A topologically twisted index for three dimensional gauge theories

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[work in collaboration with F. Benini]

Introduction

In recent years we have seen many progresses in the exact computation of partition functions of supersymmetric field theories on curved spaces

- sphere partition functions
- superconformal indices
- many others

Introduction

Here we consider a very simple example, a 3d gauge theory on $S^2 \times S^1$ where susy is preserved by a twist on S^2

$$(\nabla_\mu - iA^R_\mu)\epsilon \equiv \partial_\mu \epsilon = 0\,, \qquad\qquad \int_{S^2} F^R = 1$$

[Witten '88]

The result becomes interesting when supersymmetric backgrounds for the flavor symmetry multiplets (A^F_μ, σ^F, D^F) are turned on:

$$u^F = A^F_t + i\sigma^F\,, \qquad\qquad q^F = \int_{S^2} F^F = iD^F$$

and the path integral becomes a function of a set of magnetic charges q^F and chemical potentials u^F. We can also add a refinement for angular momentum.

[Benini-AZ; arXiv 1504.03698]

Introduction

Notice: we are not computing the superconformal index of the 3d gauge theory.

It is rather a twisted index:a trace over the Hilbert space $\mathcal{H}$ of states on a sphere in the presence of a magnetic background for the R and the global symmetries,

$$\mathrm{Tr}_{\mathcal{H}}\left((-1)^F e^{iJ_F A^F} e^{-\beta H}\right)$$

$Q^2 = H - \sigma^F J_F$

holomorphic in u^F

where J_F is the generator of the global symmetry.

Introduction

We can go up and down in dimension

- In a $(2,2)$ theory in 2d on S^2 we are computing amplitudes in gauged linear sigma models
- In a $\mathcal{N}=1$ theory on $S^2 \times T^2$ we are computing an elliptically generalized twisted index

We can explore the large N limit of these partition functions

- Holographic duals to AdS_4 static BPS black holes

The background

Consider an $\mathcal{N}=2$ gauge theory on $S^2 \times S^1$

$$ds^2 = R^2\big(d\theta^2 + \sin^2\theta\, d\varphi^2\big) + \beta^2 dt^2$$

with a background for the R-symmetry proportional to the spin connection:

$$A^R = -\frac{1}{2}\cos\theta\, d\varphi = -\frac{1}{2}\omega^{12}$$

so that the Killing spinor equation

$$D_\mu\epsilon = \partial_\mu\epsilon + \frac{1}{4}\omega_\mu^{ab}\gamma_{ab}\epsilon - iA_\mu^R\epsilon = 0 \qquad \Longrightarrow \qquad \epsilon = \text{const}$$

The partition function

The path integral for an $\mathcal{N}=2$ gauge theory on $S^2 \times S^1$ with gauge group G localizes on a set of BPS configurations specified by data in the vector multiplets $V = (A_\mu, \sigma, \lambda, \lambda^\dagger, D)$

- A magnetic flux on S^2, $\mathfrak{m} = \frac{1}{2\pi}\int_{S^2} F$ in the co-root lattice
- A Wilson line A_t along S^1
- The vacuum expectation value σ of the real scalar

Up to gauge transformations, the BPS manifold is

$$(u = A_t + i\sigma, \mathfrak{m}) \in \mathcal{M}_{\text{BPS}} = \big(H \times \mathfrak{h} \times \Gamma_{\mathfrak{h}}\big)/W$$

The partition function

The path integral reduces to a the saddle point around the BPS configurations

$$\sum_{\mathfrak{m}\in\Gamma_{\mathfrak{h}}} \int du d\bar{u}\, \mathcal{Z}^{\text{cl } +\text{1-loop}}(u, \bar{u}, \mathfrak{m})$$

- The integrand has various singularities where chiral fields become massless
- There are fermionic zero modes

The two things nicely combine and the path integral reduces to an r-dimensional contour integral of a meromorphic form

$$\boxed{\frac{1}{|W|} \sum_{\mathfrak{m}\in\Gamma_{\mathfrak{h}}} \oint_C Z_{\text{int}}(u, \mathfrak{m})}$$

The partition function

The classical and 1-loop contribution give a meromorphic form

$$Z_{\text{int}}(u,\mathfrak{m}) = Z_{\text{class}} Z_{\text{1-loop}}$$

in each sector with gauge flux $\mathfrak{m}$, where

$$\boxed{Z^{\text{CS}}_{\text{class}} = x^{k\mathfrak{m}}} \qquad x = e^{iu}$$

$$\boxed{Z^{\text{chiral}}_{\text{1-loop}} = \prod_{\rho\in\mathfrak{R}} \left[\frac{x^{\rho/2}}{1-x^{\rho}}\right]^{\rho(\mathfrak{m})-q+1}} \qquad q = \text{R charge}$$

$$\boxed{Z^{\text{gauge}}_{\text{1-loop}} = \prod_{\alpha\in G} (1-x^{\alpha})\,(i\,du)^r}$$

The partition function

The magnetic flux on S^2 generates Landau levels. Massive bosons and fermions cancel in pairs, while zero modes give

$$|\rho(\mathfrak{m}) - q + 1| \ \text{Fermi multiplets on } S^1 \qquad \rho(\mathfrak{m}) - q + 1 < 0$$

$$|\rho(\mathfrak{m}) - q + 1| \ \text{Chiral multiplets on } S^1 \qquad \rho(\mathfrak{m}) - q + 1 > 0$$

reduces to Witten index of (0,2) Quantum Mechanics

$$\mathrm{Tr}_{\mathcal{H}} \left((-1)^F e^{iJ_F A_t^F} e^{-\beta H} \right) \Big|_{\mathfrak{R}} = \prod_{\rho \in \mathfrak{R}} \left[\frac{x^{\rho/2}}{1 - x^\rho} \right] \quad , \quad \prod_{\rho \in \mathfrak{R}} \left[\frac{1 - x^\rho}{x^{\rho/2}} \right]$$

Chiral Fermi

[compare with Hori,Kim,Yi '14]

The partition function

The magnetic flux on S^2 generates Landau levels. Massive bosons and fermions cancel in pairs, while zero modes

$|\rho(\mathfrak{m}) - q + 1|$ Fermi multiplets on S^1 $\qquad \rho(\mathfrak{m}) - q + 1 < 0$

$|\rho(\mathfrak{m}) - q + 1|$ Chiral multiplets on S^1 $\qquad \rho(\mathfrak{m}) - q + 1 > 0$

$$\boxed{Z^{\text{chiral}}_{\text{1-loop}} = \prod_{\rho \in \mathfrak{R}} \left[\frac{x^{\rho/2}}{1 - x^{\rho}} \right]^{\rho(\mathfrak{m}) - q + 1}}$$

Due to magnetic flux for R-symmetry, R charges of the fields must be integer.

The contour

$Z_{\text{int}}(u, \mathfrak{m})$ has pole singularities at

- along the hyperplanes $x^\rho = e^{i\rho(u)} = \mathbb{1}$ determined by the chiral fields
- at the boundaries of $H \times \mathfrak{h}$ $\qquad (\mathrm{Im}(u) = \pm\infty, \;\; x = e^{iu} = 0, \infty)$

Supersymmetric localization selects a particular contour of integration C and picks some of the residues of the form $Z_{\text{int}}(u, \mathfrak{m})$.

The contour

Consider a $U(1)$ theory with chiral fields with charges Q_i. We can use the prescription: sum the residues

- at the poles of fields with positive charge, at $x = 0$ if $k_{\text{eff}}(+\infty) < 0$ and at $x = \infty$ if $k_{\text{eff}}(-\infty) > 0$

where the effective Chern-Simons coupling is defined as

$$k_{eff}(\sigma) = k + \frac{1}{2}\sum_i Q_i^2 \mathrm{sign}(Q_i\sigma)$$

The contour

The prescription can be written in a compact form by using the so-called Jeffrey-Kirwan residue

$$\underset{y=0}{\text{JK-Res}}(Q,\eta)\frac{dy}{y} = \theta(Q\eta)\mathrm{sign}(Q)$$

as

$$\boxed{\frac{1}{|W|}\sum_{\mathfrak{m}\in\Gamma_{\mathfrak{h}}}\left[\sum_{x_*\in\mathfrak{M}_{\text{sing}}}\underset{x=x_*}{\text{JK-Res}}\left(\mathrm{Q}(x_*),\eta\right)\,Z_{\text{int}}(x;\mathfrak{m}) \;+\; \underset{x=0,\infty}{\text{JK-Res}}(Q_x,\eta)\,Z_{\text{int}}(x;\mathfrak{m})\right]}$$

where

$$Q_{x=0} = -k_{\text{eff}}(+\infty)\,, \qquad\qquad Q_{x=\infty} = k_{\text{eff}}(-\infty)$$

Similar to the localization of the elliptic genus for 2d theories and of the Witten index in 1d [Benini,Eager,Hori,Tachikawa '13; Hori,Kim,Yi '14]

A Simple Example: $U(1)_{1/2}$ with one chiral

The theory has just a topological $U(1)_T$ symmetry: $J_\mu = \epsilon_{\mu\nu\tau} F_{\nu\tau}$

$$Z = \sum_{\mathfrak{m}\in\mathbb{Z}} \int \frac{dx}{2\pi i x} x^{\mathfrak{t}} (-\xi)^{\mathfrak{m}} x^{\mathfrak{m}/2} \left(\frac{x^{1/2}}{1-x}\right)^{\mathfrak{m}} = \frac{\xi}{(1-\xi)^{\mathfrak{t}+1}}$$

$$k_{\text{eff}}(\sigma) = \frac{1}{2} + \frac{1}{2}\text{sign}(\sigma) \qquad \to \qquad Q_{x=0} = -1,\ Q_{x=\infty} = 0$$

Consistent with duality with a free chiral.

	$U(1)_g$	$U(1)_T$	$U(1)_R$
X	1	0	1
T	0	1	0
$\tilde{T}$	-1	-1	0

Aharony and Giveon-Kutasov dualities

The twisted index can be used to check dualities: for example, $U(N_c)$ with $N_f = N_c$ flavors is dual to a theory of chiral fields M_{ab}, T and $\tilde{T}$, coupled through the superpotential $W = T\tilde{T}\det M$

$$Z_{N_f=N_c} = \left(\frac{y}{1-y^2}\right)^{(2\mathfrak{n}-1)N_c^2}\left(\frac{\xi^{\frac{1}{2}}y^{-\frac{N_c}{2}}}{1-\xi y^{-N_c}}\right)^{N_c(1-\mathfrak{n})+\mathfrak{t}}\left(\frac{\xi^{-\frac{1}{2}}y^{-\frac{N_c}{2}}}{1-\xi^{-1}y^{-N_c}}\right)^{N_c(1-\mathfrak{n})-\mathfrak{t}}$$

Aharony and Giveon-Kutasov dual pairs for generic (N_c, N_f) have the same partition function.

Refinement by angular momentum

Adding a fugacity $\zeta = e^{i\varsigma/2}$ for the angular momentum on S^2: the Landau zero-modes on S^2 form a representation of $SU(2)$.

$$\boxed{Z^{\text{chiral}}_{\text{1-loop}} = \prod_{\rho\in\mathfrak{R}} \prod_{j=-\frac{|B|-1}{2}}^{\frac{|B|-1}{2}} \left(\frac{x^{\rho/2}\zeta^j}{1-x^\rho\zeta^{2j}}\right)^{\text{sign}\, B}}\,, \qquad B = \rho(\mathfrak{m}) - q_\rho + 1$$

As noticed in other contexts: the refined partition function factorizes into the product of two vortex partition functions

$$Z = Z_{\text{1-loop}}\, Z_{\text{vortex}}(\zeta)\, Z_{\text{vortex}}(\zeta^{-1})$$

[Pasquetti '11;Beem-Dimofte-Pasquetti '12;Cecotti-Gaiotto-Vafa '13,$\cdots$]

Other dimensions

We can consider other dimensions too: $(2,2)$ theories in 2d on S^2

The BPS manifold is now $\mathfrak{M} = (\mathfrak{h} \times \mathfrak{h})/W$ and the 1-loop determinants

$$\boxed{Z_{\text{1-loop}}^{\text{chiral}} = \prod_{\rho \in \mathfrak{R}} \left[\frac{1}{\rho(\sigma)} \right]^{\rho(\mathfrak{m}) - q + 1}}$$

$$\boxed{Z_{\text{1-loop}}^{\text{gauge}} = (-1)^{\sum_{\alpha>0} \alpha(\mathfrak{m})} \prod_{\alpha \in G} \alpha(\sigma) \, (d\sigma)^r}$$

Other dimensions

We are just repackaging results about the A-twist of gauged linear sigma models

For examples, for $U(1)$ with N flavors, 2d amplitudes compute the quantum cohomology of $\mathbb{P}^{N-1}$

$$\langle \sigma_1 \cdots \sigma_n \rangle = \sum_{\mathfrak{m}} \int \frac{dx}{2\pi i} \frac{1}{x^{(\mathfrak{m}+1)N}} q^{\mathfrak{m}} x^n = \sum_{\mathfrak{m}} q^{\mathfrak{m}} \delta_{N(\mathfrak{m}+1)-n-1,0}$$

$$\boxed{\sigma^N = q}$$

$$\boxed{\prod_{j=1}^{N} (\sigma - \mu_j) = q}$$

$\Omega-$background and non abelian G can be considered [related work by Cremonesi,Closset,Park '15]

Other dimensions

We can consider other dimensions too: $\mathcal{N}=1$ theories in 4d on $S^2\times T^2$, and obtain an elliptical generalization of our index.

The BPS manifold is now $\mathfrak{M}=(H\times H)/W$ and the 1-loop determinants

$$\boxed{Z^{\text{chiral}}_{\text{1-loop}}=\prod_{\rho\in\mathfrak{R}}\prod_{j=-\frac{|B|-1}{2}}^{\frac{|B|-1}{2}}\left(\frac{i\eta(q)}{\theta_1(q,x^\rho\zeta^{2j})}\right)^{\text{sign}(B)}}$$

$$\boxed{Z^{\text{gauge, off}}_{\text{1-loop}}=(-1)^{\sum_{\alpha>0}\alpha(\mathfrak{m})}\prod_{\alpha\in G}\frac{\theta_1\left(q,x^\alpha\zeta^{|\alpha(\mathfrak{m})|}\right)}{i\eta(q)}(du)^r}$$

[also Closset-Shamir '13;Nishioka-Yaakov '14]

[related work by Yoshida-Honda '15]

Other dimensions

The index on $S^2 \times T^2$ reduces to the elliptic genus of a flux dependent collection of $(0,2)$ multiplets on T^2

$$|\rho(\mathfrak{m}) - q + 1| \quad \text{Fermi multiplets on } T^2 \qquad \rho(\mathfrak{m}) - q + 1 < 0$$

$$|\rho(\mathfrak{m}) - q + 1| \quad \text{Chiral multiplets on } T^2 \qquad \rho(\mathfrak{m}) - q + 1 > 0$$

- Again R-charges should be integer.
- It can be tested against Seiberg's dualities.
- It adds to and complete the list of existing tools (superconformal indices, sphere partition functions) for testing 4d dualities.

Conclusions

We gave a general formula for the topologically twisted path integral of 2d (2,2), 3d $\mathcal{N} = 2$ and 4d $\mathcal{N} = 1$ theories.

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- Many generalizations: higher genus Riemann surfaces, $\cdots$

Conclusions

We gave a general formula for the topologically twisted path integral of 2d (2,2), 3d $\mathcal{N}=2$ and 4d $\mathcal{N}=1$ theories.

- Many generalizations: higher genus Riemann surfaces, $\cdots$
- Many questions also

Question: role of gauge fluxes?

Sometime they cancel

- Particular choices of background fluxes; reduction $4d$ $\mathcal{N}=1$ to 2d $(0,2)$
- Particular choices of background fluxes: 3d index reduces to Hilbert series for Higgs branch of $\mathcal{N}=4$ theories [Gadde,Razamat,Willett '15]

Sometime they are crucial

- checking 3d and 4d dualities
- Particular choices of background fluxes: 3d index reduces to the formula for the Hilbert series for Coulomb branch of $\mathcal{N}=4$ theories [Cremonesi,Hanany,AZ '13]

[Observations made with N. Mekareeya]

Question: large N limit

Part of the original motivation for this work comes from holography.

[Thanks for many related discussions to K. Hristov, A. Tomasiello]

CFTs on curved space-times described by dual regular asymptotically AdS backgrounds

$$ds_4^2 = \frac{dr^2}{r^2} + (r^2 ds^2_{M_3} + O(r)) \qquad A = A_{M_3} + O(1/r)$$

Classifications of M_3 supersymmetric backgrounds (transverse holomorphic foliations)

[Klare-Tomasiello-AZ '12; Closset-Dumitrescu-Festuccia-Komargodski '12]

Question: large N limit

Twisted $M_3 = S^2 \times S^1$ leads to 1/4 BPS asymptotically AdS_4 static black holes

- solutions asymptotic to *magnetic* AdS_4 and with horizon $\mathrm{AdS}_2 \times S^2$
- Characterized by a collection of magnetic charges $\int_{S^2} F$
- preserving supersymmetry via a twist

$$(\nabla_\mu - iA_\mu)\epsilon = \partial_\mu \epsilon \qquad \Longrightarrow \qquad \epsilon = \text{cost}$$

Various regular solutions with horizon, some embeddable in $\mathrm{AdS}_4 \times S^7$.

[Cacciatori, Klemm '09; Gnecchi, Dall'Agata '10; Hristov, Vandoren '10];

Question: large N limit

Should we expect that the partition function of ABJM on twisted $S^2 \times S^1$ knows about

- black hole free energy?
- black hole entropy?

Preliminary analysis shows that it scale indeed as $N^{3/2}$ [F. Benini, K. Hristov, AZ]

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Hope to report more in the future $\cdots$