

Strings 2015
ICTS-TIFR, Bengaluru
22-26 June 2015



Scanning New Horizons: Entanglement & Holography



Holography



Entanglement

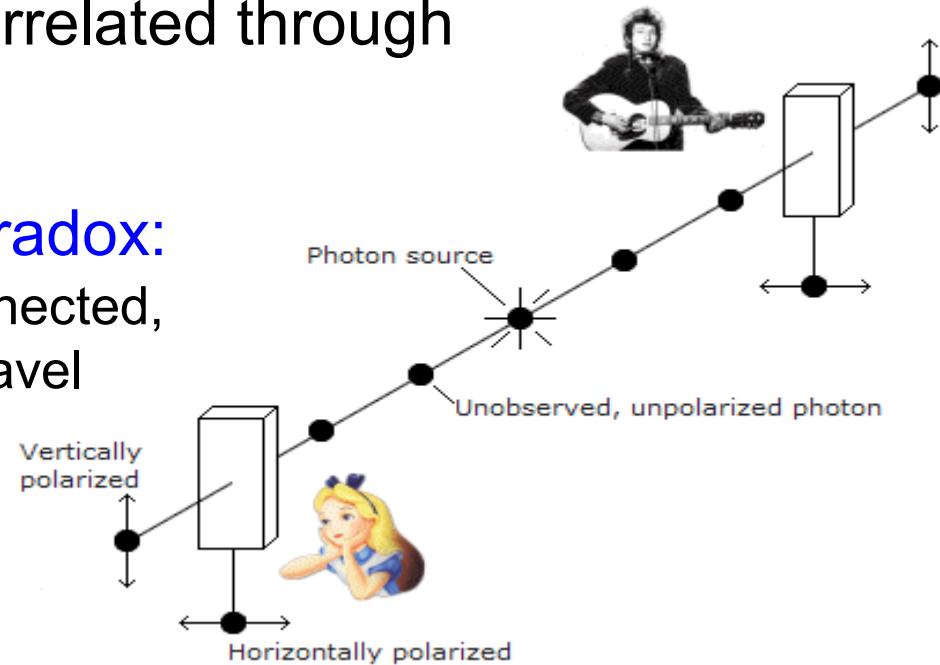
Quantum Entanglement

- different subsystems are correlated through global state of full system

Einstein-Podolsky-Rosen Paradox:

- properties of pair of photons connected, no matter how far apart they travel

“*spukhafte Fernwirkung*” =
spooky action at a distance



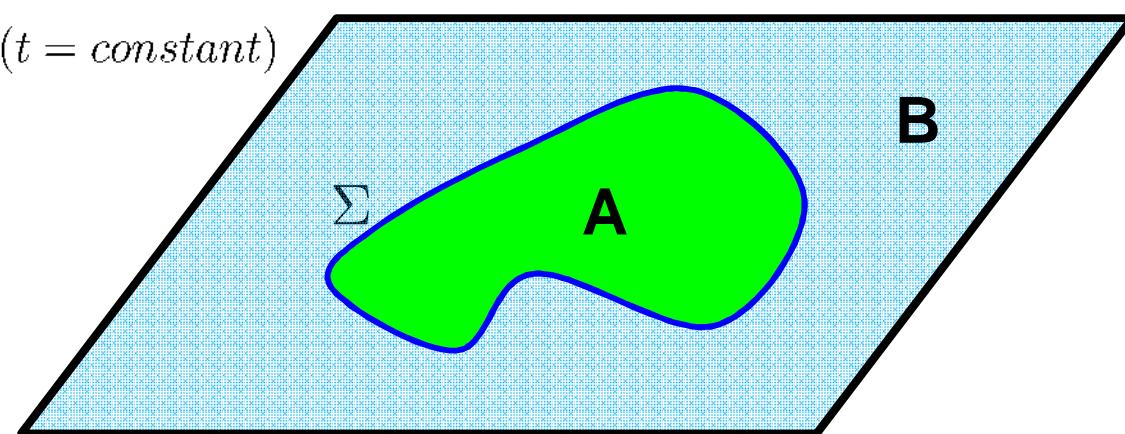
$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$$

Quantum Information: entanglement becomes a resource for (ultra)fast computations and (ultra)secure communications

Condensed Matter: key to “exotic” phases and phenomena, e.g., quantum Hall fluids, unconventional superconductors, quantum spin fluids,

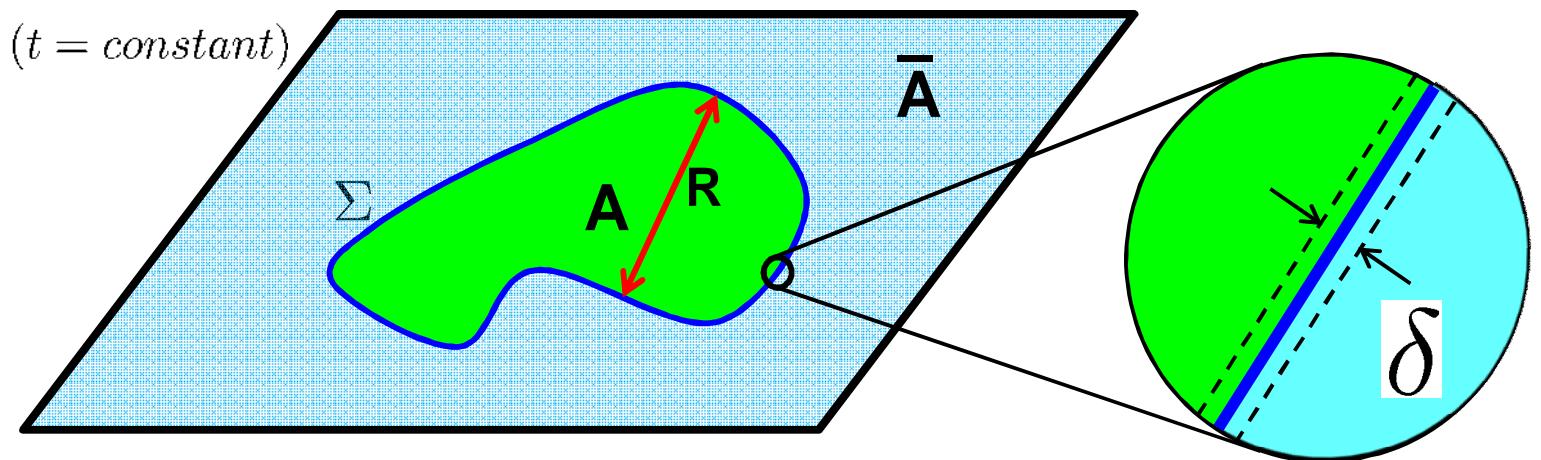
Entanglement Entropy:

- very general diagnostic: divide quantum system into two parts and use entropy as measure of correlations between subsystems
 - in QFT, typically introduce a (smooth) boundary **or entangling surface** Σ which divides the space into two separate regions
 - integrate out degrees of freedom in “outside” region
 - remaining dof are described by a density matrix ρ_A
- calculate **von Neumann entropy**: $S_{EE} = -Tr [\rho_A \log \rho_A]$



Entanglement Entropy in QFT

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- calculate von Neumann entropy: $S_{EE} = -Tr [\rho_A \log \rho_A]$



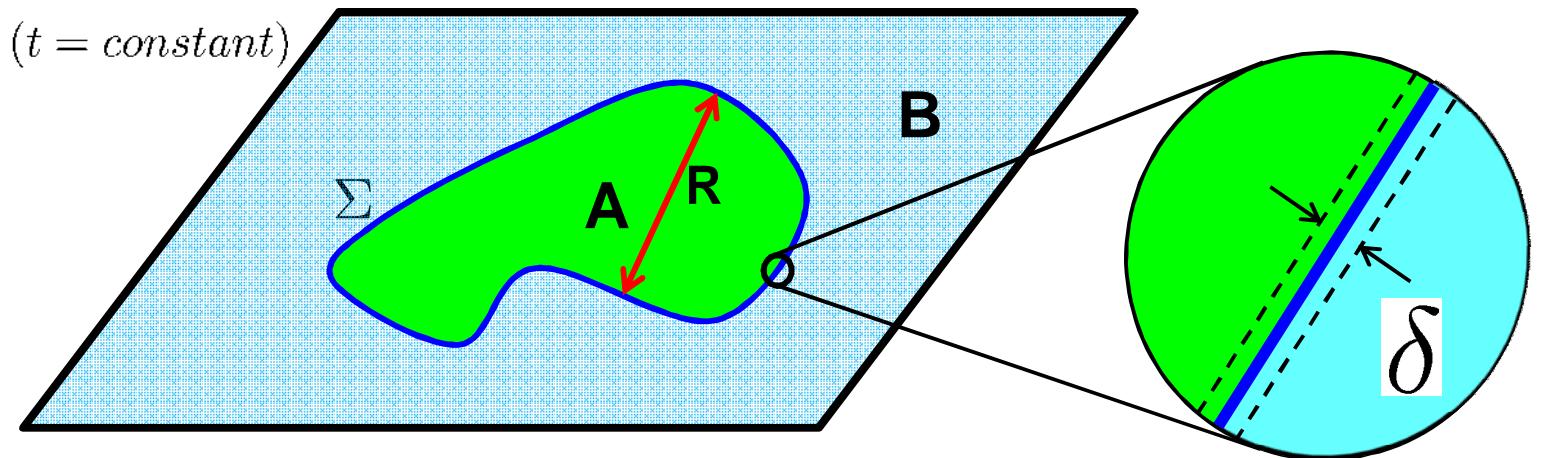
- result is **UV divergent!** dominated by short-distance correlations
- must regulate calculation: $\delta =$ **short-distance cut-off**

$$S = c_0 \frac{R^{d-2}}{\delta^{d-2}} + c_2 \frac{R^{d-4}}{\delta^{d-4}} + \dots \quad d = \text{spacetime dimension}$$

→ geometric structure, eg, $S = \tilde{c}_0 \frac{\mathcal{A}_\Sigma}{\delta^{d-2}} + \tilde{c}_2 \frac{\oint_\Sigma \text{"curvature"} }{\delta^{d-4}} + \dots$

Entanglement Entropy in QFT:

- remaining dof are described by a density matrix ρ_A
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- must regulate calculation: δ = short-distance cut-off
- $$S = c_0 \frac{R^{d-2}}{\delta^{d-2}} + c_2 \frac{R^{d-4}}{\delta^{d-4}} + \dots \quad d = \text{spacetime dimension}$$
- leading coefficients sensitive to details of regulator, eg, $\delta \rightarrow 2\delta$
 - find universal information characterizing underlying QFT in subleading terms, eg,
- $$S = \dots + c_d \log(R/\delta) + \dots$$

Beyond Entanglement Entropy:

- many other useful diagnostics/measures of entanglement:

Renyi entropy:
$$S_\alpha = \frac{1}{1-\alpha} \log \text{Tr} [\rho_A^\alpha]$$
$$\left(S_{EE} = \lim_{\alpha \rightarrow 1} S_\alpha = -\text{Tr} [\rho_A \log \rho_A] \right)$$

Relative entropy: $S(\rho_1 | \rho_0) = \text{tr}(\rho_1 \log \rho_1) - \text{tr}(\rho_1 \log \rho_0)$

Entanglement negativity: $\mathcal{E} = \log \text{Tr} |\rho^{T_2}|$

Mutual information: $I(A, B) = S(A) + S(B) - S(A \cup B)$

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- bipartite entanglement versus multipartite entanglement

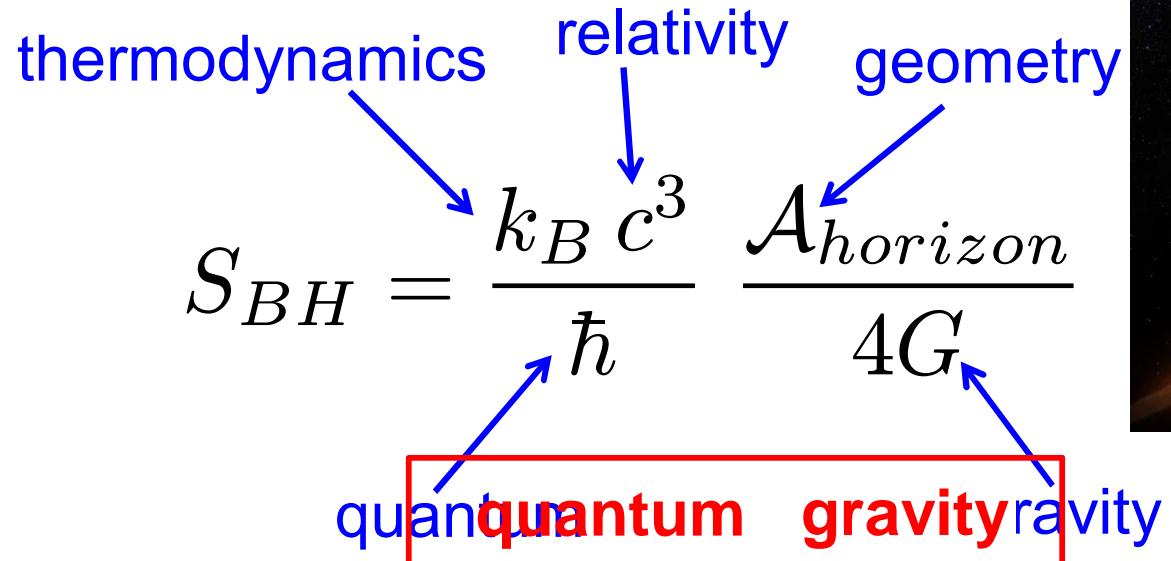
Holography v1.0: Black Holes

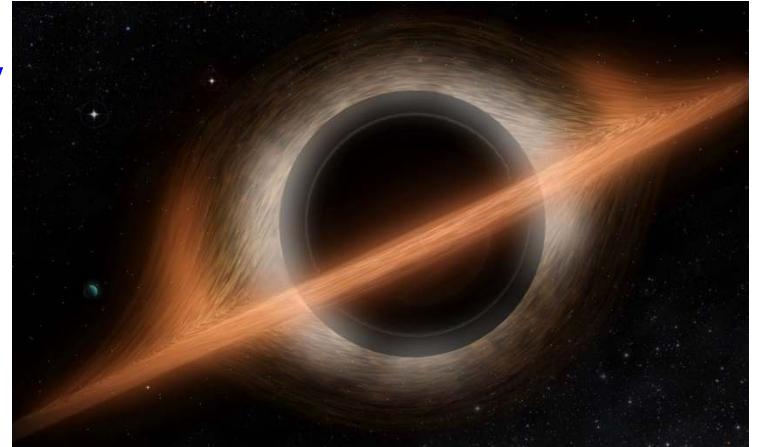
- Bekenstein and Hawking: “black holes have entropy!”

$$S_{BH} = \frac{k_B c^3}{\hbar} \frac{A_{horizon}}{4G}$$

thermodynamics relativity geometry

quantum gravity





- window into the quantum theory of gravity?!?
- quantum gravity provides a fundamental scale

$$\ell_P^2 = 8\pi G \hbar/c^3$$

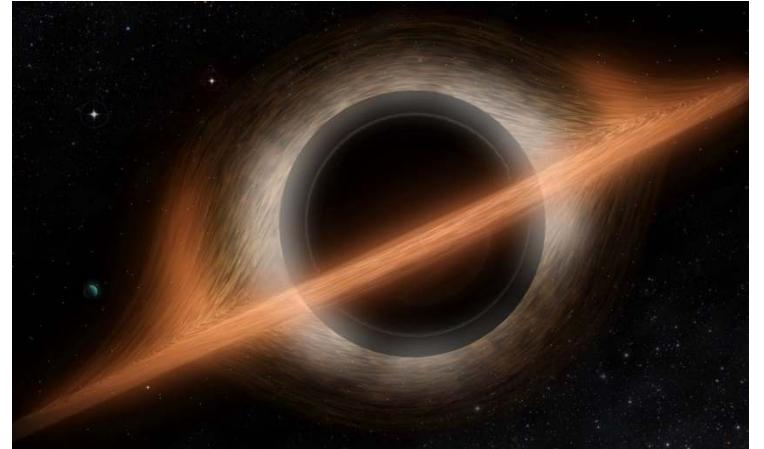
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microstates

$$S_{BH} = 2\pi \frac{\mathcal{A}_{horizon}}{\ell_P^2} k_B$$

geometry



- quantum gravity provides a fundamental scale

$$\ell_P^2 = 8\pi G \hbar/c^3$$

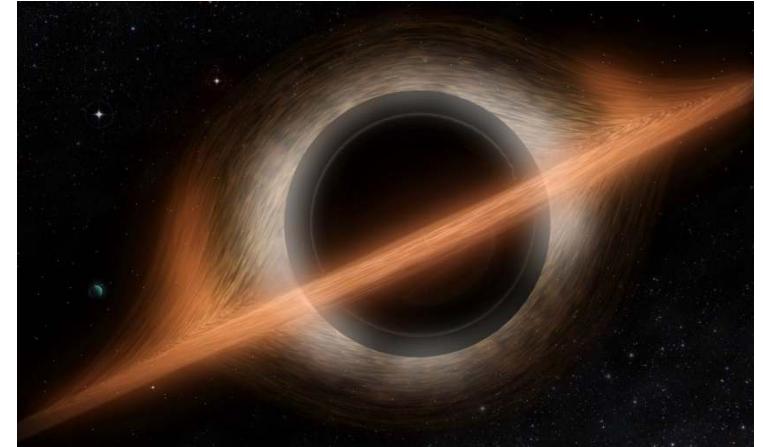
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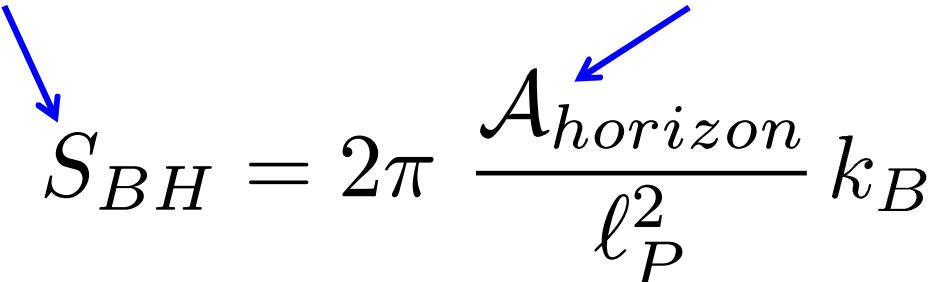
- entropy is not extensive; grows with **area** rather than volume
- Sorkin: “black hole entropy is entanglement entropy”
 $S_{EE} = \tilde{c}_0 \frac{\mathcal{A}_\Sigma}{\delta^{d-2}} + \dots \longrightarrow$ “area law” suggestive of BH formula if $\delta \simeq \ell_P$
(Sorkin '84; Bombelli, Koul, Lee & Sorkin; Srednicki; Frolov & Novikov; . . .)
- entanglement entropy of QF's contributes to gravitational entropy
(Susskind & Uglum; ...)

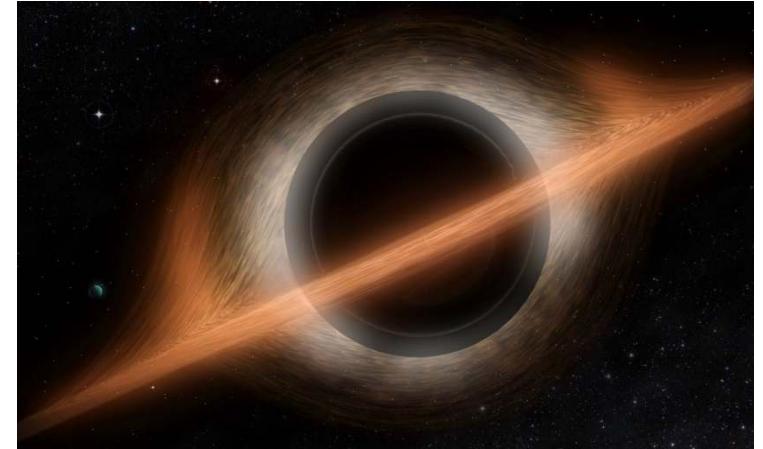
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- Bekenstein and Hawking: “black holes have entropy!”

$$S_{BH} = 2\pi \frac{\mathcal{A}_{horizon}}{\ell_P^2} k_B$$

microstates geometry





- “**holography**” suggests that not only does the horizon encode information about microstates but also that the evolution and dynamics of the black hole can be described in terms of a “dual” theory living on the horizon

(‘t Hooft; Susskind)

Holography v2.0: AdS/ CFT correspondence

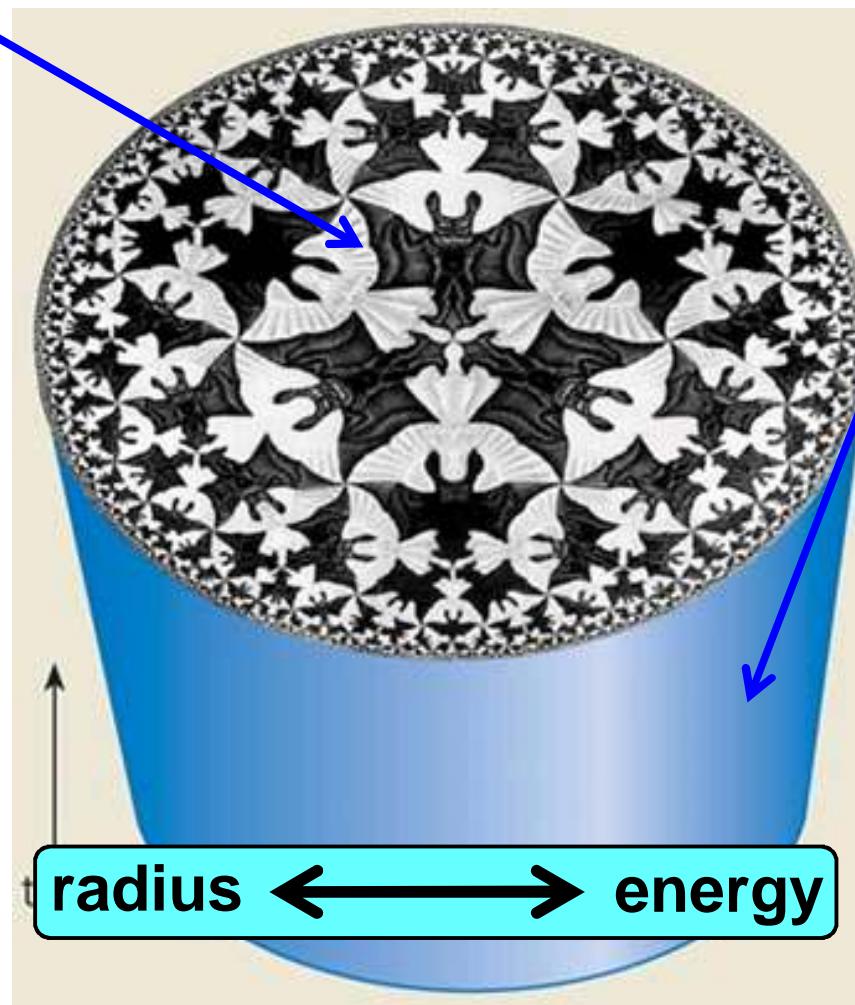
Bulk: gravity with negative Λ
in **d+1** dimensions

anti-de Sitter
space

Boundary: quantum field theory
without intrinsic scales
in **d** dimensions

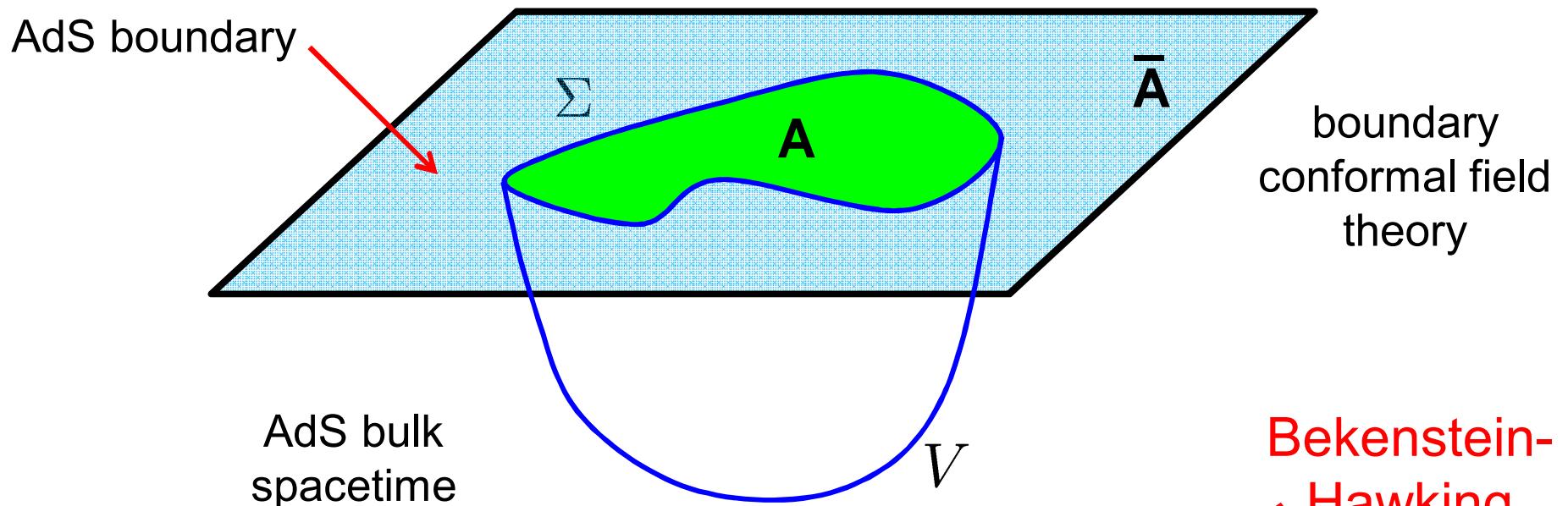
conformal
field theory

↔
“holography”



(Ryu & Takayanagi '06)

Holographic Entanglement Entropy:



$$S(A) = \underset{V \sim A}{\text{???n}} \frac{A_V}{4G_N}$$

Bekenstein-
Hawking
formula

- conjecture \longrightarrow many detailed consistency tests

(Ryu, Takayanagi, Headrick, Hung, Smolkin, RM, Faulkner, . . .)

(Ryu & Takayanagi '06)

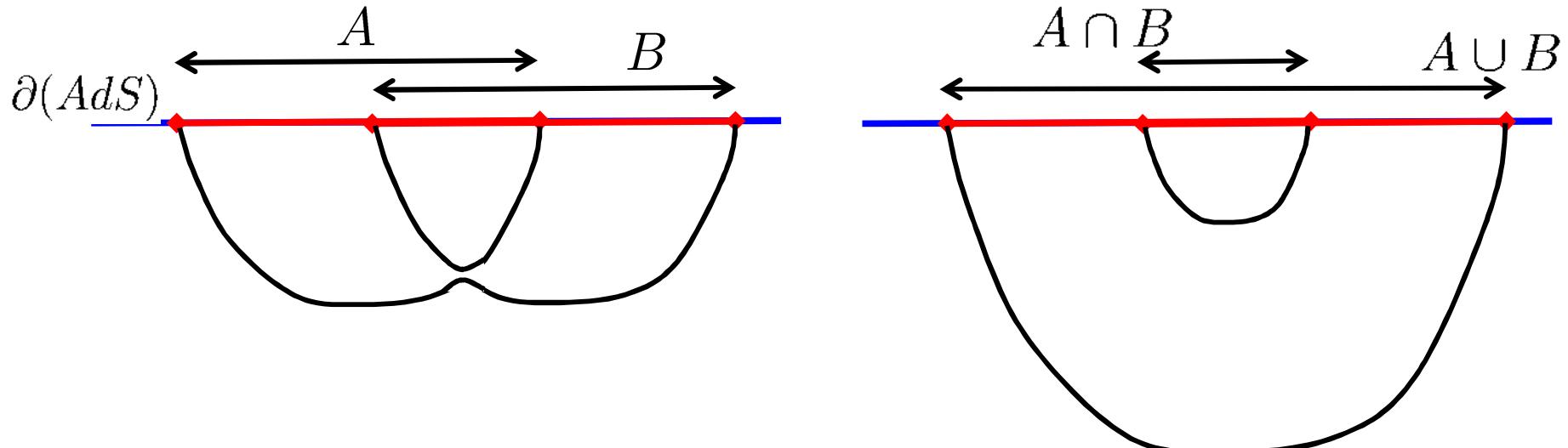
Holographic Entanglement Entropy:

$$S(A) = \min_{V \sim A} \frac{A_V}{4G_N}$$

Extensive consistency tests:

(Headrick & Takayanagi)

strong sub-additivity: $S(A \cup B) + S(A \cap B) \leq S(A) + S(B)$



[extended to dynamical setting: Wall]

[further monogamy relations: Hayden, Headrick & Maloney]

[“holography is special”: Bao, Nezami, Ooguri, Stoica, Sully & Walter]

Holographic Entanglement Entropy:

$$S(A) = \min_{V \sim A} \frac{A_V}{4G_N}$$

Extensive consistency tests: \longrightarrow proof (Lewkowycz & Maldacena '13)

- generalization of Euclidean path integral calc's for S_{BH} , extended to “periodic” bulk solutions without Killing vector
- for AdS/CFT, translates replica trick for boundary CFT to bulk

$$\begin{aligned} \Delta\tau = 2\pi \rightarrow 2\pi n &\longrightarrow \log Z(n) = \log \text{Tr} [\rho^n] = -I_{grav}(n) \\ &\longrightarrow S = -n\partial_n [\log Z(n) - n \log Z(1)] \Big|_{n=1} \end{aligned}$$

- at $n \sim 1$, linearized gravity eom demand: $K^\alpha = h^{ij} K_{ij}^\alpha = 0$
- $\longrightarrow \tau$ shrinks to zero on an extremal surface in bulk
- evaluating Einstein action yields $S = A/4G_N$ for extremal surface

Holographic Entanglement Entropy:

$$S(A) = \min_{V \sim A} \frac{A_V}{4G_N}$$

- implicitly bulk theory described by Einstein gravity
- generalization for higher curvatures? $S(A) = \min_{V \sim A} S_{\text{grav}}(V)$

eg, for (stationary) BH's: $S_{\text{Wald}} = -2\pi \int_V d^{d-1}\sigma \sqrt{h} \frac{\partial \mathcal{L}}{\partial R^{\mu\nu}}{}_{\rho\sigma} \hat{\varepsilon}^{\mu\nu} \hat{\varepsilon}_{\rho\sigma}$

- ~~extending Wald's formula to higher-order terms~~ (Dong; Chang, Ryu & Smolkin)

$$S_{\text{grav}}(V) = S_{\text{Wald}}(V) + 4\pi \int_V d^{d-1}\sigma \sqrt{h} \frac{\partial^2 \mathcal{L}}{\partial R_{\mu_1\nu_1\rho_1\sigma_1} \partial R_{\mu_2\nu_2\rho_2\sigma_2}} K_{\nu_1\sigma_1}^{\lambda_1} K_{\nu_2\sigma_2}^{\lambda_2}$$

$$[(h_{\mu_1\mu_2} h_{\rho_1\rho_2} - \hat{\varepsilon}_{\mu_1\mu_2} \hat{\varepsilon}_{\rho_1\rho_2}) h_{\lambda_1\lambda_2} + (h_{\mu_1\mu_2} \hat{\varepsilon}_{\rho_1\rho_2} + \hat{\varepsilon}_{\mu_1\mu_2} h_{\rho_1\rho_2}) \hat{\varepsilon}_{\lambda_1\lambda_2}] + \dots$$

- not yet understood beyond K^2 (Miao, Guo & Huang; Astaneh, Patrushev & Solodukhin; Bhattacharyya & Sharma; Bhattacharjee, Sarkar & Wall; Dong; Smolkin)

Recent topics trending in **Holographic S_{EE}**:

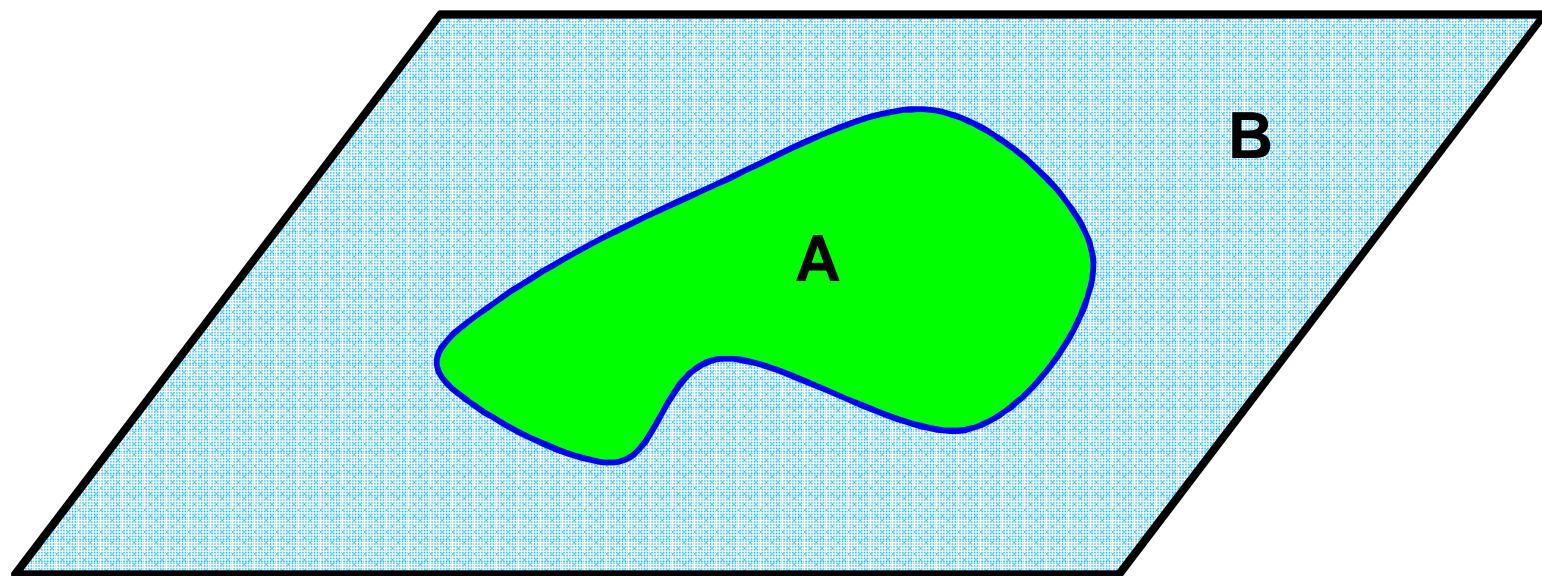
Ryu & Takayanagi, hep-th/06030016 → total cites 710 (219, past year)

- emergence of bulk geometry for large-c CFT's
(Faulkner; Asplund, Bernamonti, Galli & Hartman;)
- holographic entanglement entropy and internal spaces
(Mollabashi, Shiba & Takayanagi; Karch & Uhlemann)
- universal response of EE to variations
(Allais & Mezei; Lewkowycz & Perlmutter; Rosenhaus & Smolin;
Bueno, RCM & Witczak-Krempa;)
- differential entropy and kinematic space
(Czech, Dong, Hayden, Lamprou, Lashkari, McCandlish, Sully & Swingle;
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- holography & tensor networks
(Swingle; Miyaji, Numasawa, Ryu, Shiba, Takayanagi, Watanabe & Wen;
Pastawski, Yoshida, Harlow & Preskill; Lee & Qi;)
- energy conditions from entanglement inequalities
(Lashkari, Rabideau, Sabella-Garnier & Van Raamsdonk; Lin, Marcolli, Ooguri
& Stoica; Bhattacharya, Hubeny, Rangamani & Takayanagi)

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Entanglement Entropy for (2+1)-dimensional CFT

$$S_{EE} = -Tr [\rho_A \log \rho_A] = c_1 \frac{\mathcal{A}}{\delta} - 2\pi c_0$$

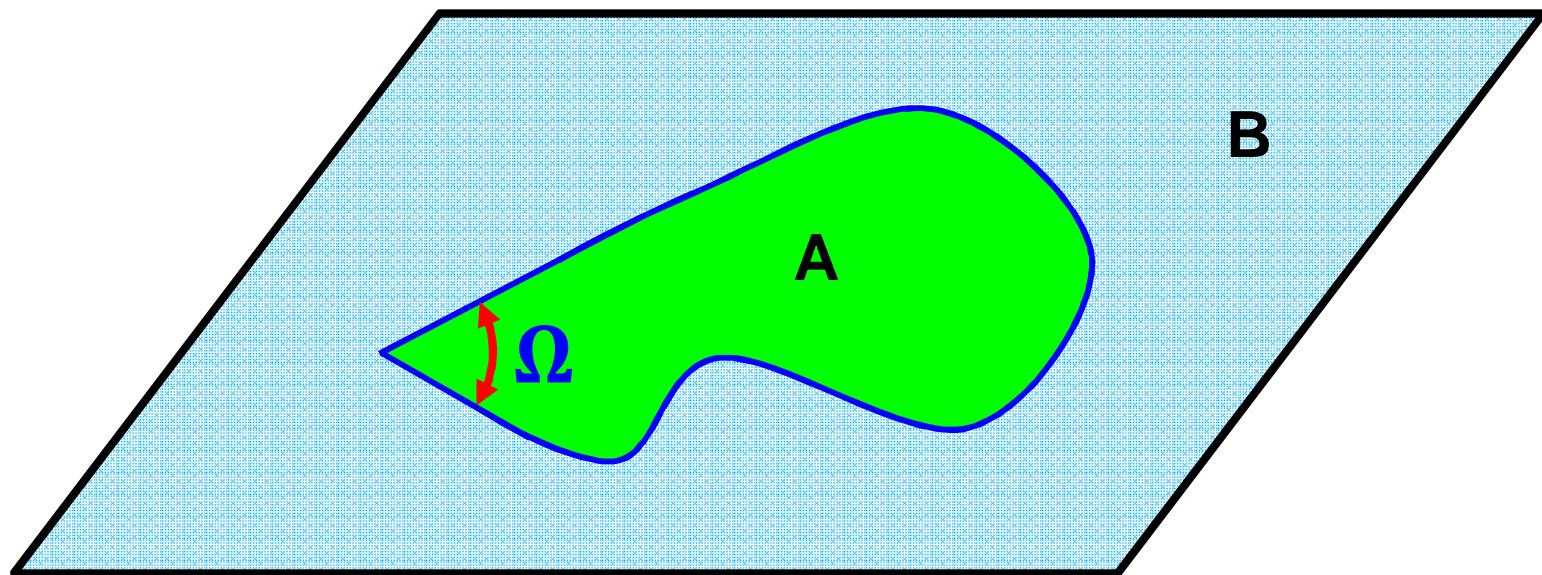


(Bueno, RCM & Witczak-Krempa)

Entanglement Entropy for (2+1)-dimensional CFT

$$S_{EE} = c_1 \frac{\mathcal{A}}{\delta} - \boxed{a(\Omega) \log (H/\delta)} - 2\pi \tilde{c}_0$$

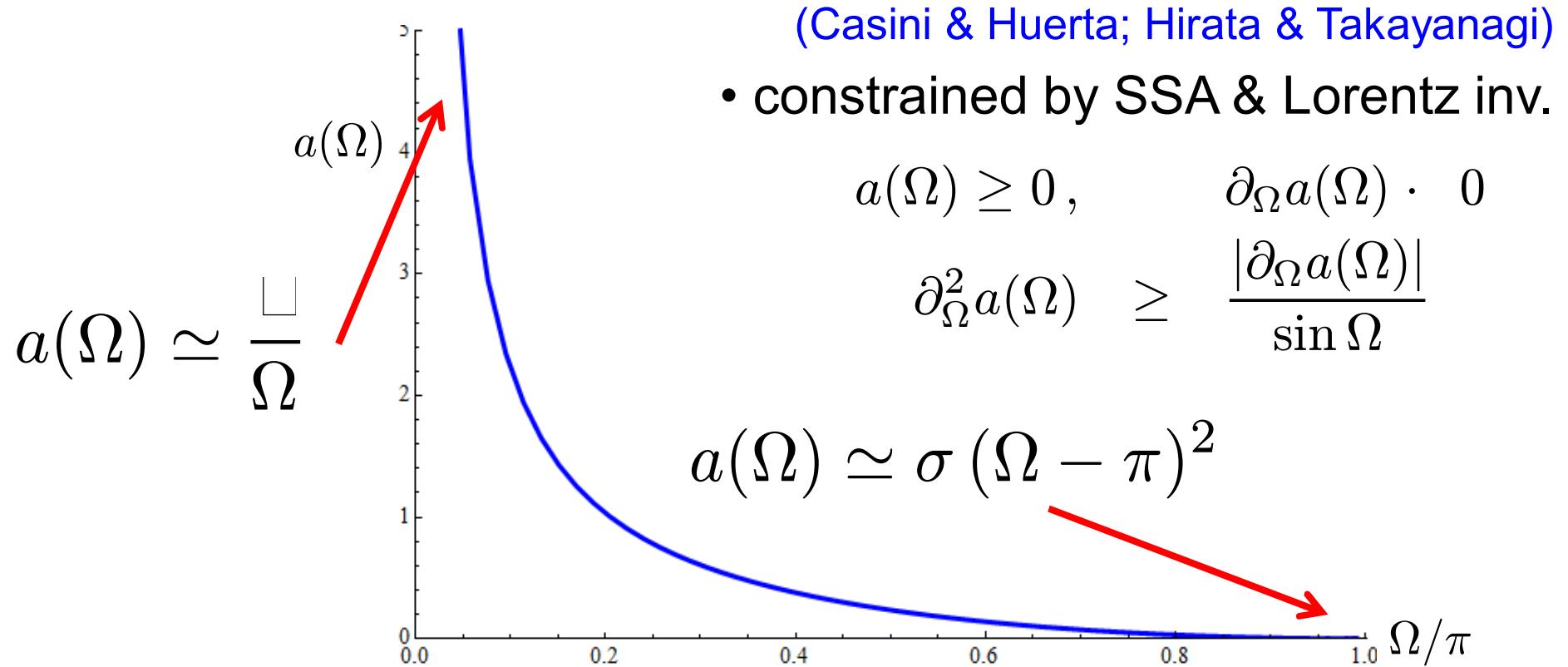
 “Universal corner contribution”



Entanglement Entropy for (2+1)-dimensional CFT

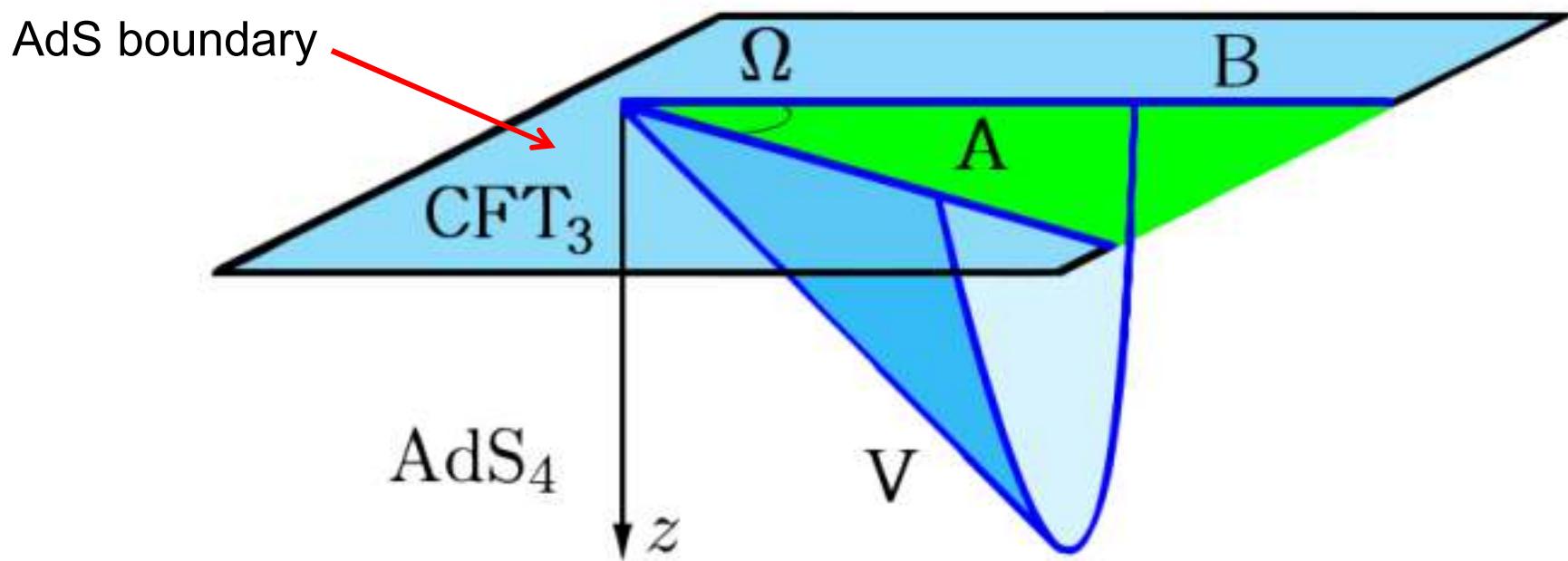
$$S_{EE} = c_1 \frac{\mathcal{A}}{\delta} - a(\Omega) \log(H/\delta) - 2\pi \tilde{c}_0$$

“Universal corner contribution”



(Hirata & Takayanagi)

Holographic Corner Term:

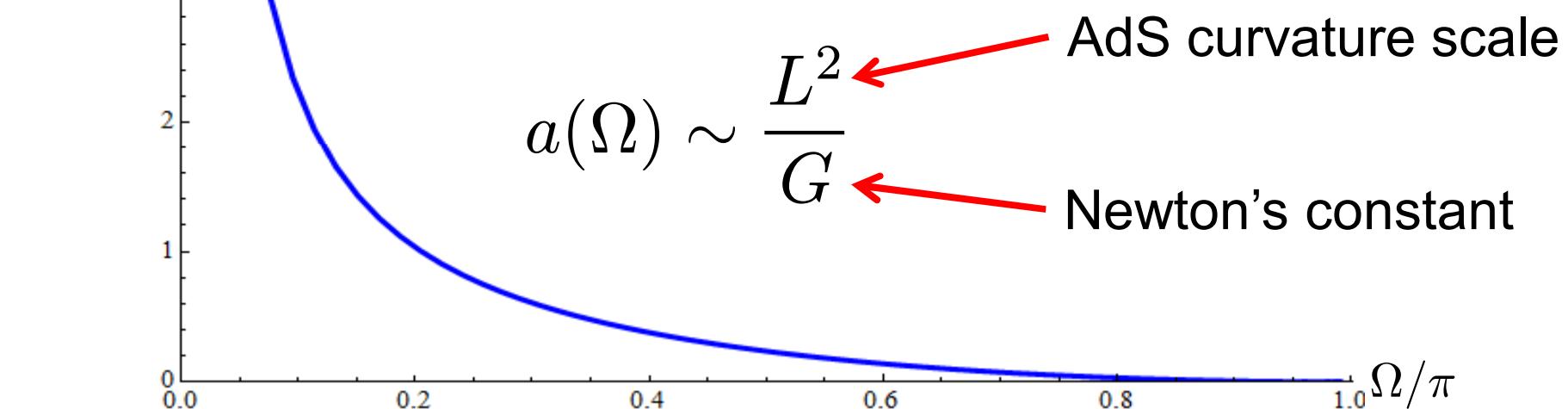
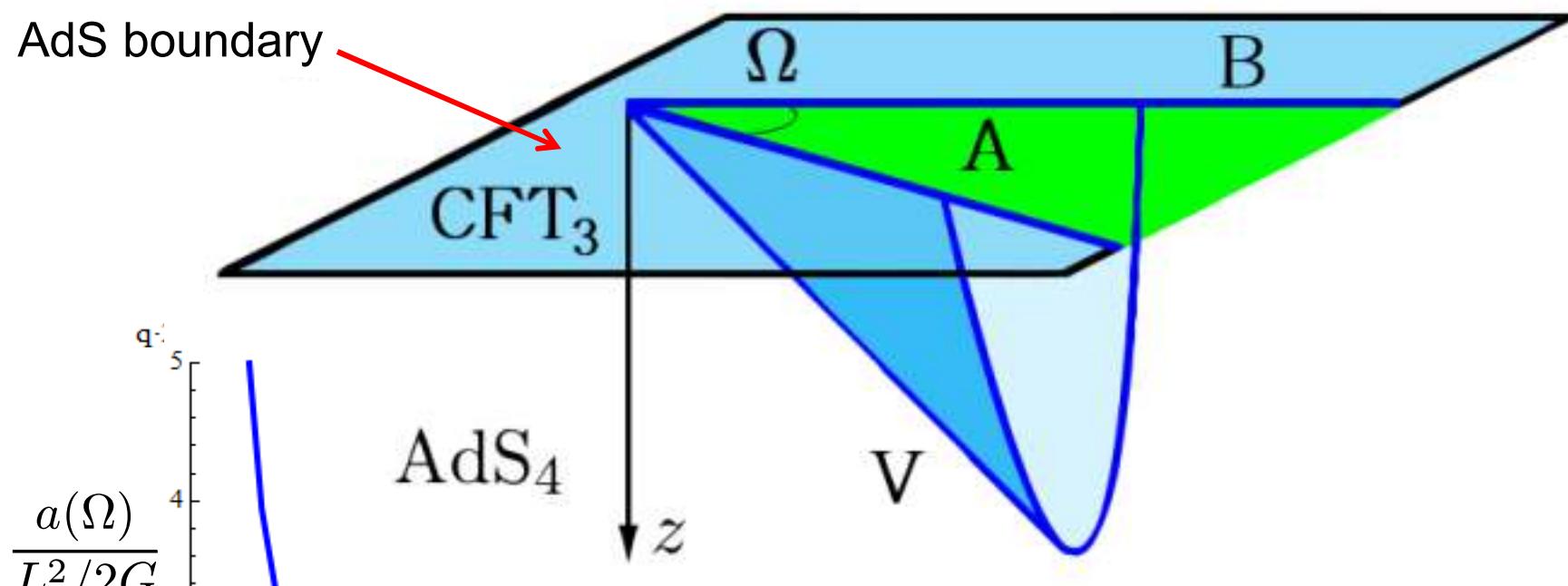


$$a(\Omega) = \frac{L^2}{2G} \int_0^\infty dy \left[1 - \sqrt{\frac{1 + h_0^2(1 + y^2)}{2 + h_0^2(1 + y^2)}} \right]$$

$$\Omega = \int_0^{h_0} dh \frac{2h^2 \sqrt{1 + h_0^2}}{\sqrt{1 + h^2} \sqrt{(h_0^2 - h^2)(h_0^2 + (1 + h_0^2)h^2)}}$$

(Hirata & Takayanagi)

Holographic Corner Term:



L^2/G is ubiquitous in holography!

- Universal term in circle entropy (appears in F-theorem):

$$S_{EE} = c_1 \frac{\mathcal{A}}{\delta} - 2\pi c_0 \xrightarrow{\text{red arrow}} c_0 = \frac{1}{4\pi} \frac{L^2}{G}$$

- Thermal entropy (horizon entropy of planar black hole):

$$s = c_s T^2 \xrightarrow{\text{red arrow}} c_s = \frac{4\pi^2}{9} \frac{L^2}{G}$$

- Central charge (graviton propagator):

$$\langle T_{ab}(x) T_{cd}(0) \rangle = \frac{C_T}{x^6} \mathcal{I}_{ab,cd}(x) \xrightarrow{\text{red arrow}} C_T = \frac{3}{\pi^3} \frac{L^2}{G}$$

Does the corner term provide a “new” count of dof?

- enrich gravitational theory with higher curvature interactions

$$I = \frac{1}{16\pi G} \int d^4x \sqrt{g} \left[\frac{6}{L^2} + R + L^2 (\lambda_1 R^2 + \lambda_2 R_{\mu\nu} R^{\mu\nu} + \lambda_{\text{GB}} \mathcal{X}_4) + L^4 (\lambda_{3,0} R^3 + \lambda_{1,1} R \mathcal{X}_4) + L^6 (\lambda_{4,0} R^4 + \lambda_{2,1} R^2 \mathcal{X}_4 + \lambda_{0,2} \mathcal{X}_4^2) \right]$$

where $\mathcal{X}_4 = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4R_{\mu\nu} R^{\mu\nu} + R^2$

(Dong; Camps; Sinha et al;)

- Holographic Entanglement Entropy: $S(A) = \min_{V \sim A} S_{\text{grav}}(V)$

$$S_{\text{grav}} = \frac{\mathcal{A}(V)}{4G} + \frac{L^2}{4G} \int_V d^2y \sqrt{\gamma} \left[2\lambda_1 R + \lambda_2 \left(R^{\hat{a}}_{\hat{a}} - \frac{1}{2} K^{\hat{a}} K_{\hat{a}} \right) + 2\lambda_{\text{GB}} \mathcal{R} + \dots \right]$$

- Holographic Entanglement Entropy: $a(\Omega) = \alpha a_E(\Omega)$

Einstein gravity result 

$$\alpha = 1 - 24\lambda_1 - 6\lambda_2 + 432\lambda_{3,0} + 24\lambda_{1,1} - 6912\lambda_{4,0} - 576\lambda_{2,1} + \mathcal{O}(\lambda^2)$$

Compare other parameters to corner term:

- Universal term in circle entropy (appears in F-theorem):

$$\frac{q(\Omega)}{c_0} = \frac{q_E(\Omega)}{c_{0,E}} [1 - 2\lambda_{\text{GB}} - 24\lambda_{1,1} + 288\lambda_{2,1} + 96\lambda_{0,2} + \mathcal{O}(\lambda^2)]$$

- Thermal entropy (horizon entropy of planar black hole):

$$\frac{q(\Omega)}{c_s} = \frac{q_E(\Omega)}{c_{s,E}} [1 - 16\lambda_{0,2} + \mathcal{O}(\lambda^2)]$$

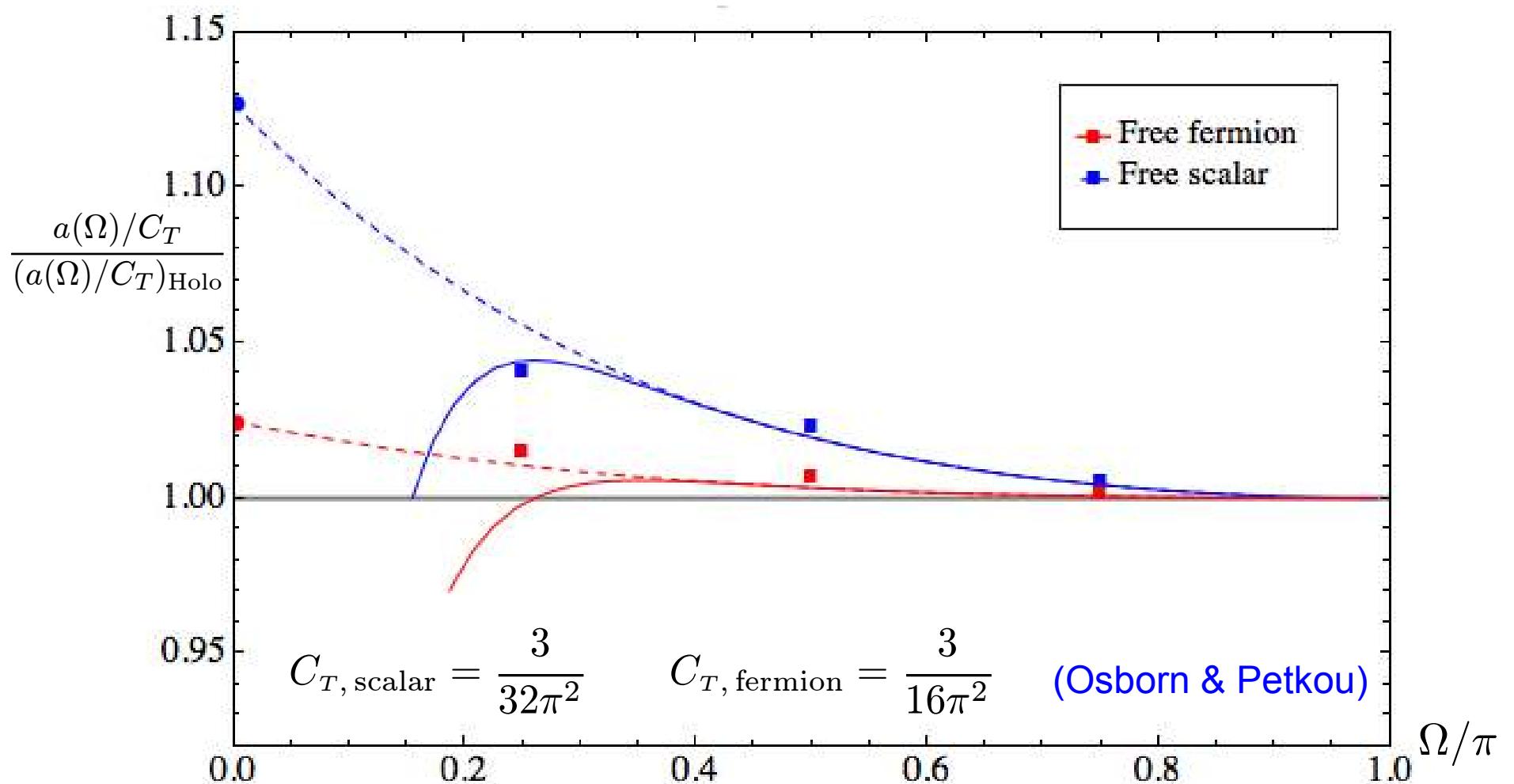
- Central charge (graviton propagator):

$$\frac{q(\Omega)}{C_T} = \frac{q_E(\Omega)}{C_{T,E}}$$



**for broad class of holographic CFT's,
corner term contains same info as central charge!**

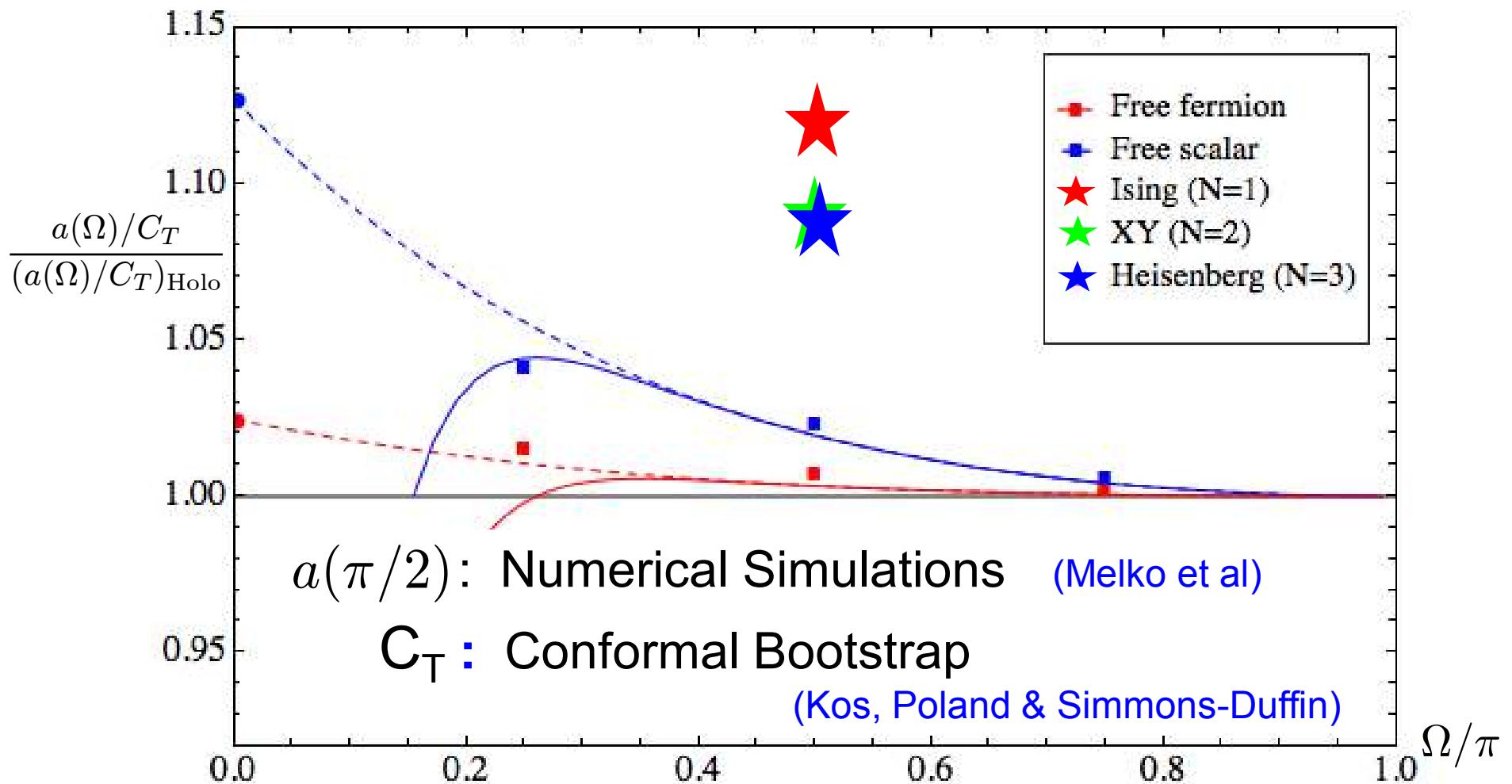
- holography suggests C_T provides an interesting normalization for corner term if we want to compare different theories
- $$a(\Omega)/C_T = a_E(\Omega)/C_{T,E}$$
- so lets compare: free massless scalar and fermion (Casini & Huerta)



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$$a(\Omega)/C_T = a_E(\Omega)/C_{T,E}$$

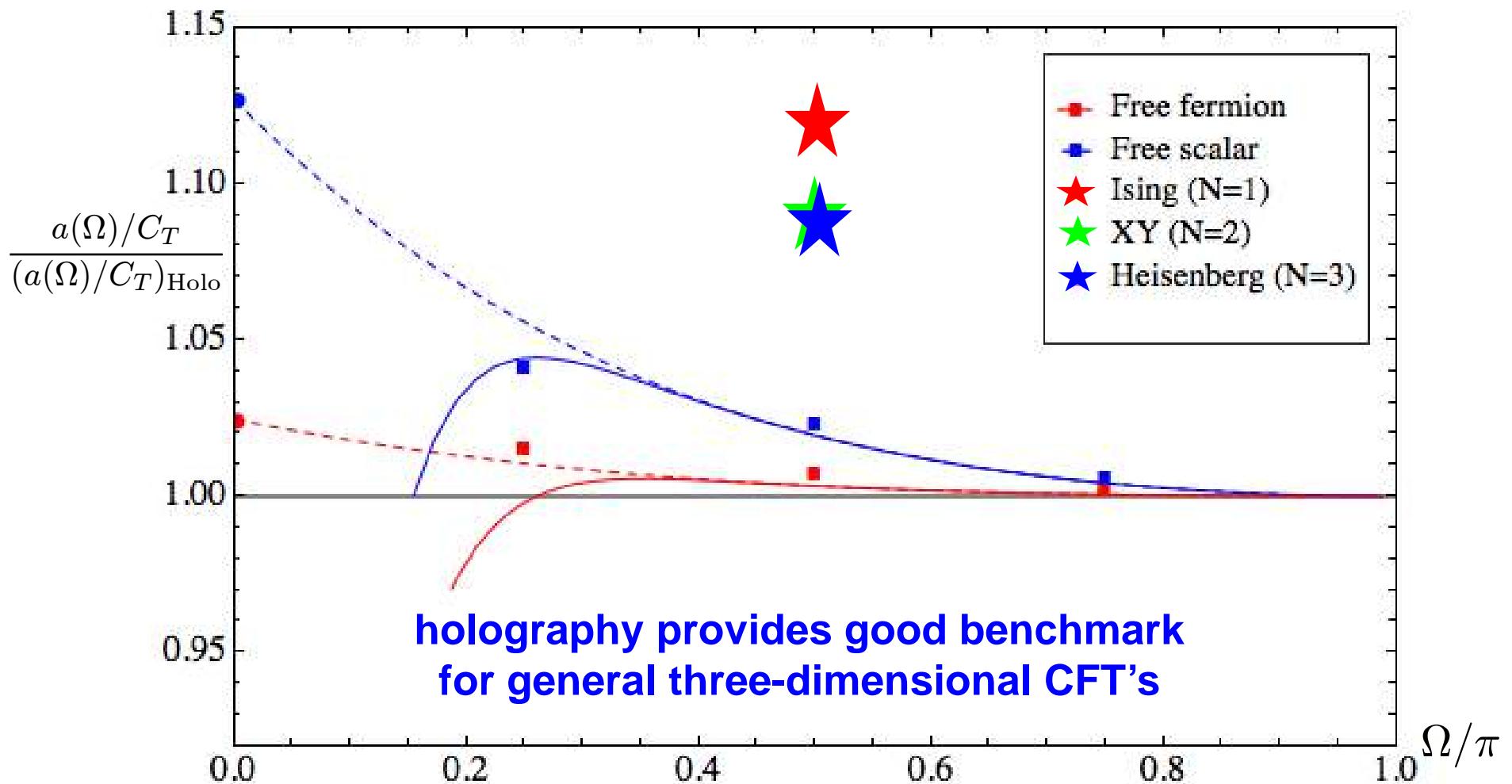
- so lets compare: Wilson-Fisher fixed pts for $O(N)$ vector model



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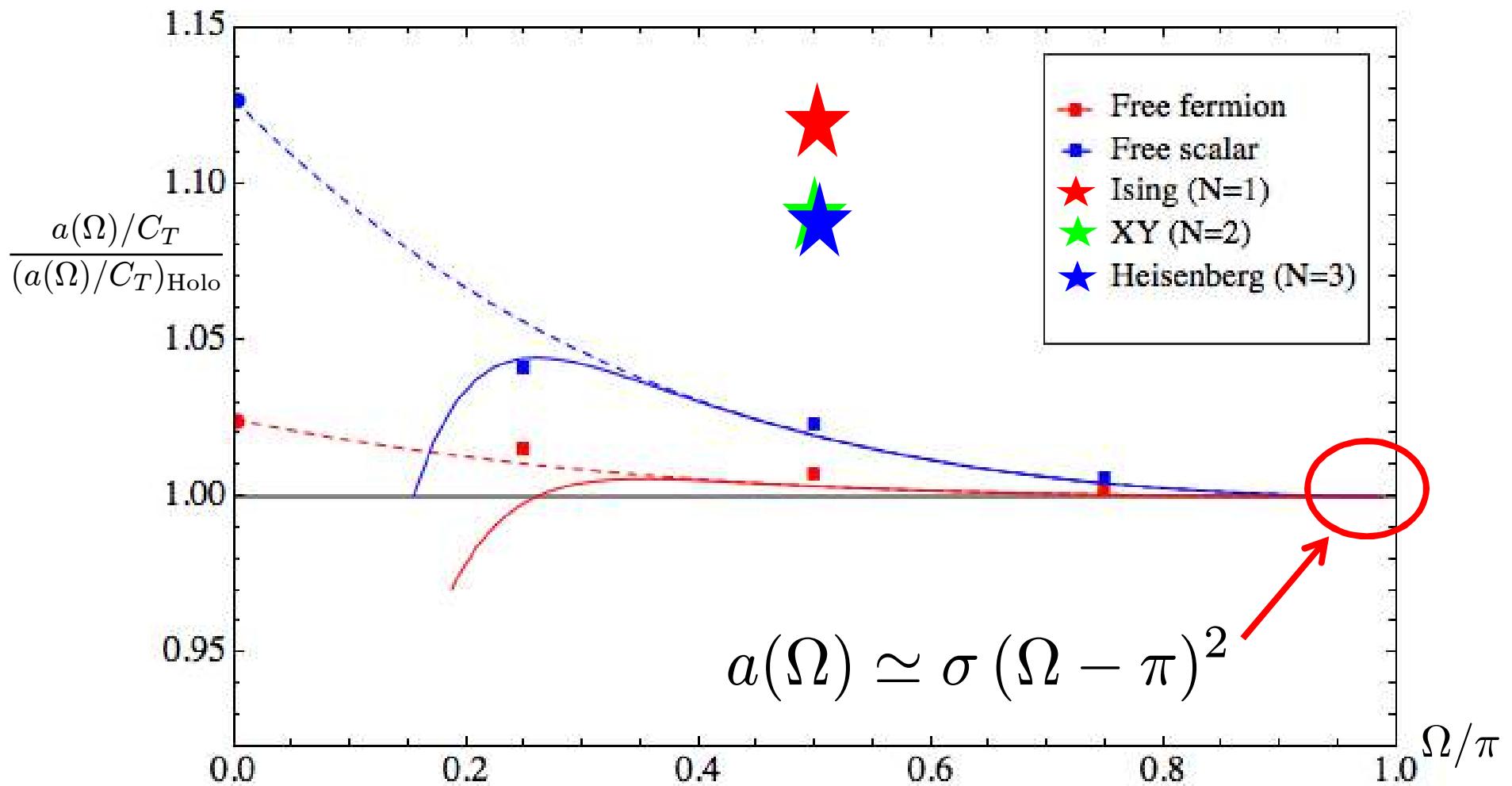
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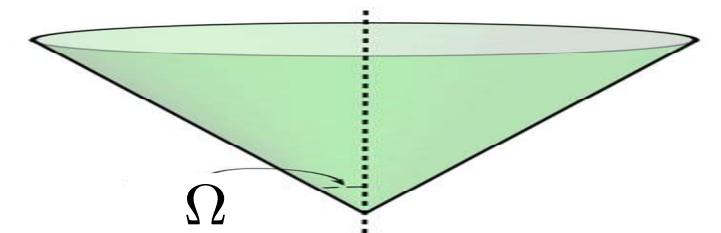
- holographic corner term is good benchmark for general CFT's
- Holography & free fields suggests σ/C_T is a universal constant

$$a(\Omega) \simeq \sigma (\Omega - \pi)^2 \quad \xrightarrow{\text{red arrow}} \quad \frac{\sigma}{C_T} = \frac{\pi^2}{24}$$

[analytic results by [Elvang & Hadjiantonis](#)]

- Renyi corner terms: $h_n/\sigma_n = (n-1)\pi$
- higher dimensions: $a_d(\Omega) \simeq \sigma_d (\Omega - \pi/2)^2$

$$\frac{\sigma_d}{C_{T,d}} = \frac{2(d-1)\pi^{d-1} i \left(\frac{d-1}{2}\right)^2}{(d-2) i \left(\frac{d-2}{2}\right)^2 i (d-2)}$$



[use holographic result of [Mezei](#)]

- proof?? (work in progress)

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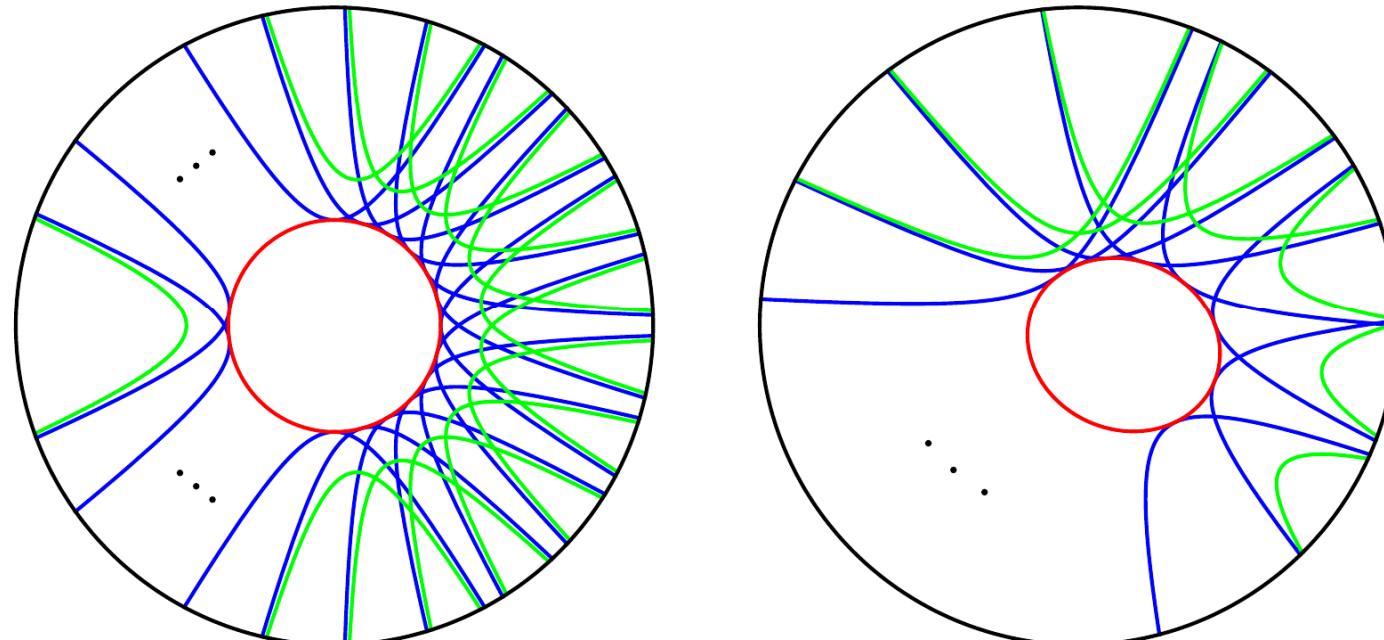
(Balasubramanian, McDermott & van Raamsdonk)

- meaning of $S_{BH} = \mathcal{A}/(4G_N)$ on a constant-r surface in bulk?
→ entanglement between high and low scales

(Balasubramanian, Chowdhury, Czech, de Boer & Heller)

- differential entropy is an observable in the boundary CFT that measures the BH entropy of general surfaces in bulk geometry

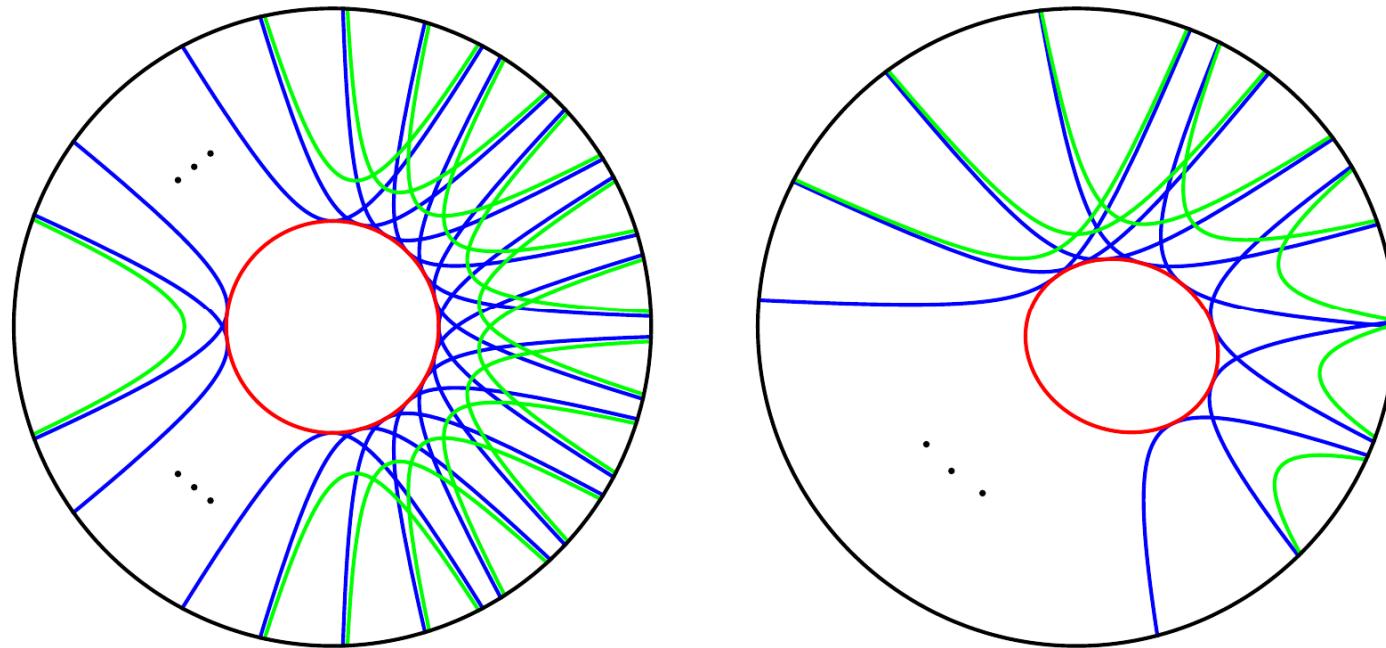
$$\frac{\mathcal{A}}{4G_N} = \sum_{k=1}^{\infty} [S(I_k) - S(I_k \setminus I_{k+1})] = - \oint d\lambda \left. \frac{\partial S(\gamma_L(\lambda'), \gamma_R(\lambda))}{\partial \lambda'} \right|_{\lambda'=\lambda}$$



(Balasubramanian, Chowdhury, Czech, de Boer & Heller)

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- generalizes to general bkgds, higher dim's, higher curvatures, . . .
(RM, Rao & Sugishita; Czech Dong & Sully; Headrick, RM & Wein)
- interpreted as QI task: “constrained state swapping”
(Czech, Hayden, Lashkari & Swingle)

(Czech, Lamprou; Czech, Lamprou, McCandlish & Sully)

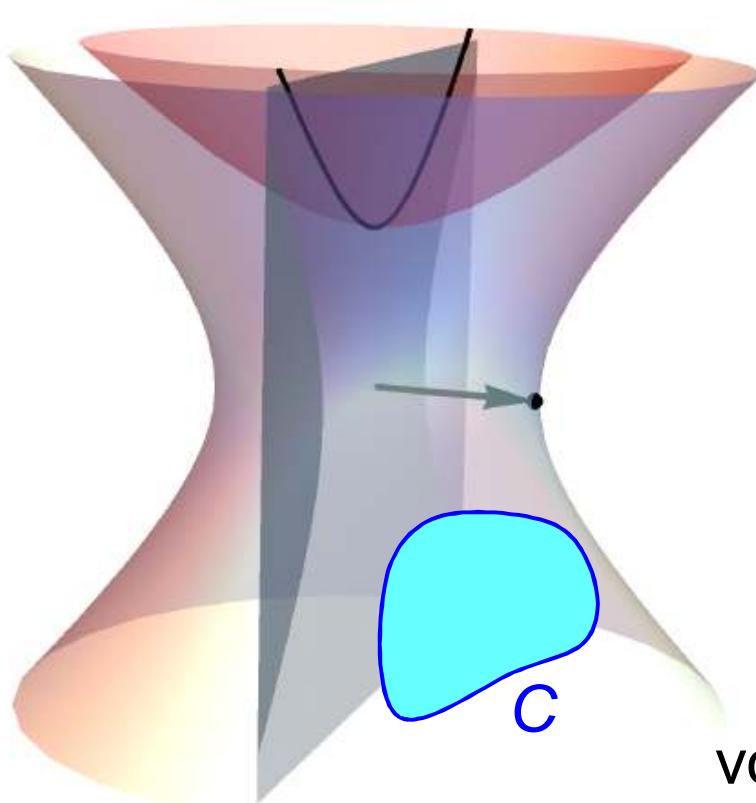
Kinematic Space:

- consider space of intervals $u < x < v$ on time slice of 2d CFT

\longleftrightarrow
holography

space of geodesics on 2d slice of AdS_3

\longleftrightarrow points in 2d de Sitter



$$ds^2 = \lambda \frac{du dv}{(v - u)^2}$$

scale?

interesting choice: $\lambda = \frac{c}{3}$

$$\longrightarrow ds^2 = \partial_u \partial_v S_0 du dv$$

with $S_0 = \frac{c}{3} \log \frac{v - u}{\delta}$

volume in dS_2 = differential entropy

$$\int_C \sqrt{-g} dudv = \int_C \partial_u \partial_v S_0 dudv = \oint_{\partial C} \partial_u S_0|_{v=v(u)} du$$

(Czech, Lamprou; Czech, Lamprou, McCandlish & Sully)

Kinematic Space:

- 2d de Sitter-like geometry: $ds^2 = \partial_u \partial_v S du dv$
with $S = S(v, u)$ on intervals $u < x < v$ in state of 2d CFT
- can leave behind original holographic motivations and study kinematic space for general CFT's, eg, de Sitter geometry in vacuum holds for any CFT



See talk by: Bartek Czech
“Integral geometry: from tensor networks to holography”

- could be kinematic space is a new forum for holography??

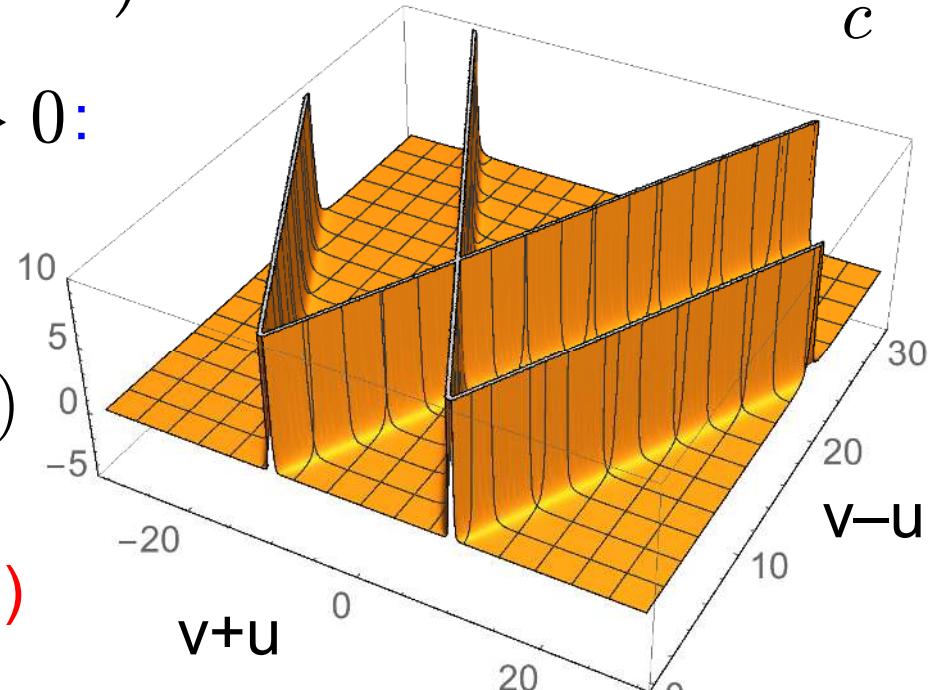
(Heller & RM; Czech & Lamprou)

Kinematic Space as Holographic Geometry (Hint?):

- 2d de Sitter-like geometry: $ds^2 = \partial_u \partial_v S du dv$
with $S = S(v, u)$ on intervals $u < x < v$ in state of 2d CFT
- consider excited state: $S = S_0(u, v) + \delta S(u, v)$
- “first law of EE”: $\delta S = \delta \langle H \rangle$
- wave equation on dS_2 : $(\nabla_0^2 + m^2) \delta S = 0$ with $m^2 L^2 = \frac{24}{c}$
- boundary data as $\ell = v - u \rightarrow 0$:

$$\delta S \sim \frac{\lambda_0}{\ell} + \lambda_2 \ell^2$$

with $\lambda_0 = 0$, $\lambda_2 = \frac{4\pi}{3} T_{00}(x)$
- applies for any CFT (no large N)



Entanglement & Holography:

- holographic entanglement entropy is an interesting forum for **dialogue** between string theory and other fields (eg, quantum information, condensed matter theory , . . .)

Dialogue:

- potential to learn lessons about issues in boundary theory
 - investigate entanglement structure of QFT's
 - insights into RG flows and c-theorems
 - challenge QI with “unconventional” questions
- potential to learn lessons about issues in quantum gravity in bulk
 - emergence of spacetime and gravitational dynamics?
 - geometric understanding of new measures, eg, negativity?
 - understand why there are no firewalls?
 - “**put your favourite question here**”?

[Last Slide]