}_S^2} =$$

$$= \left|e^{\pi i\tau\mathrm{Tr}\hat{a}_N^2}\right|^2$$

Step 3. Compute $Z_{\text{1-loop}}(a)$

organize fields in Q-multiplets

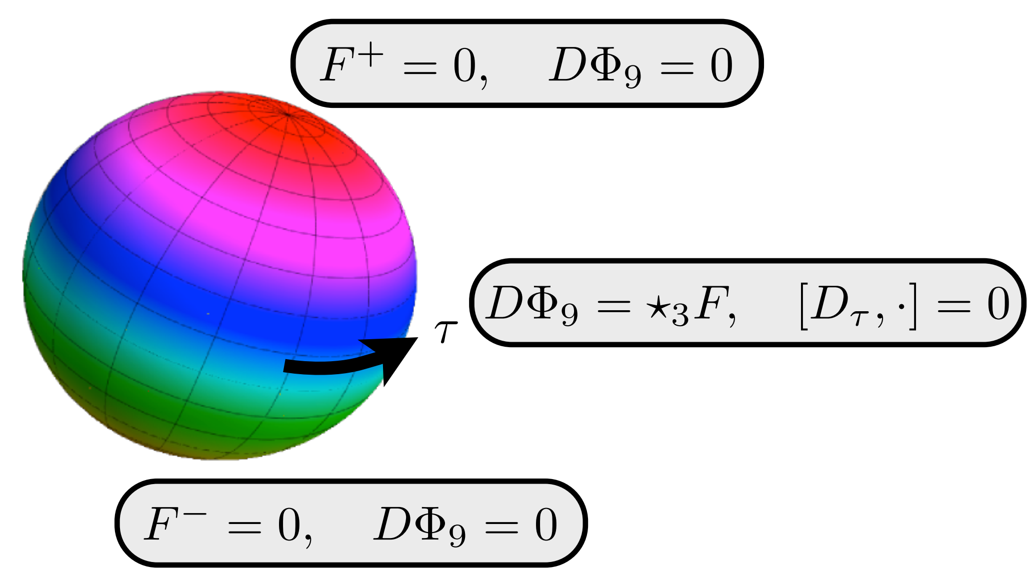
$$Q \cdot \varphi_{\mathsf{b,f}} = \varphi'_{\mathsf{b,f}}$$
$$Q \cdot \varphi'_{\mathsf{b,f}} = \mathcal{R} \cdot \varphi_{\mathsf{b,f}}$$

so $Q^2 = \mathcal{R}$ with $\mathcal{R} = J + R + G_a$

$$Z_{\text{1-loop}}(a) = \int_{\mathcal{N}_{\mathcal{Y}}} e^{tQV} = \frac{\det_{\text{cokerD}} \mathcal{R}}{\det_{\ker D} \mathcal{R}}, \quad t \to \infty$$

D – transverally elliptic differential operator

from the linearized equations



$$D : \Gamma(E_0) \to \Gamma(E_1)$$

$$\Gamma(E_0) = \{\phi_{\rm b}\}$$
$$\Gamma(E_1) = \{\phi'_{\rm f}\}$$

compute

$$Z_{\text{1-loop}}(a) = \frac{\det_{\rm cokerD} \mathcal{R}}{\det_{{\rm ker}D} \mathcal{R}}$$

from the index

$$\mathrm{ind}\, D = \mathrm{tr}_{{\rm Ker}D} e^{\mathcal{R}} - \mathrm{tr}_{{\rm Coker}D} e^{\mathcal{R}}$$

using the rule

$$\sum m_j e^{w_j} \to \prod w_j^{m_j}$$

w_j -weights of R

m_j -multiplicities

To compute $\mathrm{ind}\, D$ we slice sphere into

- neighborhood of the north pole
- neighborhood of the equator
- neighborhood of the south pole

and use Atiyah-Singer index formula

$$\boxed{\mathrm{ind} D(\mathcal{R}) = \sum_{p \in F} \frac{\mathrm{tr}_{E_0(p)} \mathcal{R} - \mathrm{tr}_{E_1(p)} \mathcal{R}}{\det_{TM_p}(1 - \mathcal{R})}}$$

F -set of fixed points of $\mathcal{R}$ action on M

Each pole region gives

$$Z^{\text{pole}}_{\text{1-loop}}(\hat{a}) = \frac{\prod_\alpha \left[G\left(\frac{\alpha\cdot\hat{a}}{\epsilon}\right) G\left(2+\frac{\alpha\cdot\hat{a}}{\epsilon}\right)\right]^{1/2}}{\prod_{f=1}^{N_{\mathrm{F}}} \prod_{w\in R} \left[G\left(1+\frac{w\cdot\hat{a}}{\epsilon}-\frac{im_f}{\epsilon}\right) G\left(1-\frac{w\cdot\hat{a}}{\epsilon}+\frac{im_f}{\epsilon}\right)\right]^{1/2}}$$

where $\quad \hat{a}_N = ia - \frac{\epsilon}{2}B \qquad \hat{a}_S = ia + \frac{\epsilon}{2}B$

Equator region gives

$$Z^{\text{eq}}_{\text{1-loop}}(a) = \frac{\prod_{f=1}^{N_{\mathrm{F}}} \prod_{w\in R} \left[\sin\left(\pi w\cdot\left(\frac{ia}{\epsilon}+\frac{B}{2}\right) - \pi\frac{im_f}{\epsilon}\right)\right]^{|w\cdot B|/2}}{\prod_{\alpha>0}\left[\sin\left(\pi\alpha\cdot\left(\frac{ia}{\epsilon}+\frac{B}{2}\right)\right)\right]^{|\alpha\cdot B|}}$$

R - hypermultiplet rep

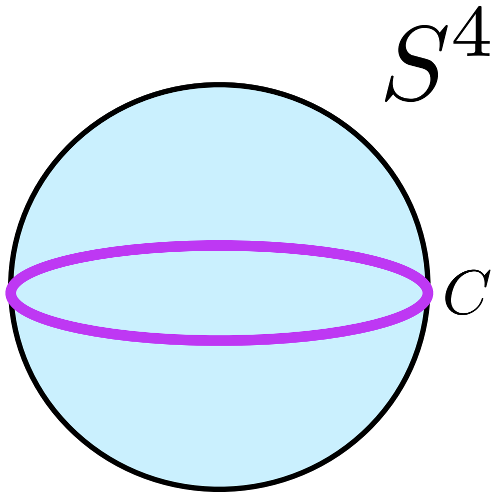
G - Barnes G-function /related to q-dilog/

$$G(1+z) = (2\pi)^{z/2} e^{-((1+\gamma z^2)+z)/2} \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right)^n e^{-z+\frac{z^2}{2n}}$$

Y_0 locus gives exact result up to nonperturbative corrections

$$\langle T_B \rangle \overset{Y_0}{=} \int da \left| Z_N(ia - \frac{\epsilon}{2}B) \right|^2 Z^{\text{eq}}_{\text{1-loop}}(a)$$

$$\langle W_R \rangle \overset{Y_0}{=} \int da \, |Z_N(ia)|^2 \, \text{tr}_R e^{2\pi i a}$$



where

$$Z_N(\hat{a}) = e^{\pi i \tau \hat{a}^2} Z^N_{\text{1-loop}}(\hat{a})$$

notice connection with Nekrasov's Z

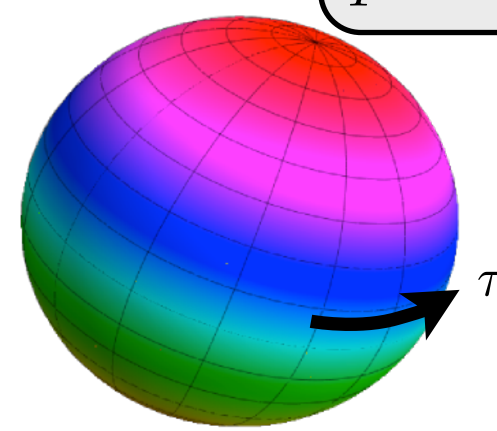
$$Z_N(\hat{a}) = \left[Z^{\Omega\text{-bg}}_{\mathbb{R}^4_{\epsilon_1,\epsilon_2}} \left(\hat{a}; \epsilon_1 = \tfrac{1}{r}, \epsilon_2 = \tfrac{1}{r}\right) \right]^{\text{pert}}$$

Step 4. Non-perturbative corrections

(from higher Y_α loci)

- point instantons at North pole
- point monopoles at the equator
- point anti-instantons at South pole

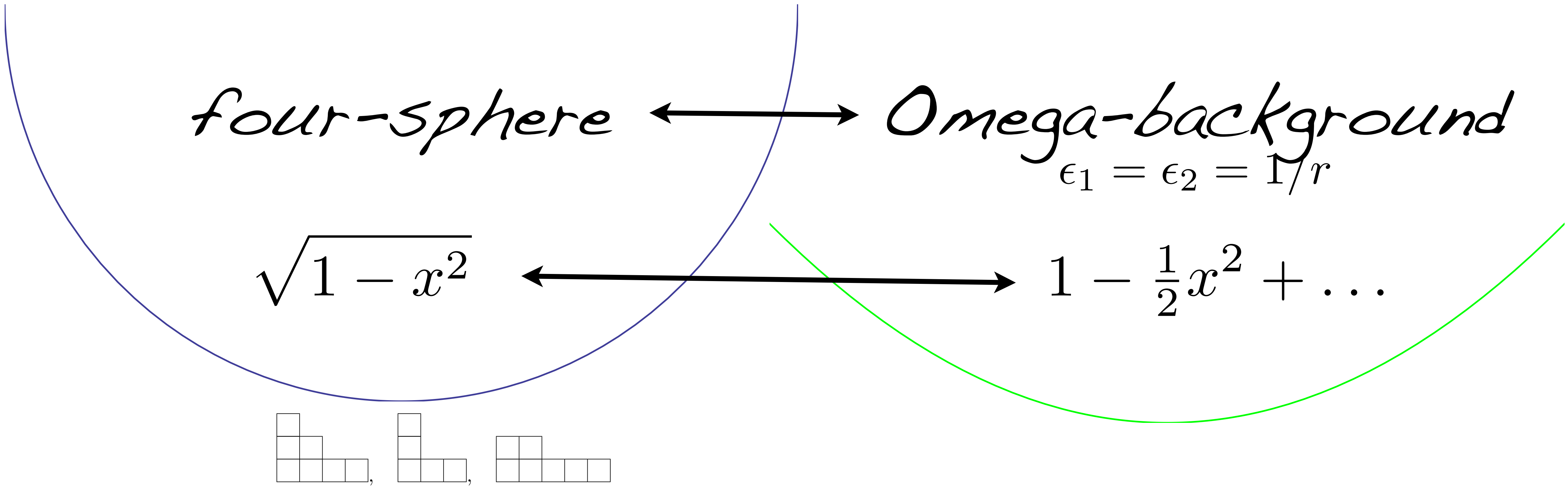
$$F^+ = 0, \quad D\Phi_9 = 0$$



$$D\Phi_9 = \star_3 F, \quad [D_\tau, \cdot] = 0$$

$$F^- = 0, \quad D\Phi_9 = 0$$

Instanton corrections



gauge theory in Ω-background /Losev, Moore, Nekrasov, Shatashvili /
approximates OSp(2|4) theory up to $O(x^2)$ at $x = 0$

point instantons at x=0
contribute in the same way

with instanton corrections the result is

$$\langle T_B\rangle = \int da \left| Z_N(ia - \frac{\epsilon}{2}B)\right|^2 Z^{\mathrm{eq}}_{\text{1-loop}}(a)$$

$$\langle W_R\rangle = \int da\, |Z_N(ia)|^2\, \mathrm{tr} e^{2\pi i a}$$

$$Z_N(\hat{a}) = e^{\pi i \tau \hat{a}^2} Z^N_{\text{1-loop}}(\hat{a}) Z_{\mathrm{inst}}(\hat{a}|\epsilon_1 = \epsilon_2 = r^{-1}; im)$$

if B is miniscule rep, the result is final

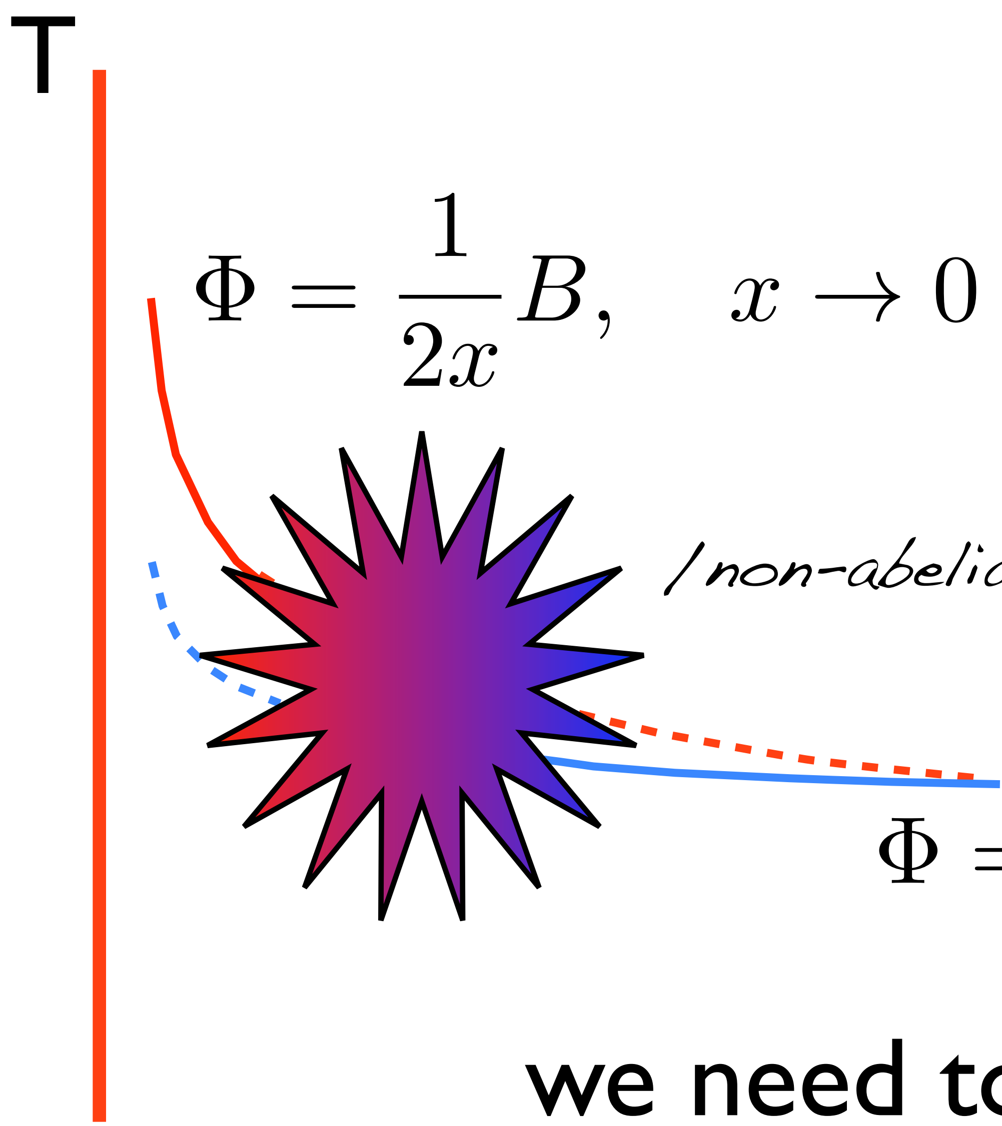
if not, expect screening corrections

Monopole screening

Kronheimer
Kapustin, Witten
Cherkis, Durkan

Non-abelian monopoles can screen the singularity and reduce effective magnetic charge seen at infinity

$$\boxed{D_A\Phi = \star F_A}$$



$$\Phi = \frac{1}{2x}B, \qquad x \to 0$$ /abelian/

$$B' \subset \mathrm{rep}(B)$$

$$\Phi = \frac{1}{2x}B', \qquad x \to \infty$$ /abelian/

we need to localize on moduli space $\mathcal{M}(B, B')$

The final exact result

$$\langle T_B \rangle = \int da \sum_{B' \in \mathrm{rep}(B)} \left| Z_N(ia - \frac{\epsilon}{2} B') \right|^2 Z^{\mathrm{eq}}_{\text{1-loop}}(a; B, B')$$

Gomis, Okuda, V.P. '11

$$\langle W_R \rangle = \int da \, |Z_N(ia)|^2 \sum_{R' \in \mathrm{rep}(R)} e^{2\pi i R' \cdot a}$$

V.P. '07

- S-duality for SU(2) N=2* checked /numerically/
- results agree with conjectures for AGT dual loop observables in Liouville (Toda) theories

Drukker, Gomis, Okuda, Teschner'09,
Alday, Gaiotto, Gukov, Tachikawa, Verlinde'09
Gomis, Floch'10 .

Summary/Discussion

- Exact results for vevs of susy 't Hooft and Wilson loops in OSp(2|4) theories: all perturbative and non-perturbative corrections.
- Agreement with conjectured loop observables in AGT dual Liouville/Toda CFTs.
- Precision S-duality test
- similar localization techniques are used
  - more complicated loops in N=4 SYM V.P., Giombi
  - $S^3 \times S^1, S^3, S^2 \times S^1$ Seiberg's review talk
  - gravity/black holes calculations OSV, Dabholkar-Gomes-Murthy-Sen

Thank you!