## A Cardy-like Formula in D = 4

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with Lorenzo Di Pietro, in progress

## Introduction

Given a  $CFT_{D=d+1}$  we can study

$$Z(\beta) \equiv \sum_{ops.} \exp(-\beta \Delta) \; .$$

We can represent this as a path integral over the cylinder  $S^d \times S^1_\beta$ , with anti-periodic boundary conditions for fermions.

$$Z(eta) = \int [\mathcal{D}X] \exp\left(-\int_{S^d imes S^1_eta} \mathcal{L}(X)
ight)$$

## Introduction

We expect that

$$Z(eta 
ightarrow 0) \sim \exp\left(\kappa rac{Vol(S^d)}{eta^d}
ight)$$

In D = 2 it follows from modular invariance that

$$\kappa = \frac{\pi}{12}(c_L + c_R)$$

In D = 4,  $\kappa$  depends on exactly marginal parameters, so such a simple formula for  $\kappa$  is impossible.

We will combine methods and ideas from **hydrodynamics** and from **supersymmetry** in order to understand the  $\beta \rightarrow 0$  limit of some partition functions in D = 4.

SUSY partition functions on  $\mathcal{M}_3 imes S^1$  compute

$$Z(\beta) = \sum_{\mathcal{H}} \exp(-\beta H) (-1)^F$$
,

with  $\mathcal{H}$  the Hilbert space on  $\mathcal{M}_3$  and H the generator of translations of the circle.

The limit  $Z(\beta \to 0)$  of the SUSY partition function **does not contain** the volume term  $\exp\left(\kappa \frac{Vol(S^d)}{\beta^d}\right)$ , i.e.  $\kappa = 0$ . This is essentially because this term originates from a cosmological constant, which vanishes in SUSY theories.

What is the leading term then?

We will show that the leading asymptotic behavior is of the type

$$Z(eta o 0) \sim \exp\left(\kappa' rac{L(\mathcal{M}_3)}{eta}
ight) \; ,$$

where  $L(\mathcal{M}_3)$  is a length scale of  $\mathcal{M}_3$  that I will explain how to compute. We will see that for SCFTs

$$\kappa' \sim c - a$$

### Consider a QFT with a U(1) symmetry

$$\partial^{\mu}j_{\mu}=0$$

#### and introduce temperature

$$T \equiv 1/\beta \equiv 1/2\pi r$$

Introduce a background metric  $g_{\mu\nu}$  and a background gauge field  $A_{\mu}$ .

Below the KK scale 1/r, the theory on  $\mathcal{M}_3 \times S^1$ reduces to a **local** theory on  $\mathcal{M}_3$ . The effective action depends on

$$(A_0, A_i; g_{00}, a_i, g_{ij}^{(3)}; T)$$

With zero derivatives we have

$$\int d^3x \sqrt{g^{(3)}} \mathcal{F}(A_0;T)$$
 .

With one derivative we have Chern-Simons terms of the type

$$i\int \left(rac{c_1}{T}A_0A\wedge dA+rac{c_2}{T}A_0^2A\wedge da+c_3TA\wedge da
ight)$$

and there is one interesting Chern-Simons term with three derivatives

$$i\int c_4A\wedge R^{(3)ij}f_{ij}$$

$$i\int \left(\frac{c_1}{T}A_0A \wedge dA + \frac{c_2}{T}A_0^2A \wedge da + c_3TA \wedge da\right)$$

The field-dependent CS terms  $c_{1,2}$  are non-gauge invariant. Their coefficients are fixed to reproduce the four-dimensional anomaly

$$\delta_{\alpha}W = -\frac{iTr(U(1)^3)}{24\pi^2}\int \alpha F \wedge F$$

[Son et al, Banerjee et al.]

$$i\int c_4A\wedge R^{(3)ij}f_{ij}$$

This is again non-gauge invariant. It is fixed to reproduce the U(1) gravitational anomaly

$$\delta_{\alpha}W = -\frac{iTr(U(1))}{192\pi^2}\int \alpha Tr(R \wedge R)$$

An equivalent expression was found by Jensen et al.

# Thermal Field Theory

The Chern-Simons term

$$ic_3T\int A\wedge da$$

is gauge invariant under small transformations. Computing  $c_3$  in examples, *Landsteiner et al.* were led to the conjecture

$$c_3=-\frac{1}{48\pi}Tr(U(1))$$

The conjecture has been generalized to other dimensions by *Loganayagam*.

Example: A massless D = 4 chiral fermion. On  $\mathbb{R}^3$  we have an infinite tower of massive fermions with charges (n, 1) under (a, A). Integrating out the n-th state gives

$$-rac{i}{4\pi}n \, sgn(n)\int d^3x A\wedge da \;, \qquad n\in\mathbb{Z}+1/2$$

Summing over n,

$$\sum_{\mathsf{n}\in\mathbb{Z}+1/2}|\mathsf{n}|=1/12$$

See Golkar et al. for a similar approach.

For Lagrangian theories,  $c_3 = -\frac{1}{48\pi}Tr(U(1))$  is easy to establish:

- c<sub>3</sub> cannot depend on continuous coupling constants (for otherwise, promoting them to continuous functions, we would violate gauge invariance – see *Closset et al.*)
- 2. Therefore, we tune the couplings to the free-field point. By anomaly matching,  $c_3 = -\frac{1}{48\pi}Tr(U(1))$  is thus true at all values of the couplings.

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Assuming the smoothness of the path integral in some singular geometries, *Jensen et al.* have been able to derive the conjectured relation. The difficulties arising when considering QFT in singular geometries include localized/delocalized states, decoupling, etc.

I hope to have time to explain a new approach to the problem of proving  $c_3 = -\frac{1}{48\pi}Tr(U(1))$  that bypasses these issues.

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The discussion so far concerned with thermal field theories, but it also has applications for supersymmetric compactifications on  $\mathcal{M}_3 \times S^1$ .

The main novelty is that now we have a massless sector on  $\mathcal{M}_3$ , so the full effective action is nonlocal.

However, let us ignore the nonlocality for a second and supersymmetrize the  $c_3$  contact term in D = 3 $\mathcal{N} = 2$  supergravity. One finds

$$\frac{\operatorname{Tr}(U(1)_R)}{24\beta}\int \left(\frac{1}{4}R^{(3)}-\frac{1}{2}H^2-(da)^2+iA^{(R)}\wedge da\right)$$

where  $A^{(R)}$  is the *R*-symmetry gauge field and *H* is a field in the supergravity multiplet.

A physicist studying D = 3 SUSY theories on  $\mathcal{M}_3$  would have said that

$$\int \left(\frac{1}{4}R^{(3)}-\frac{1}{2}H^2-(da)^2+A^{(R)}\wedge da\right)$$

is the counter-term he/she needs to add to cancel an unphysical linear divergence.

But since our theory is really four-dimensional, this term is **calculable** via the rules we explained. Its coefficient is linear in  $T = 1/2\pi r$  and proportional to the  $Tr(U(1)_R)$  anomaly.

We can now evaluate the SUSY contact term on any admissible  $\mathcal{M}_3$  (any Seifert manifold is admissible – see *Klare et al., Closset et al.*). This shows that for  $\beta \rightarrow 0$ 

$$\sum_{\mathcal{H}} \exp\left(-\beta H\right) \left(-1\right)^{\mathsf{F}} \to \exp\left(-\frac{Tr(U(1)_{\mathsf{R}})}{6\beta}L\right) \;,$$

with L a length scale that is calculated by evaluating the contact term on  $\mathcal{M}_3$ ,  $L \sim \int d^3x \sqrt{g^{(3)}}R + ...$ 

The special case  $\mathcal{M}_3 = S^3$  is of interest. In this case, for the conformally coupled theory, we find for  $\beta \to 0$ 

$$\sum_{short-reps} \exp\left(-eta \left(\Delta + rac{1}{2}R
ight)
ight)(-1)^F \ \longrightarrow \exp\left(-rac{16\pi^2(a-c)}{3eta}R_{S^3}
ight) \ .$$

This is the asymptotic Cardy-like behavior of the superconformal index of *Kinney et al.* 

The asymptotic formula has a simple generalization to the situation when we add a chemical potential for angular momentum.

Chemical potential for angular momentum corresponds to  $\mathcal{M}_3 = S_b^3$ , with b a squashing parameter. We find in this case a similar asymptotic formula with  $R_{S^3} \rightarrow \frac{1}{2}R_{S^3}(b+b^{-1})$ .

# $\mathcal{N} = 1$ SUSY in D = 4

- *a c* is computable from the spectrum of short representations of the superconformal group.
- This is consistent with one-loop  $g_s \sim 1/N^2$  corrections to a = c in  $AdS_5$ , see e.g. Arabi Ardehali et al.
- In many specific examples where the superconformal index is known, one can verify that our asymptotic formula is correct (compare e.g. with *Imamura*, *Niarchos*, *Spiridonov et al.*, *Aharony et al.*).

# $\mathcal{N}=1$ SUSY in $\mathit{D}=4$

- In general, infinitely many short representations with F = ±1.
- If c a > 0, then  $\infty_B \infty_F = \pm \infty$ , which is what we would expect generically.
- If a = c then  $\infty_B \infty_F = finite$ . This is the case of  $\mathcal{N} = 4$  and minimal supergravity in  $AdS_5$ .
- If c a < 0, then  $\infty_B \infty_F = 0$ , which naively seems like an unlikely accident. This is consistent with the fact that models with c - a < 0 are extremely rare.

- If a c ≠ 0, then a modification to the spectrum of pure bulk Einstein super-gravity is in order. Perhaps can be directly related to Camanho, Edelstein, Maldacena, Zhiboedov.
- (Subtlety: We have assumed above that the partition function of the massless sector on S<sup>3</sup> is finite. This is the case in most of the interesting examples, but not all.)

# Related Open Problems

- Subleading terms in the expansion in  $1/R_{S^3}T$ . Likely to contain (all?) the other anomalies...
- Generalization to SUSY on  $\mathcal{M}_5 \times S^1$  should provide a new way to extract anomalies in 6d!
- Can one show that c − a ≥ 0 under some circumstances?
- The connection between the S<sup>3</sup> × S<sup>1</sup> partition function and the superconformal index appears to be possibly subtle, as recently pointed out by Assel et al. This needs to be understood.

### Many More ...

### Thank you for the attention!!

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We emphasize that we are discussing a physical property of the partition function. Various other connections between the superconformal index and anomalies were discussed under the names 'SL(3, Z)' and 'total ellipticity' (Spiridonov et al.). Typically, these are interesting properties of the Coloumb branch integrand. There is no known analog of these properties for the integrated expression.

## An Outline of an Approach to $c_3$

The  $S^3 \times S^1_\beta$  partition function with chemical potential  $A = \frac{\mu}{T} d\theta$  is real by reflection positivity.

We can reduce on the Hopf fiber instead of  $S^1_{\beta}$ :

$$S^1_{Hopf} o S^3 o S^2$$

In that case

$$\mathcal{M}_3 = S^2 imes S^1_eta \;, \qquad \int_{S^2} da = 2\pi$$

# An Outline of an Approach to $c_3$

This is rarely a useful reduction to consider, because there is no hierarchy of scales between  $S^2$  and  $S^1_{Hopf}$ . But since the effective action has only **finitely many** imaginary terms that could contribute, it is under control.

# An Outline of an Approach to $c_3$

One finds that the two CS terms

$$ic_3 T \int d^3x \sqrt{g^{(3)}} A \wedge da + ic_4 \int d^3x \sqrt{g^{(3)}} c_4 A \wedge R^{(3)ij} f_{ij}$$

are activated, and they must produce a result consistent with reflection positivity.