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Quantum Entanglement and Local Operators

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arXiv:1403.0702, arXiv:1405.5946
(see also arXiv: arXiv:1302.5703 [JHEP05(2013)080])

with Pawel Caputa (YITP), Song He (YITP), Masahiro Nozaki (YITP)
Tokiro Numasawa (YITP) and Kento Watanabe (YITP).

① Introduction

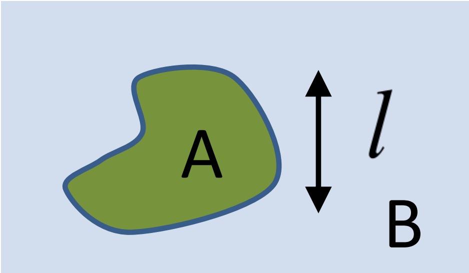
In **QFTs**, the entanglement entropy (EE) provides us a **universal physical quantity (~order parameter)**.

For example, we can characterize the degrees of freedom of CFTs (~central charges) from the EE for ground states.

(i) 2d CFT $$S_A = \frac{c}{3}\log\frac{l}{\varepsilon}.$$ [Holzhey-Larsen-Wilczek 94, Calabrese-Cardy 04,..]

(ii) 3d CFT $$S_{A(=S^1)} = \gamma\cdot\frac{l}{\varepsilon} - F.$$ [F-th: Jafferis-Klebanov-Pufu-Safdi 11, Entropic proof: Casini-Huerta 12]

(iii) 4d CFT $$S_{A(=S^2)} = \gamma\cdot\frac{l^2}{\varepsilon^2} - 4a\cdot\log\frac{l}{\varepsilon} + s.$$ [Ryu-TT 06, Solodukhin 08, Sinha-Myers 10, Casini-Huerta-Myers 11,…]



$$H_{tot} = H_A \otimes H_B \quad .$$

$$\rho_A = \mathrm{Tr}_B |\Psi\rangle\langle\Psi| \ .$$

It is also helpful to look at **(n-th) Renyi entanglement entropy (REE)** which generalizes the EE :

$$S_A^{(n)} = \frac{1}{1-n} \cdot \log \mathrm{Tr}\left[(\rho_A)^n\right].$$

$$\lim_{n\to 1} S_A^{(n)} = -\mathrm{Tr}[\rho_A \log \rho_A] = S_A \quad . \qquad (\mathrm{Tr}[\rho_A] = 1).$$

If we know all of $S_A^{(n)}$, we find all eigenvalues of ρ_A.

(so called entanglement spectrum)

In **gravity**, we might expect that quantum entanglement gives a quantum bit of spacetime (~ a plank size unit) .

(i) BH entropy

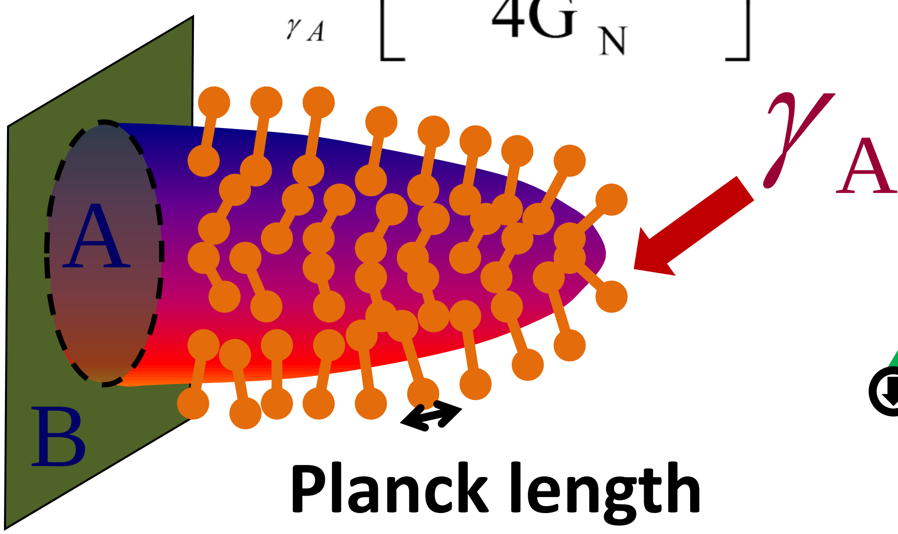
[Bekenstein 73, Hawking 75,...]

$$S_{BH} = \frac{\text{Area}\,(\Sigma)}{4G_N} \approx \frac{\text{Area}\,(\Sigma)}{(l_p)^{D-2}}.$$

(ii) Holographic EE (HEE)

[Ryu-TT 06, Hubeny-Rangamani-TT 07,...]

$$S_A = \underset{\gamma_A}{\text{Min}}\left[\frac{\text{Area}\,(\gamma_A)}{4G_N}\right]$$

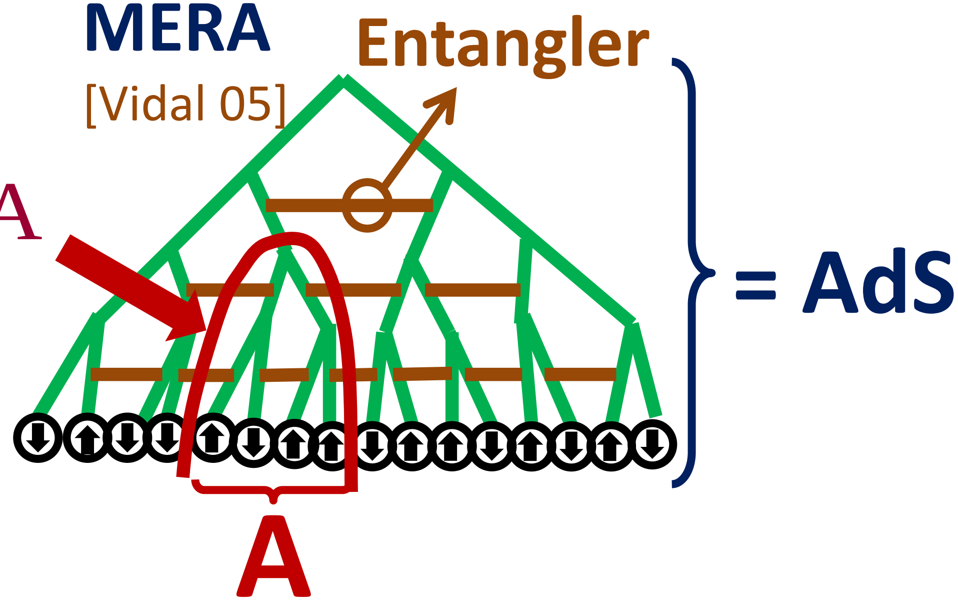


(iii) Entanglement/Gravity

[Swingle 09, Raamsdonk 09, Myers 12, ...]

MERA

[Vidal 05]



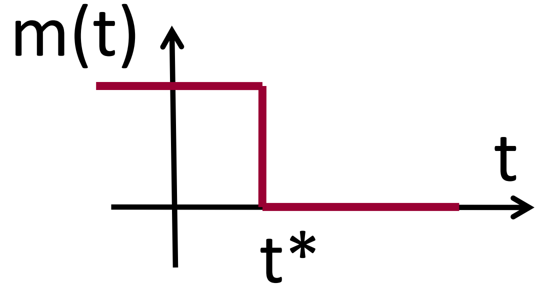
The entanglement entropy is also a useful quantity to characterize **excited states**.

Well-studied examples are quantum quenches:

[Calabrese-Cardy 05, 07, ..., Liu's talk]

(a) Global quantum quenches

$$S_A \propto c \cdot (t/\varepsilon) \quad .$$



(b) Local quantum quenches

$$S_A^{2d} \propto c \cdot \log(t/\varepsilon) \; .$$

Here we want to focus on more elementary excited states:

(c) Local operator insertions at a time

$\Rightarrow \quad S_A = ?$ (The main aim of this talk)

Consider excited states defined by local operators:

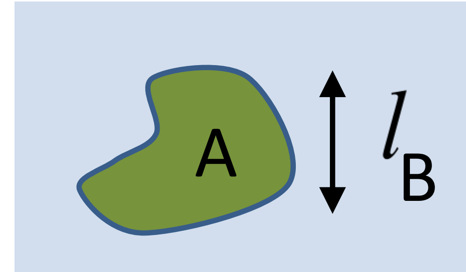
$$\left|O(x)\right\rangle \equiv O(x)\left|0\right\rangle.$$

We study $\boxed{\Delta S_A^{(n)} \equiv S_A^{(n)}\left[\left|O(x)\right\rangle\right] - S_A^{(n)}\left[\left|0\right\rangle\right].}$

$S_A^{(n)}$ ~ Loss of information when we assume that the region B is invisible.

~ ``degrees of freedom'' of the operator O.

Two limits



(1) $l \to 0$ limit ($\approx$ small energy limit)

In this case, we find a property analogous to the first law of thermodynamics:

$$\Delta S_A^{(n)}\left[\left|O\right\rangle\right] \propto \Delta E_A$$

[Bhattacharya-Nozaki-Ugajin-TT 12, Blanco-Casini-Hung-Myers 13, Wong-Klich-Pando Zayas-Vaman 13 …, Raamsdonk's talk]

(2) $l \to \infty$ limit ($\approx$ large energy limit)

This leads to a very `entropic' quantity !

⇒ **The main purpose of this talk.**

[Nozaki-Numasawa-TT 14, He-Numasawa-Watanabe-TT 14, Caputa-Nozaki-TT 14]

Contents

② Replica Calculations of EE for locally excited states

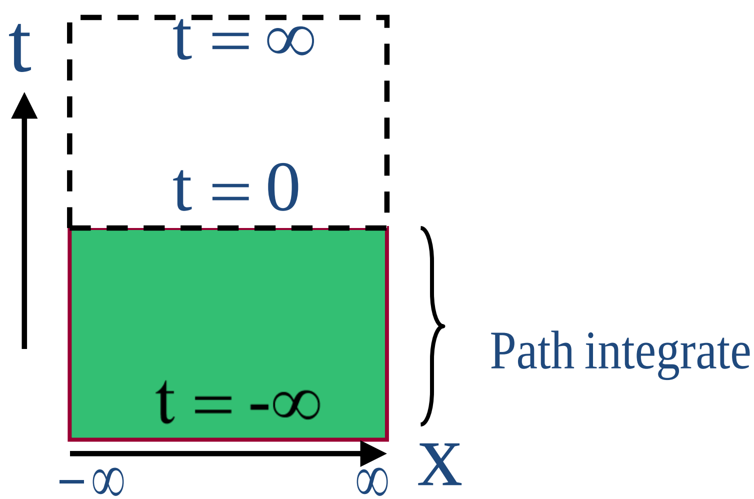
(2-1) Replica method for ground states

A basic method to find EE in QFTs is the **replica method**.

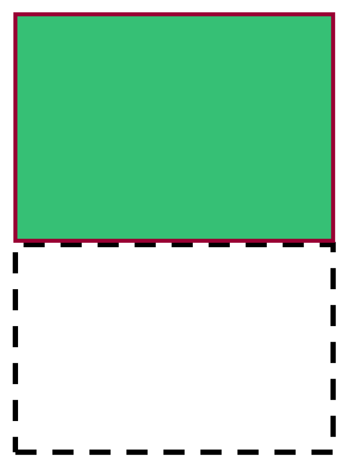
$$S_A = -\frac{\partial}{\partial n}\log \mathrm{Tr}_A\left(\rho_A\right)^n\Big|_{n=1} \quad .$$

In the path-integral formalism, the ground state wave function $|\Psi\rangle$ can be expressed as follows:

$|\Psi\rangle =$

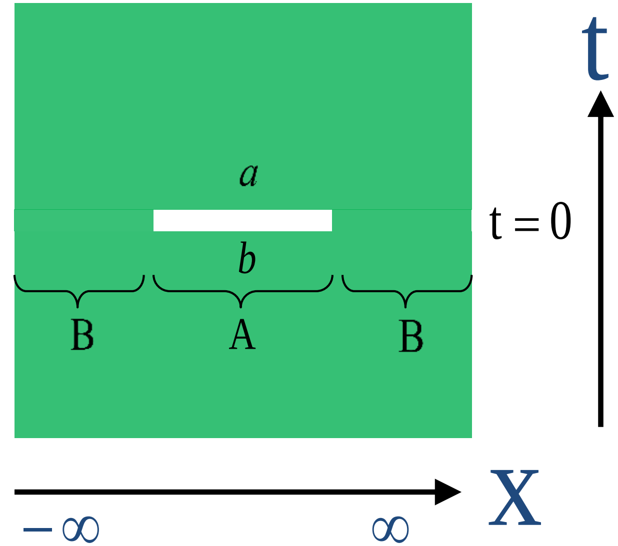


, $\langle\Psi| =$



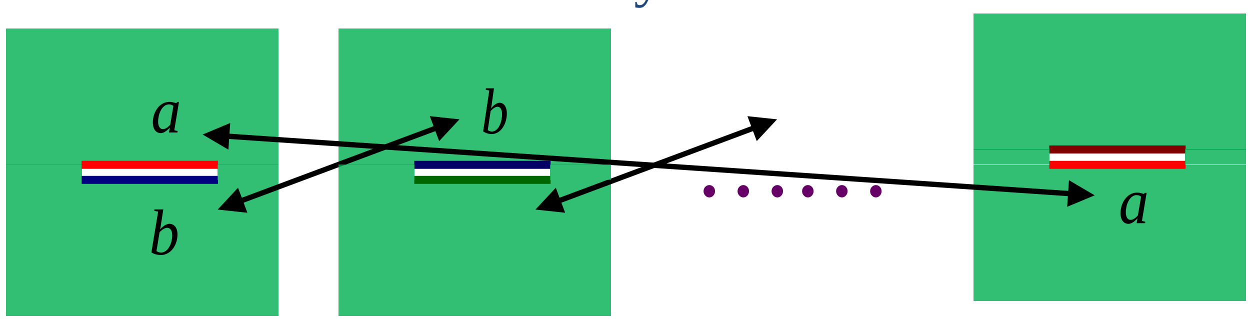
Then we can express

$\rho_A = \mathrm{Tr}_B|\Psi\rangle\langle\Psi|$ as follows: $[\rho_A]_{ab} =$



Glue each boundaries successively.

$\mathrm{Tr}\,(\rho_A)^n =$

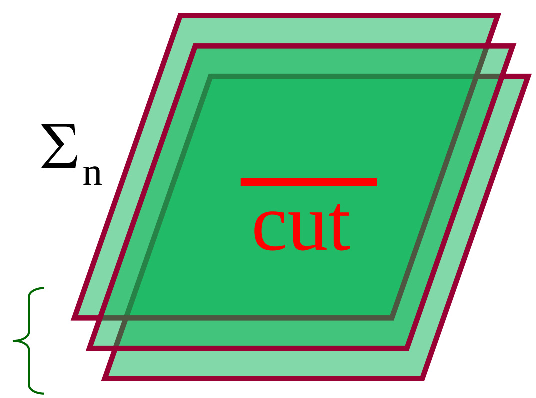


$$= \frac{Z(\Sigma_n)}{Z(\Sigma_1)^n}.$$

n-sheeted

Riemann surface Σ_n

n sheets



(2-2) Replica Method for Excited States

We want to calculate $\mathrm{Tr}\left(\rho_A\right)^n$ for

$$\boxed{\begin{aligned}\rho_A(t,x) &= e^{-iHt}e^{-\varepsilon H}O(x)\left|0\right\rangle\left\langle 0\right|O(x)e^{-\varepsilon H}e^{iHt}\\ &= O(\tau_e,x)\left|0\right\rangle\left\langle 0\right|O(\tau_l,x),\\ &(\tau_e\equiv -\varepsilon - it, \qquad \tau_l \equiv -\varepsilon + it),\end{aligned}}$$

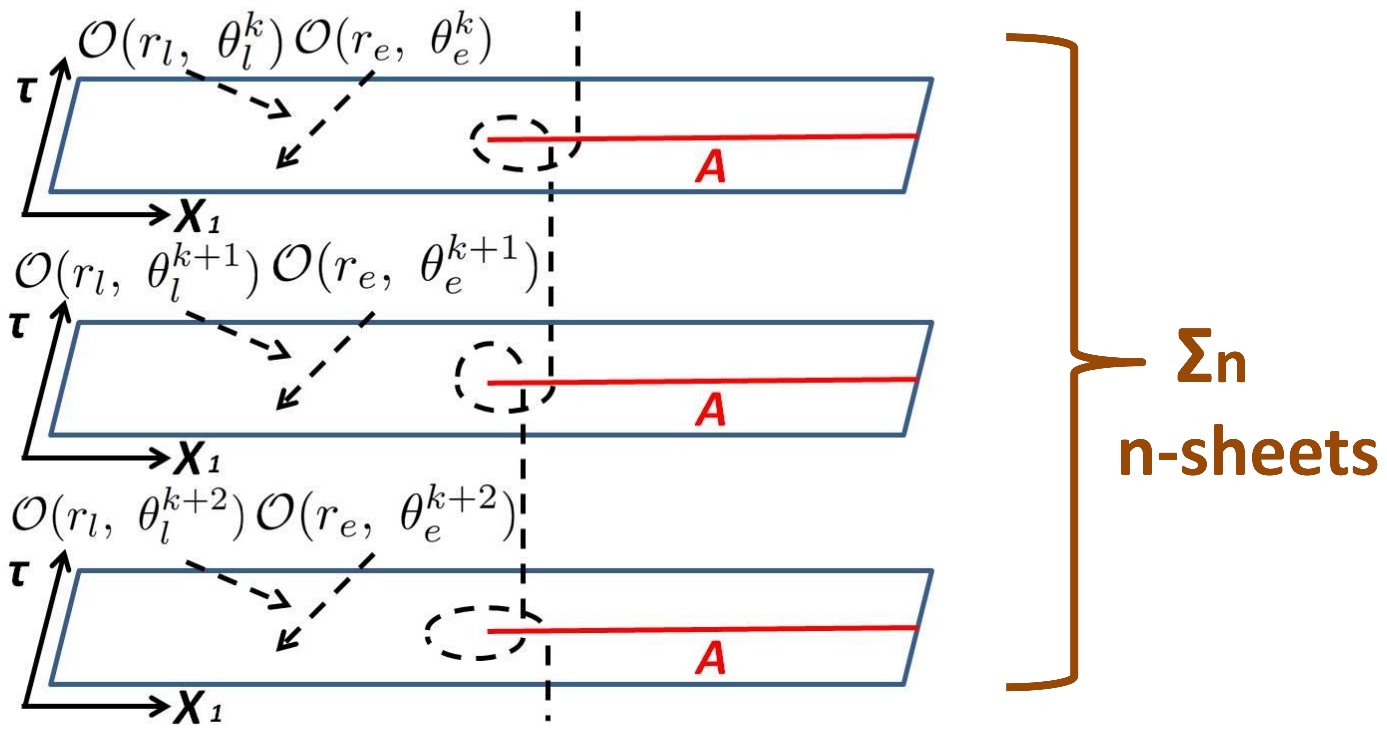
where ε is the UV regulator for the operator.

Here we consider a $d+1$ dim. CFT on R^{d+1}.

$(\tau, x_1, x_2, \cdots, x_d) \in \mathrm{R}^{d+1} \;\; \Rightarrow \;\;$ We set $x_1 + i\tau = re^{i\theta}$.

In this way, the Renyi EE can be expressed in terms of correlation functions (2n-point function etc.) on Σn :

$$\Delta S_A^{(n)} = \frac{1}{1-n}\cdot\Big[\log\left\langle O(r_l,\theta_l^n)O(r_e,\theta_e^n)\cdots O(r_l,\theta_l^1)O(r_e,\theta_e^1)\right\rangle_{\Sigma_n} - n\cdot\log\left\langle O(r_l,\theta_l)O(r_e,\theta_e)\right\rangle_{\Sigma_1}\Big].$$



③ Case 1: Free scalar CFTs in any dimensions

[Numasawa-Nozaki-TT 14]

We focus on the free massless scalar field theory on Σn

$$S=\int d^{d+1}x\left[\partial_\mu\phi\partial^\mu\phi\right]$$

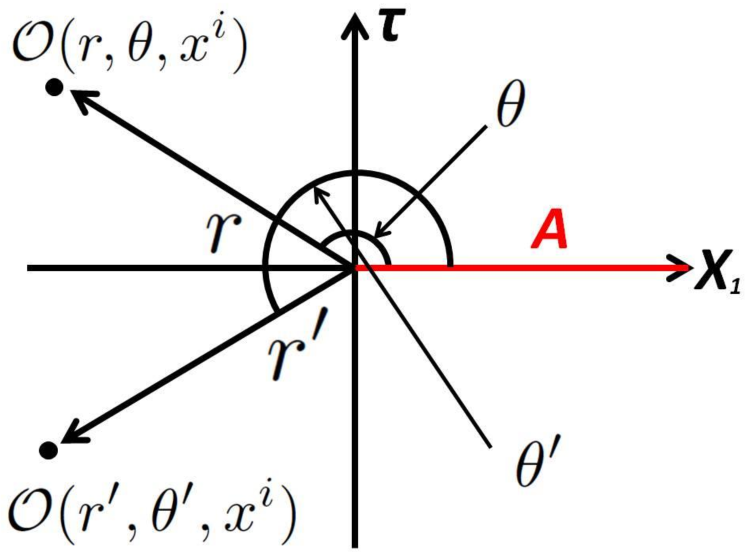
and calculate 2n-pt functions using the Green function:

$$G^{d=3}_{\Sigma n}[(r,\theta,\vec{x});(s,\varphi,\vec{y})]=\frac{1}{4n\pi^2 rs(a-1/a)}\cdot\frac{a^{1/n}-a^{-1/n}}{a^{1/n}+a^{-1/n}-2\cos((\theta-\varphi)/n)},$$

where $$\frac{a}{1+a^2}\equiv\frac{rs}{|\vec{x}-\vec{y}|^2+r^2+s^2}.$$

The operator O is chosen as

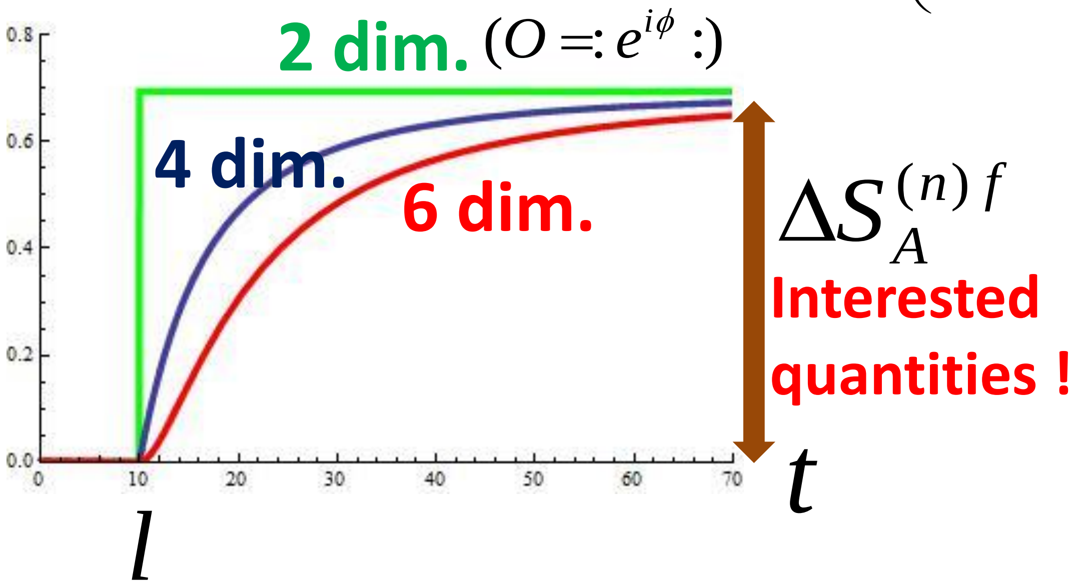
$$O_k=:\phi^k: \quad .$$

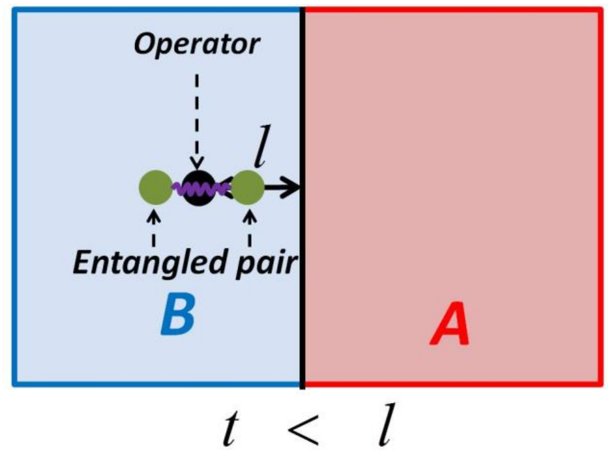


Time evolution in free massless scalar theory

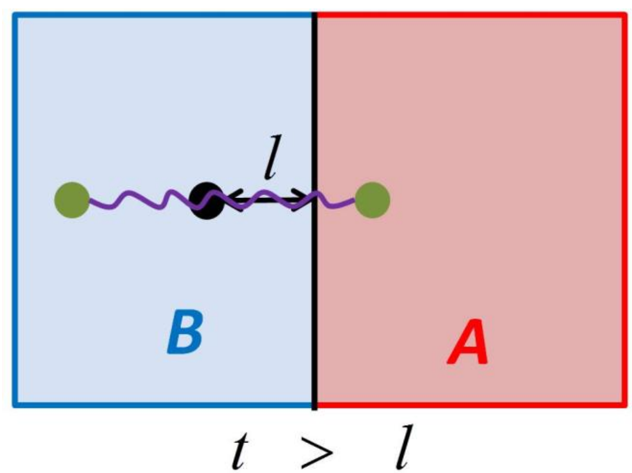
$\Delta S_A^{(2)}$ for $O =: \phi :$ (i.e. $k = 1$)

$\left(\begin{array}{l} \text{We chose } x_1 = -l \text{ with } l = 10 \\ \text{and } x_2 = \cdots = x_d = 0. \end{array} \right)$





$t < l$



$t > l$

$$E.g. \quad \Delta S_{A(4\,\mathrm{dim})}^{(2)} = \log\left(\frac{2t^2}{t^2 + l^2}\right).$$

Note :

$\Delta S_A^{(n)f}$ is `topologically invariant' under deformations of A.

$$\boxed{\Delta S_A^{(n)f} \quad \text{for} \quad O=\phi^k \quad \text{in} \;\; \mathrm{d}+1>2\,\mathrm{dim.}}$$

TABLE I. $\Delta S_A^{(n)f}$ and $\Delta S_A^{f}\left(=\Delta S_A^{(1)f}\right)$ for free massless scalar field theories in dimensions higher than two ($d>1$).

	n	$k=1$	$k=2$	$\cdots$	$k=l$
$\Delta S_A^{(n)f}$	2	$\log 2$	$\log \frac{8}{3}$	$\cdots$	$-\log\left(\frac{1}{2^{2l}}\sum_{j=0}^{l}\left({}_lC_j\right)^2\right)$
	3	$\log 2$	$\frac{1}{2}\log\frac{32}{5}$	$\cdots$	$\frac{-1}{2}\log\left(\frac{1}{2^{3l}}\sum_{j=0}^{l}\left({}_lC_j\right)^3\right)$
Renyi Entropy	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	m	$\log 2$	$\frac{1}{m-1}\log\frac{2^{2m-1}}{2^{m-1}+1}$	$\cdots$	$\frac{1}{1-m}\log\left(\frac{1}{2^{ml}}\sum_{j=0}^{l}\left({}_lC_j\right)^m\right)$
ΔS_A^{f}	1	$\log 2$	$\frac{3}{2}\log 2$	$\cdots$	$l\log 2-\frac{1}{2^l}\sum_{j=0}^{l}{}_lC_j\log{}_lC_j$

EE

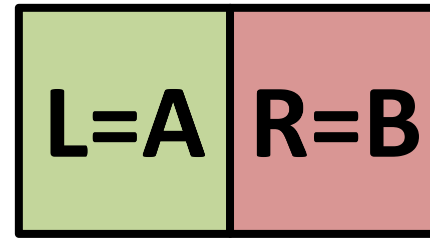
EPR state !

[For a proof: Nozaki 14]

Heuristic Explanation

First , notice that in free CFTs, there are definite (quasi) particles moving at the speed of light.

$$\Rightarrow \quad \phi \approx \underset{\text{left-moving}}{\phi_L} + \underset{\text{right-moving}}{\phi_R} \quad .$$



$$\phi^k |\mathrm{vac}\rangle \approx \sum_{j=0}^{k} {}_kC_j \cdot (\phi_L)^j \cdot (\phi_R)^{k-j} |\mathrm{vac}\rangle$$

$$= 2^{-k/2} \sum_{j=0}^{k} \sqrt{{}_kC_j}\, |j\rangle_L |k-j\rangle_R .$$

$$\Rightarrow \quad \Delta S_A^{(n)f} = \frac{1}{1-n} \log\left[2^{-nk} \sum_{j=0}^{k} ({}_kC_j)^n\right],$$

$$\Delta S_A^{f} = k \log 2 - 2^{-k} \sum_{j=0}^{k} {}_kC_j \cdot \log[{}_kC_j].$$

Agree with replica Calculations !

④ Case 2: Rational 2d CFTs [He-Numasawa-Watanabe-TT 14]

(4-1) Free Scalar CFT in 2d

Consider following two operators in the free scalar CFT:

(i) $$O_1 =: e^{i\alpha\phi} : \quad \Rightarrow \quad \Delta S_A^{(n)f} = 0.$$

$$|O_1\rangle = e^{i\alpha\phi_L}|0\rangle_L \otimes e^{i\alpha\phi_R}|0\rangle_R$$ ⇒Direct product state

(ii) $$O_2 =: e^{i\alpha\phi} : + : e^{-i\alpha\phi} : \quad \Rightarrow \quad \Delta S_A^{(n)f} = \log 2.$$

$$|O_2\rangle = e^{i\alpha\phi_L}|0\rangle_L \otimes e^{i\alpha\phi_R}|0\rangle_R + e^{-i\alpha\phi_L}|0\rangle_L \otimes e^{-i\alpha\phi_R}|0\rangle_R$$

$$\approx |\uparrow\rangle_L|\uparrow\rangle_R + |\downarrow\rangle_L|\downarrow\rangle_R$$ ⇒ EPR state

(4-2) General Results for 2d Rational CFTs

First, focus on n=2 REE and assume O = a primary op.

We can employ the following conformal map: $\Sigma_2 \to \Sigma_1$

$$z = \sqrt{w} = \sqrt{re^{i\theta}} \quad .$$

It is straightforward to rewrite the n=2 REE in terms of 4-pt functions on $\Sigma_1 = \mathrm{C}$.

$$\left\langle O(w_1,\overline{w}_1)O(w_2,\overline{w}_2)O(w_3,\overline{w}_3)O(w_4,\overline{w}_4)\right\rangle_{\Sigma_2}$$

$$= | z_{13}z_{24} |^{-4\Delta_O} \cdot G_O(z,\overline{z}).$$

$$w_1 = i(\varepsilon - it) - l, \quad \overline{w}_1 = -i(\varepsilon - it) - l,$$
$$w_2 = -i(\varepsilon + it) - l, \quad \overline{w}_2 = i(\varepsilon + it) - l.$$
$$w_3 = e^{2\pi i} w_1, \quad w_4 = e^{2\pi i} w_2.$$

$$\left(z \equiv \frac{z_{12}z_{34}}{z_{13}z_{24}}, \qquad z_{ij} \equiv z_i - z_j . \right)$$

We can show that the limit $\varepsilon \to 0$ leads to

(i) Early time: $0 < t < l$

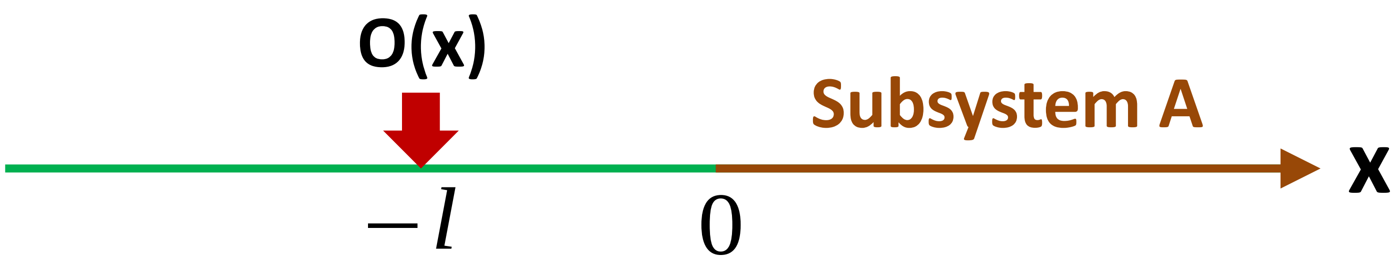
$$(z, \bar{z}) \approx (O(\varepsilon^2), O(\varepsilon^2)) \to (0,0).$$

Chiral Fusion Transformation

(ii) Late time: $t \geq l$

$$(z, \bar{z}) \approx (1 + O(\varepsilon^2), O(\varepsilon^2)) \to (1,0).$$

z→1-z

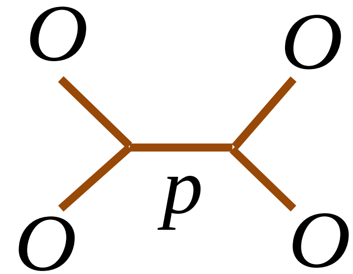


Note: It is straightforward to confirm

$\Delta S_A^{(n)} = 0$ at early time (i).

In terms of conformal block, we find at late time:

$$G_O(z,\bar{z}) = \sum_p C_{OO}^{p} \cdot F_O(p\,|\,z) \cdot \bar{F}_O(p\,|\,\bar{z})$$

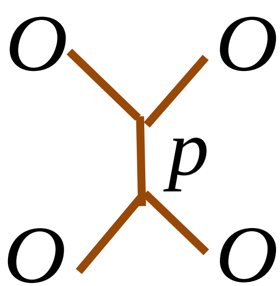
$\sum_p$ 

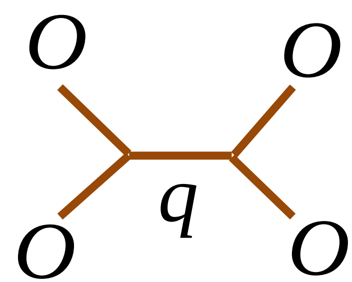
$$\underset{\substack{(z,\bar{z})\\ \to(1,0)}}{\cong} F_O(I\,|\,z) \cdot \bar{F}_O(I\,|\,\bar{z})$$

$$\cong F_{I,I}[O] \cdot (1-z)^{-2\Delta_O} \cdot \bar{z}^{-2\Delta_O},$$

where $F_{p,q}[O]$ is so called the fusion matrix, defined by

$$F_O(p\,|\,1-z) = \sum_q F_{p,q}[O] \cdot F_O(q\,|\,z).$$





Then the n=2 REE is simply expressed at late time:

$$\Delta S_A^{(2)f} = -\log F_{I,I}[O].$$

In rational 2d CFTs, we can rewrite this in term of the quantum dimension d_O

$$d_O \equiv \frac{S_{I,O}}{S_{I,I}} = \frac{1}{F_{I,I}[O]},$$

[Moore-Seiberg 89]

as follows: $\Delta S_A^{(2)f} = \log d_O$.

Actually, more generally we can prove $\Delta S_A^{(n)f} = \log d_O$ for any n.

Example: Ising model

3 conformal blocks: $[I],\ [\sigma],\ [\varepsilon].$

$$[I]\otimes[I]=[I],\quad [\varepsilon]\otimes[\varepsilon]=[I].\quad \Rightarrow d_I=d_\varepsilon=1.$$

$$[\sigma]\otimes[\sigma]=[I]\oplus[\varepsilon].$$

$$\Rightarrow[\sigma]^{2N}=([I]\oplus[\varepsilon])^N=\underbrace{2^{N-1}[I]\oplus 2^{N-1}[\varepsilon]}_{2^N\ \text{particles}}$$

$$\Rightarrow d_\sigma=\sqrt{2}.$$

$$\text{Thus we find}:\ \Delta S_A^{(n)}[I]=\Delta S_A^{(n)}[\varepsilon]=0,$$

$$\Delta S_A^{(n)}[\sigma]=\log\sqrt{2}.$$

⑤ Case 3: Large N CFTs and AdS/CFT [Caputa-Nozaki-TT 14]

(5-1) Free U(N) Yang-Mills at large N

We choose $O(x) = \mathrm{Tr}[\Phi(x)^J]$. ($\Phi = N \times N$ Hermitian matrix scalar)

For example, when $J = 2$, we find the exact result:

$$\Delta S_A^{(n)f} = \frac{1}{1-n}\log\left[2^{1-2n} + \underbrace{2^{-n}\cdot N^{2(1-n)}}\right].$$

can be neglected **only if n>1**

In general, we find

$$\text{if } n>1 \Rightarrow \Delta S_A^{(n)f} = \frac{Jn-1}{n-1}\log 2 + O(N^{-2}).$$

$$\text{if } n=1 \Rightarrow \Delta S_A^{(1)f} = \frac{J}{2}\log N + O(N^{-2}).$$

Enhance at n=1

~deconfinement ?

Actually, the behavior $\Delta S_A^{(1)\,f} \propto J \log N$ is easy to explain.

$$\mathrm{Tr}\left[(\Phi_{\mathrm{L}} + \Phi_R)^J\right]|0\rangle$$
$$= (\Phi_L)_{a_1 a_2} \cdots (\Phi_L)_{a_{J-1} a_J} |0\rangle_L \otimes (\Phi_R)_{a_1 a_2} \cdots (\Phi_R)_{a_{J-1} a_J} |0\rangle_R + \cdots$$

EE ~ J▪Log[N^2]

cf. Log [N] behavior for a heavy quark

[Lewkowycz-Maldacena 13]

(5-2) Holographic Results from AdS/CFT

$\Delta S_A^{(n>2)}$ in d dim CFTs

⇒ Holographic 2n-point functions in (d+1) dim. topological AdS BH

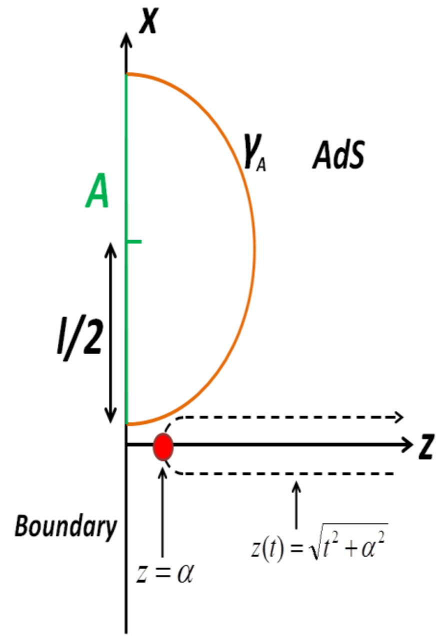
$$\Delta S_A^{(n)} \approx \frac{4n\Delta_O}{d(n-1)} \log\left(\frac{t}{\varepsilon}\right) .$$

This calculation is based on naïve large N limit.
Thus the n=1 limit and the late time limit t=∞ are not trustable.

For n=1 (EE), we can employ the HEE formula to find $\Delta S_A^{(1)}$ directly. [Nozaki-Numasawa-TT 13]

For 2d CFT (AdS_3/CFT_2),

$$\Delta S_A^{(1)} \approx \frac{c}{6} \log\left(\frac{t}{\varepsilon}\right) .$$



⑥ Conclusions

In the large limit of A, the (Renyi) EEs $\Delta S_A^{(n)}$ for a locally excited state describe the `degrees of freedom' of a given local operator.

- Monotonic time evolution describes entangled pair propagation.
- The final values $\Delta S_A^{(n)f}$ can be explained by entanglement of finite number of states such as EPR states.
- They are topological invariant against deformations of A.
- In 2d rational CFTs, $\Delta S_A^{(n)f}$ is given by the log of quantum dimension. [cf. Topological EE: Kitaev-Preskill, Levin-Wen 05]
- In large N CFTs, 1/N subleading terms get important at n=1. The von-Neumann EE sees N^2 degrees of freedom, while REE not.
- In strongly coupled large N CFTs, we find a logarithmic time evolution. (Does it approach to finite value or not ? –future problem.)

One lesson:

The Renyi EE and von-Neumann EE behave differently !

~Low temp. ~High temp.

In QFTs, the Renyi EE (REE) is easier to compute.

In Gravity, the von-Neumann EE (EE) is simpler.

$\Rightarrow$ **Why ??**