The Vernacular of the S-Matrix

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Niels Bohr International Academy
and Discovery Center

The Niels Bohr International Academy

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June 25, 2015

Organization and Outline

1. *Spiritus Movens*: the Discovery of On-Shell Physics
   - Using *Generalized Unitarity* to Compute One-Loop Amplitudes
2. Upgrading Unitarity at One-Loop: the *Chiral* Box Expansion
   - *Chiral* Boxes Expansion for One-Loop *Integrands*
   - Making *Manifest* the Finiteness of All Finite Observables
3. Generalizing Unitarity for Two-Loop Amplitudes & Integrands
   - The Two-Loop Chiral *Integrand* Expansion
   - *Novel* Contributions at Two-Loops and *Transcendentality*
   - Local, Integrand-Level Representations of All Two-Loop Amplitudes

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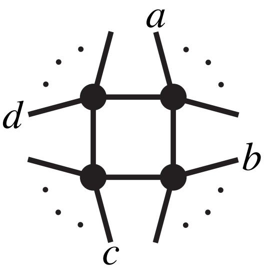
$I_{a,b,c,d} \equiv$ 

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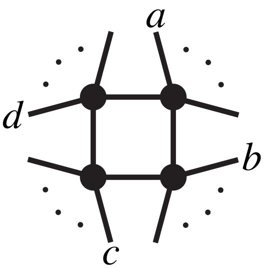

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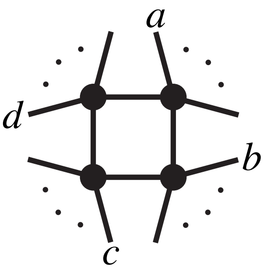
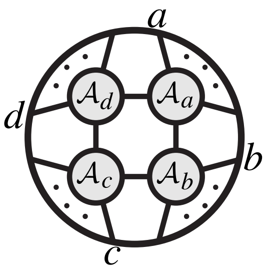
$I_{a,b,c,d} \equiv$ 

$f_{a,b,c,d} \equiv$ 

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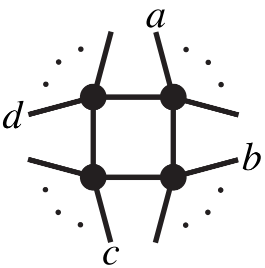
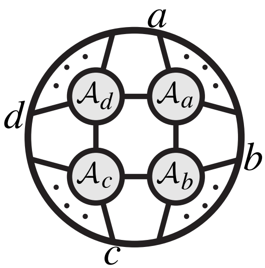
$I_{a,b,c,d} \equiv$ 

$f_{a,b,c,d} \equiv$ 

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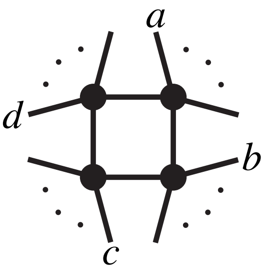

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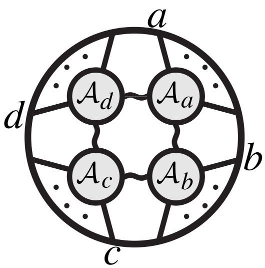

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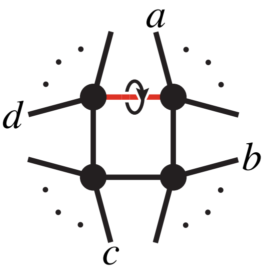

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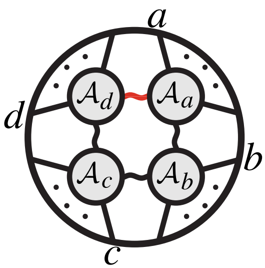

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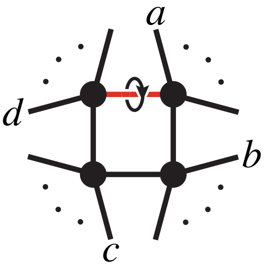
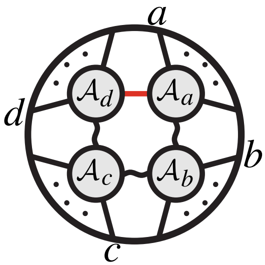
$I_{a,b,c,d} \equiv$ 

$f_{a,b,c,d} \equiv$ 

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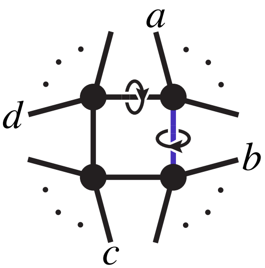

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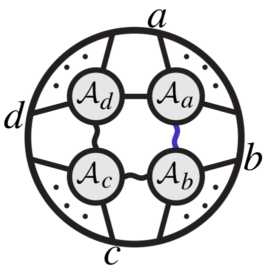

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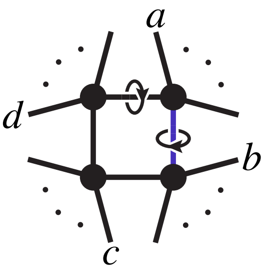

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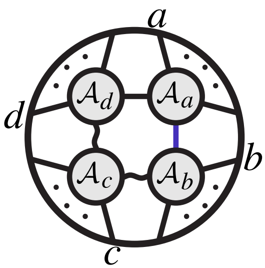

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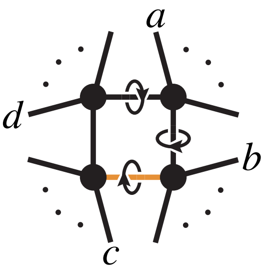

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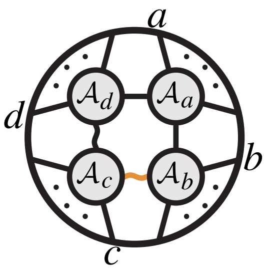

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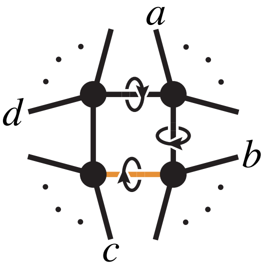

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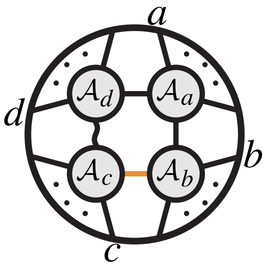

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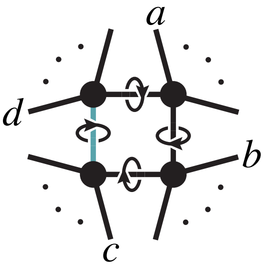
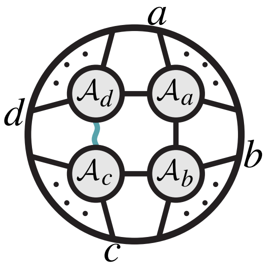
$I_{a,b,c,d} \equiv$ 

$f_{a,b,c,d} \equiv$ 

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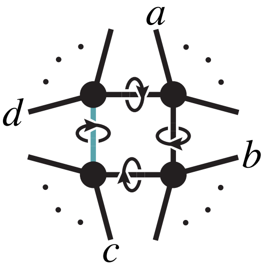

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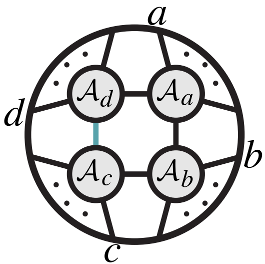

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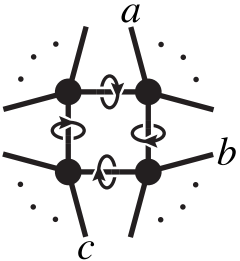

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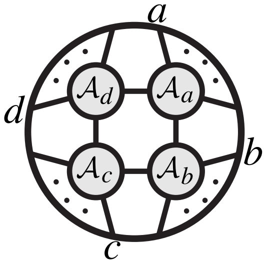

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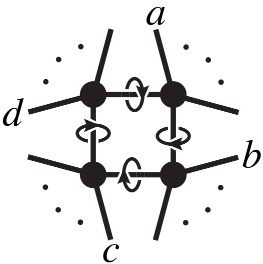

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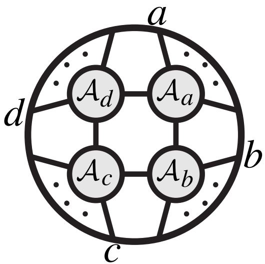

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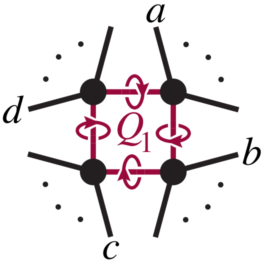

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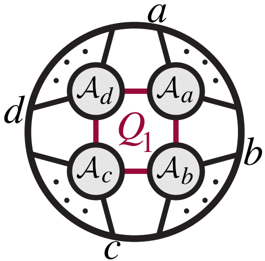

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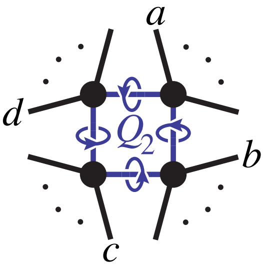
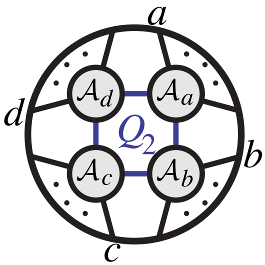
$I_{a,b,c,d} \equiv$ 

$f^2_{a,b,c,d} \equiv$ 

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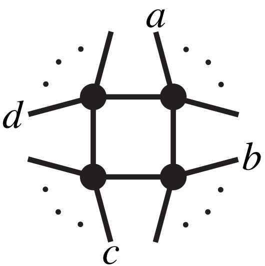
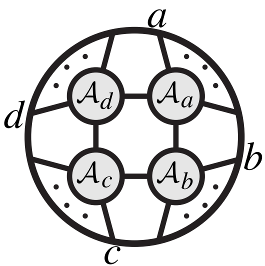
$I_{a,b,c,d} \equiv$ 

$f_{a,b,c,d} \equiv$ 

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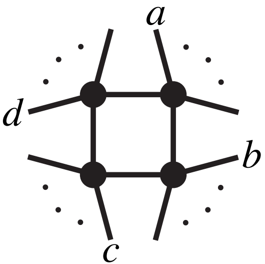
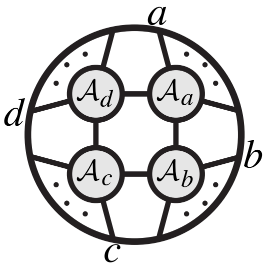
$I_{a,b,c,d} \equiv$ 

$f_{a,b,c,d} \equiv$ 

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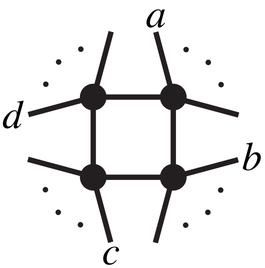
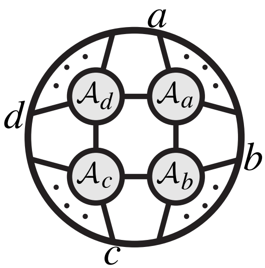
$I_{a,b,c,d} \equiv$ 

$f_{a,b,c,d} \equiv$ 

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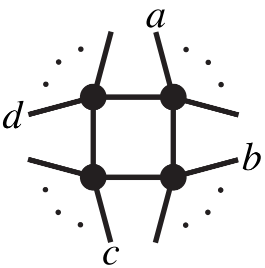
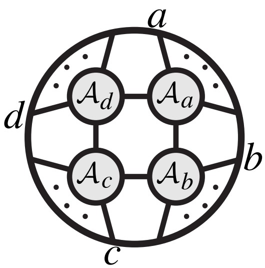
$I_{a,b,c,d} \equiv$ 

$f_{a,b,c,d} \equiv$ 

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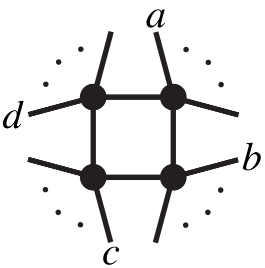
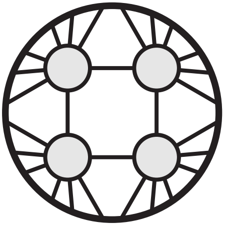
$I_{a,b,c,d} \equiv$ 

$f_{a,b,c,d} \Leftrightarrow$ 

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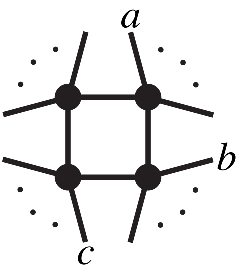

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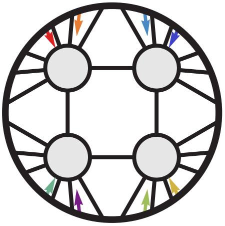

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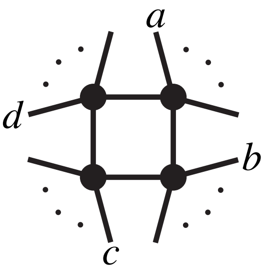
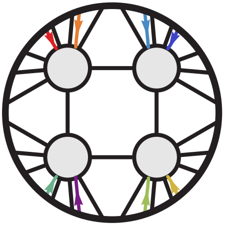
$I_{a,b,c,d} \equiv$ 

$f_{a,b,c,d} \Leftrightarrow$ 

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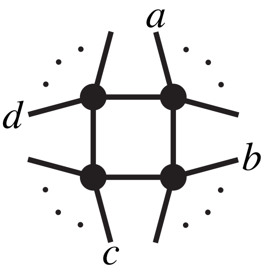
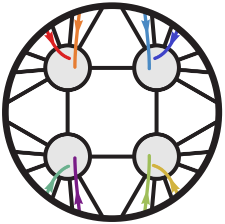
$I_{a,b,c,d} \equiv$ 

$f_{a,b,c,d} \Leftrightarrow$ 

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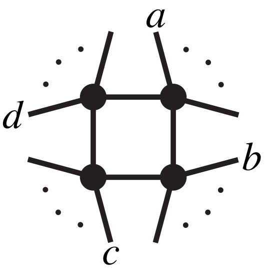
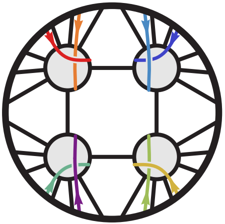
$I_{a,b,c,d} \equiv$ 

$f_{a,b,c,d} \Leftrightarrow$ 

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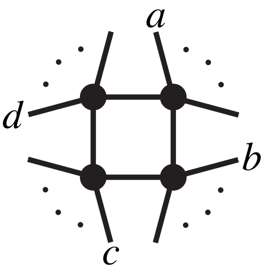
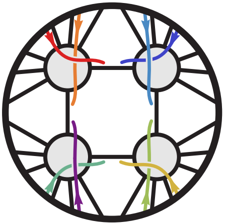
$I_{a,b,c,d} \equiv$ 

$f_{a,b,c,d} \Leftrightarrow$ 

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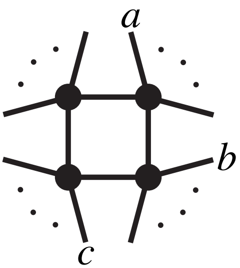
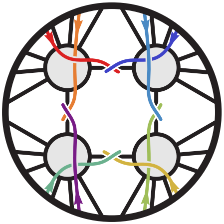
$I_{a,b,c,d} \equiv$ 

$f_{a,b,c,d} \Leftrightarrow$ 

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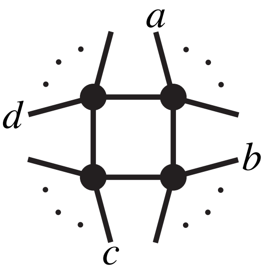
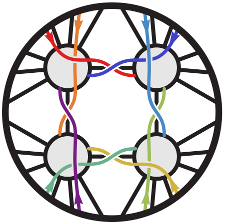
$I_{a,b,c,d} \equiv$ 

$f_{a,b,c,d} \Leftrightarrow$ 

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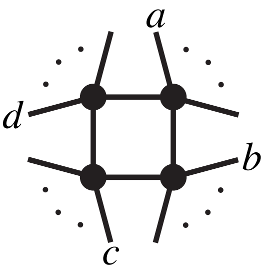
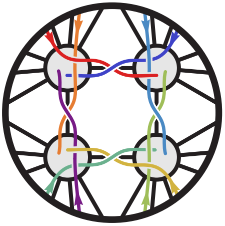
$I_{a,b,c,d} \equiv$ 

$f_{a,b,c,d} \Leftrightarrow$ 

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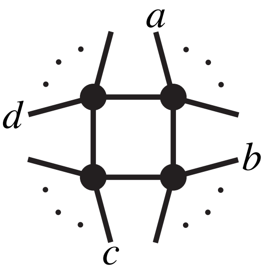
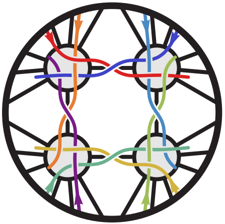
$I_{a,b,c,d} \equiv$ 

$f_{a,b,c,d} \Leftrightarrow$ 

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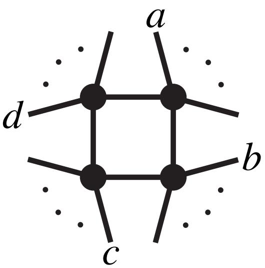
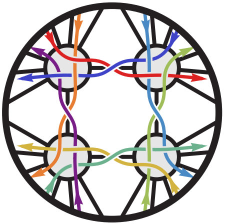
$I_{a,b,c,d} \equiv$ 

$f_{a,b,c,d} \Leftrightarrow$ 

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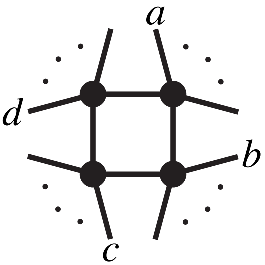
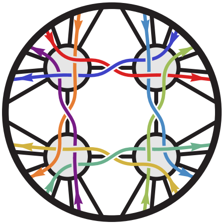
$I_{a,b,c,d} \equiv$ 

$f_{a,b,c,d} \Leftrightarrow$ 

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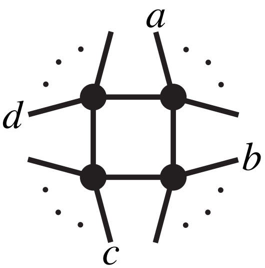
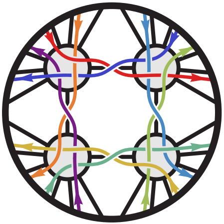
$I_{a,b,c,d} \equiv$ 

$f_{a,b,c,d} \Leftrightarrow$ 

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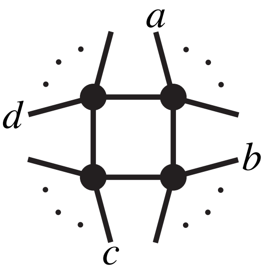
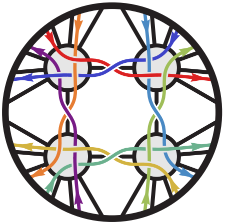
$I_{a,b,c,d} \equiv$ 

$f_{a,b,c,d} \Leftrightarrow$ 

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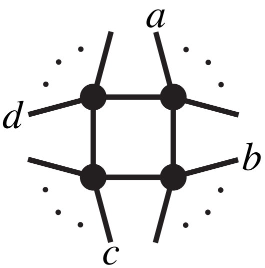
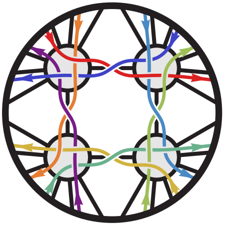
$I_{a,b,c,d} \equiv$ 

$f_{a,b,c,d} \Leftrightarrow$ 

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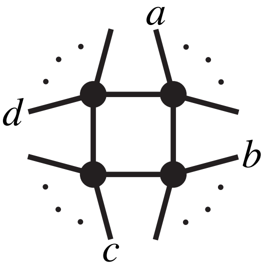

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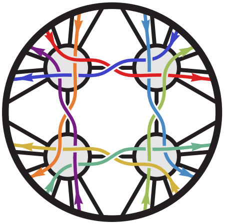

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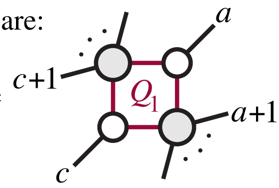

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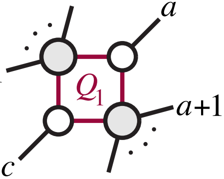

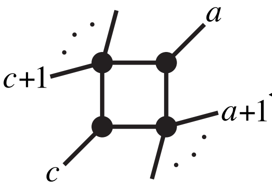

$$\Leftrightarrow \int d^4\ell\, \frac{(a,c)(a,a+1)-(a,c+1)(c,a+1)}{(\ell,a)(\ell,a+1)(\ell,c)(\ell,c+1)}$$

A 'Box'-Expansion for One-Loop *Integrands*

The *Scalar* Box Expansion for the One-Loop Amplitude

$$\int d^4\ell\, \mathcal{A}_n^{(2),1} = \sum_{a,c} I_{a,a+1,c,c+1}\, f^1_{a,a+1,c,c+1}$$

Consider for example the 'MHV' amplitude $(k=2)$, for which $f^2_{a,b,c,d} = 0$, and the only non-vanishing $f^1_{a,b,c,d}$ are:

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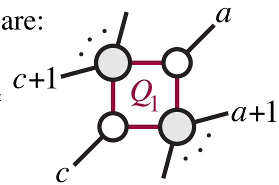
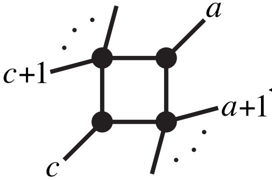




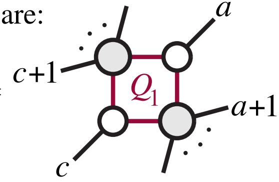
$$\Leftrightarrow \int d^4\ell\, \frac{(a,c)(a,a{+}1)-(a,c{+}1)(c,a{+}1)}{(\ell,a)(\ell,a{+}1)(\ell,c)(\ell,c{+}1)}$$

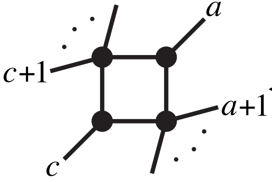
A 'Box'-Expansion for One-Loop *Integrands*

The *Scalar* Box Expansion for the One-Loop Amplitude

$$\int d^4\ell\, \mathcal{A}_n^{(2),1} = \sum_{a,c} I_{a,a+1,c,c+1}\, f^1_{a,a+1,c,c+1}$$

Consider for example the 'MHV' amplitude $(k=2)$, for which $f^2_{a,b,c,d} = 0$, and the only non-vanishing $f^1_{a,b,c,d}$ are:

$f^1_{a,a+1,c,c+1} =$ 



$$\Leftrightarrow \int d^4\ell\, \underbrace{\frac{(a,c)(a,a{+}1)-(a,c{+}1)(c,a{+}1)}{(\ell,a)(\ell,a{+}1)(\ell,c)(\ell,c{+}1)}}_{\mathcal{I}_{a,a+1,c,c+1}}$$

A 'Box'-Expansion for One-Loop *Integrands*

The *Scalar* Box Expansion for the One-Loop Amplitude

$$\int d^4\ell\,\mathcal{A}_n^{(2),1} = \sum_{a,c} I_{a,a+1,c,c+1}\, f^1_{a,a+1,c,c+1}$$

Consider for example the 'MHV' amplitude $(k=2)$, for which $f^2_{a,b,c,d}=0$, and the only non-vanishing $f^1_{a,b,c,d}$ are:

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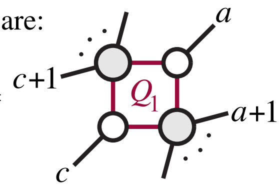

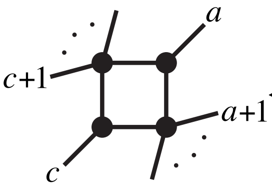

$$\Leftrightarrow \int d^4\ell\, \underbrace{\frac{(a,c)(a,a{+}1)-(a,c{+}1)(c,a{+}1)}{(\ell,a)(\ell,a{+}1)(\ell,c)(\ell,c{+}1)}}_{\mathcal{I}_{a,a+1,c,c+1}}$$

A 'Box'-Expansion for One-Loop *Integrands*

The *Scalar* Box Expansion for the One-Loop Amplitude

$$\int d^4\ell\,\mathcal{A}_n^{(2),1}=\sum_{a,c} I_{a,a+1,c,c+1}\,f^1_{a,a+1,c,c+1}$$

Consider for example the 'MHV' amplitude $(k=2)$, for which $f^2_{a,b,c,d}=0$, and the only non-vanishing $f^1_{a,b,c,d}$ are:

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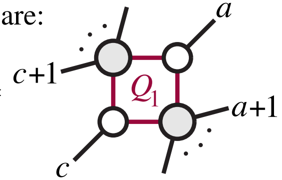

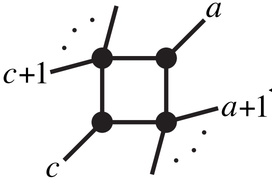

$$\Leftrightarrow \int d^4\ell\,\underbrace{\frac{(a,c)(a,a+1)-(a,c+1)(c,a+1)}{(\ell,a)(\ell,a+1)(\ell,c)(\ell,c+1)}}_{\mathcal{I}_{a,a+1,c,c+1}}$$

A 'Box'-Expansion for One-Loop *Integrands*

The *Scalar* Box Expansion for the One-Loop Amplitude

$$\int d^4\ell\,\mathcal{A}_n^{(2),1} = \sum_{a,c} I_{a,a+1,c,c+1}\, f^1_{a,a+1,c,c+1}$$

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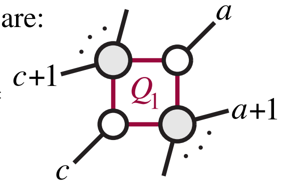

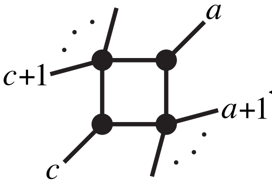

$$\Leftrightarrow \underbrace{\int d^4\ell\,\frac{(a,c)(a,a+1)-(a,c+1)(c,a+1)}{(\ell,a)(\ell,a+1)(\ell,c)(\ell,c+1)}}_{\mathcal{I}_{a,a+1,c,c+1}}$$

A 'Box'-Expansion for One-Loop *Integrands*

The *Scalar* Box Expansion for the One-Loop Amplitude

$$\int d^4\ell\,\mathcal{A}_n^{(2),1} = \sum_{a,c} I_{a,a+1,c,c+1}\, f^1_{a,a+1,c,c+1}$$

Consider for example the 'MHV' amplitude $(k=2)$, for which $f^2_{a,b,c,d}=0$, and the only non-vanishing $f^1_{a,b,c,d}$ are:

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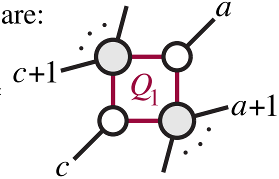

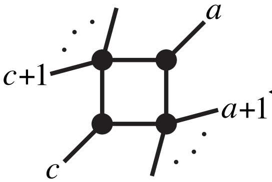

$$\Leftrightarrow \underbrace{\int d^4\ell\,\frac{(a,c)(a,a{+}1)-(a,c{+}1)(c,a{+}1)}{(\ell,a)(\ell,a{+}1)(\ell,c)(\ell,c{+}1)}}_{\mathcal{I}_{a,a+1,c,c+1}}$$

A ‘Box’-Expansion for One-Loop *Integrands*

The *Scalar* Box Expansion for the One-Loop Amplitude

$$\int d^4\ell\,\mathcal{A}_n^{(2),1} = \sum_{a,c} I_{a,a+1,c,c+1}\, f^1_{a,a+1,c,c+1}$$

Consider for example the ‘MHV’ amplitude $(k=2)$, for which $f^2_{a,b,c,d} = 0$, and the only non-vanishing $f^1_{a,b,c,d}$ are:

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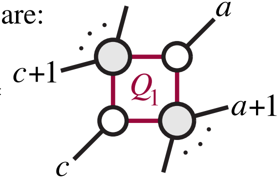

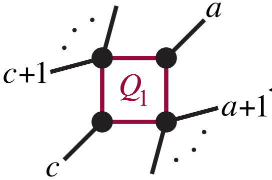

$$\Leftrightarrow \int d^4\ell\, \underbrace{\frac{(a,c)(a,a{+}1)-(a,c{+}1)(c,a{+}1)}{(\ell,a)(\ell,a{+}1)(\ell,c)(\ell,c{+}1)}\,\frac{(\ell,Q_2)(X,Q_1)}{(\ell,X)(Q_2,Q_1)}}_{\mathcal{I}^1_{a,a+1,c,c+1}}$$

A 'Box'-Expansion for One-Loop *Integrands*

The *Scalar* Box Expansion for the One-Loop Amplitude

$$\int d^4\ell\,\mathcal{A}_n^{(2),1} = \sum_{a,c} I_{a,a+1,c,c+1}\, f^1_{a,a+1,c,c+1}$$

Consider for example the 'MHV' amplitude $(k\!=\!2)$, for which $f^2_{a,b,c,d} = 0$, and the only non-vanishing $f^1_{a,b,c,d}$ are:

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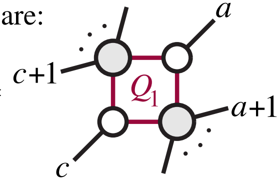

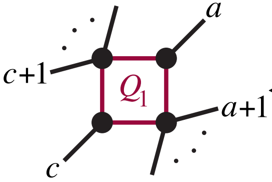

$$\Leftrightarrow \int d^4\ell\, \underbrace{\frac{(a,c)(a,a{+}1){-}(a,c{+}1)(c,a{+}1)}{(\ell,a)(\ell,a{+}1)(\ell,c)(\ell,c{+}1)}\,\frac{(\ell,Q_2)(X,Q_1)}{(\ell,X)(Q_2,Q_1)}}_{\mathcal{I}^1_{a,a+1,c,c+1}}$$

A ‘Box’-Expansion for One-Loop *Integrands*

A *Chiral* ‘Box’-Expansion for the One-Loop Amplitude Integrand

$$\mathcal{A}_n^{(2),1} \overset{¿?}{=} \sum_{a,c} \mathcal{I}^1_{a,a+1,c,c+1}\, f^1_{a,a+1,c,c+1}$$

Consider for example the ‘MHV’ amplitude $(k=2)$, for which $f^2_{a,b,c,d}=0$, and the only non-vanishing $f^1_{a,b,c,d}$ are:

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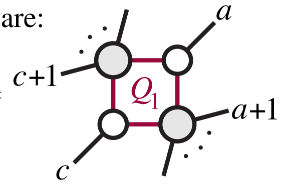

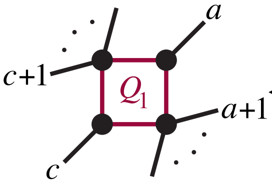

$$\Leftrightarrow \underbrace{\int d^4\ell\, \frac{(a,c)(a,a{+}1)-(a,c{+}1)(c,a{+}1)}{(\ell,a)(\ell,a{+}1)(\ell,c)(\ell,c{+}1)}\, \frac{(\ell,Q_2)(X,Q_1)}{(\ell,X)(Q_2,Q_1)}}_{\mathcal{I}^1_{a,a+1,c,c+1}}$$

A 'Box'-Expansion for One-Loop *Integrands*

A *Chiral* 'Box'-Expansion for the One-Loop Amplitude Integrand

$$\mathcal{A}_n^{(k),1} \overset{¿?}{=} \sum_{a,b,c,d} \left(\mathcal{I}^1_{a,b,c,d} f^1_{a,b,c,d} + \mathcal{I}^2_{a,b,c,d} f^2_{a,b,c,d} \right)$$

Consider for example the 'MHV' amplitude $(k=2)$, for which $f^2_{a,b,c,d} = 0$, and the only non-vanishing $f^1_{a,b,c,d}$ are:

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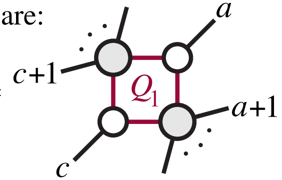

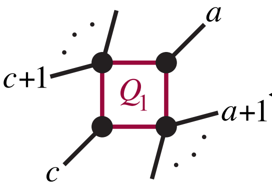

$$\Leftrightarrow \underbrace{\int d^4\ell \, \frac{(a,c)(a,a{+}1){-}(a,c{+}1)(c,a{+}1)}{(\ell,a)(\ell,a{+}1)(\ell,c)(\ell,c{+}1)} \, \frac{(\ell,Q_2)(X,Q_1)}{(\ell,X)(Q_2,Q_1)}}_{\mathcal{I}^1_{a,a+1,c,c+1}}$$

A ‘Box’-Expansion for One-Loop *Integrands*

A *Chiral* ‘Box’-Expansion for the One-Loop Amplitude Integrand

$$\mathcal{A}_n^{(k),1} \overset{¿?}{=} \sum_{a,b,c,d} \left(\mathcal{I}^1_{a,b,c,d} f^1_{a,b,c,d} + \mathcal{I}^2_{a,b,c,d} f^2_{a,b,c,d} \right)$$

A ‘Box’-Expansion for One-Loop *Integrands*

A *Chiral* ‘Box’-Expansion for the One-Loop Amplitude Integrand

$$\mathcal{A}_n^{(k),1} \overset{¿?}{=} \sum_{a,b,c,d} \left(\mathcal{I}^1_{a,b,c,d} f^1_{a,b,c,d} + \mathcal{I}^2_{a,b,c,d} f^2_{a,b,c,d} \right)$$

This ansatz matches the correct integrand on **all** co-dimension four residues

A ‘Box’-Expansion for One-Loop *Integrands*

A *Chiral* ‘Box’-Expansion for the One-Loop Amplitude Integrand

$$\mathcal{A}_n^{(k),1} \overset{¿?}{=} \sum_{a,b,c,d} \left(\mathcal{I}^1_{a,b,c,d} f^1_{a,b,c,d} + \mathcal{I}^2_{a,b,c,d} f^2_{a,b,c,d} \right)$$

This ansatz matches the correct integrand on **all** co-dimension four residues *involving four distinct propagators*.

A ‘Box’-Expansion for One-Loop *Integrands*

A *Chiral* ‘Box’-Expansion for the One-Loop Amplitude Integrand

$$\mathcal{A}_n^{(k),1} \stackrel{¿?}{=} \sum_{a,b,c,d} \left(\mathcal{I}^1_{a,b,c,d} f^1_{a,b,c,d} + \mathcal{I}^2_{a,b,c,d} f^2_{a,b,c,d}\right)$$

This ansatz matches the correct integrand on **all** co-dimension four residues *involving four distinct propagators*. **However**, each chiral box is **IR-finite**!

A 'Box'-Expansion for One-Loop *Integrands*

A *Chiral* 'Box'-Expansion for the One-Loop Amplitude Integrand

$$\mathcal{A}_n^{(k),1} \overset{¿?}{=} \sum_{a,b,c,d} \left(\mathcal{I}^1_{a,b,c,d} f^1_{a,b,c,d} + \mathcal{I}^2_{a,b,c,d} f^2_{a,b,c,d} \right)$$

This ansatz matches the correct integrand on **all** co-dimension four residues *involving four distinct propagators*. **However**, each chiral box is **IR-finite**!

There are **also** co-dimension four residues involving only three propagators:

A 'Box'-Expansion for One-Loop *Integrands*

A *Chiral* 'Box'-Expansion for the One-Loop Amplitude Integrand

$$\mathcal{A}_n^{(k),1} \overset{¿?}{=} \sum_{a,b,c,d} \left(\mathcal{I}^1_{a,b,c,d} f^1_{a,b,c,d} + \mathcal{I}^2_{a,b,c,d} f^2_{a,b,c,d} \right)$$

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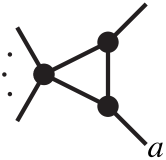

A 'Box'-Expansion for One-Loop *Integrands*

A *Chiral* 'Box'-Expansion for the One-Loop Amplitude Integrand

$$\mathcal{A}_n^{(k),1} \overset{¿?}{=} \sum_{a,b,c,d} \left(\mathcal{I}^1_{a,b,c,d} f^1_{a,b,c,d} + \mathcal{I}^2_{a,b,c,d} f^2_{a,b,c,d} \right)$$

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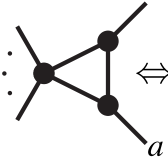

$$\Leftrightarrow \int d^4\ell \, \frac{(a{-}1, a{+}1)(a, X)}{(\ell, a{-}1)(\ell, a)(\ell, a{+}1)(\ell, X)}$$

A 'Box'-Expansion for One-Loop *Integrands*

A *Chiral* 'Box'-Expansion for the One-Loop Amplitude Integrand

$$\mathcal{A}_n^{(k),1} \overset{?}{=} \sum_{a,b,c,d} \left(\mathcal{I}^1_{a,b,c,d} f^1_{a,b,c,d} + \mathcal{I}^2_{a,b,c,d} f^2_{a,b,c,d} \right)$$

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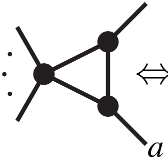
$\mathcal{I}^a_{\text{div}} \equiv$ 

$\Leftrightarrow \displaystyle\int d^4\ell \, \frac{(a{-}1, a{+}1)(a, X)}{(\ell, a{-}1)(\ell, a)(\ell, a{+}1)(\ell, X)}$

A 'Box'-Expansion for One-Loop *Integrands*

A *Chiral* 'Box'-Expansion for the One-Loop Amplitude Integrand

$$\mathcal{A}_n^{(k),1} \overset{¿?}{=} \sum_{a,b,c,d} \left(\mathcal{I}^1_{a,b,c,d} f^1_{a,b,c,d} + \mathcal{I}^2_{a,b,c,d} f^2_{a,b,c,d} \right)$$

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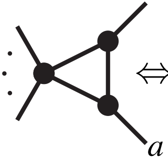
There are **also** co-dimension four residues involving only three propagators:

$$\mathcal{I}^a_{\text{div}} \equiv$$  $$\Leftrightarrow \int d^4\ell \, \frac{(a-1, a+1)(a, X)}{(\ell, a-1)(\ell, a)(\ell, a+1)(\ell, X)}$$

and the residue about the point $\ell \to x_a$ must be the tree amplitude: $\mathcal{A}_n^{(k),0}$

A 'Box'-Expansion for One-Loop *Integrands*

A *Chiral* 'Box'-Expansion for the One-Loop Amplitude Integrand

$$\mathcal{A}_n^{(k),1} \overset{¿?}{=} \sum_{a,b,c,d}\left(\mathcal{I}^1_{a,b,c,d} f^1_{a,b,c,d} + \mathcal{I}^2_{a,b,c,d} f^2_{a,b,c,d}\right)$$

This ansatz matches the correct integrand on **all** co-dimension four residues *involving four distinct propagators*. **However**, each chiral box is **IR-finite**!

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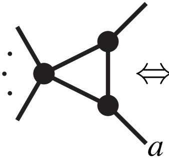
$\mathcal{I}^a_{\text{div}} \equiv$



$$\Leftrightarrow \int d^4\ell \, \frac{(a-1, a+1)(a, X)}{(\ell, a-1)(\ell, a)(\ell, a+1)(\ell, X)}$$

and the residue about the point $\ell \to x_a$ must be the tree amplitude: $\mathcal{A}_n^{(k),0}$

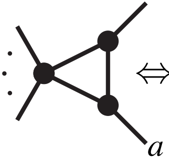
A ‘Box’-Expansion for One-Loop *Integrands*

A *Chiral* ‘Box’-Expansion for the One-Loop Amplitude Integrand

$$\mathcal{A}_n^{(k),1} \overset{¿?}{=} \sum_{a,b,c,d}\left(\mathcal{I}^1_{a,b,c,d}f^1_{a,b,c,d}+\mathcal{I}^2_{a,b,c,d}f^2_{a,b,c,d}\right)+\mathcal{A}_n^{(k),0}\sum_{a}\mathcal{I}^a_{\text{div}}$$

This ansatz matches the correct integrand on **all** co-dimension four residues *involving four distinct propagators*. **However**, each chiral box is **IR-finite**!

There are **also** co-dimension four residues involving only three propagators:

$\mathcal{I}^a_{\text{div}} \equiv$  $\displaystyle\Leftrightarrow\int d^4\ell\,\frac{(a-1,a+1)(a,X)}{(\ell,a-1)(\ell,a)(\ell,a+1)(\ell,X)}$

and the residue about the point $\ell\to x_a$ must be the tree amplitude: $\mathcal{A}_n^{(k),0}$

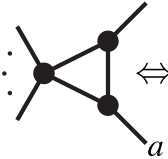
A ‘Box’-Expansion for One-Loop *Integrands*

A *Chiral* ‘Box’-Expansion for the One-Loop Amplitude Integrand

$$\mathcal{A}_n^{(k),1} \overset{¿?}{=} \sum_{a,b,c,d} \left(\mathcal{I}^1_{a,b,c,d} f^1_{a,b,c,d} + \mathcal{I}^2_{a,b,c,d} f^2_{a,b,c,d} \right) + \mathcal{A}_n^{(k),0} \sum_a \mathcal{I}^a_{\text{div}}$$

This ansatz matches the correct integrand on **all** co-dimension four residues *involving four distinct propagators*. **However**, each chiral box is **IR-finite**!

There are **also** co-dimension four residues involving only three propagators:

$$\mathcal{I}^a_{\text{div}} \equiv$$ 

$$\Leftrightarrow \int d^4\ell \, \frac{(a-1, a+1)(a, X)}{(\ell, a-1)(\ell, a)(\ell, a+1)(\ell, X)}$$

and the residue about the point $\ell \to x_a$ must be the tree amplitude: $\mathcal{A}_n^{(k),0}$

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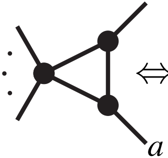
$\mathcal{I}^a_{\text{div}} \equiv$  $\Leftrightarrow \displaystyle\int d^4\ell\, \frac{(a-1,a+1)(a,X)}{(\ell,a-1)(\ell,a)(\ell,a+1)(\ell,X)}$

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Constructing Local Integrands for Two-Loop Amplitudes

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"*Merging*" One-Loop, Chiral (X-dependent) **Integrands**

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$\mathcal{I}_L(X)\bigotimes\mathcal{I}_R(X)$

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$$\mathcal{I}_L(X) \otimes \mathcal{I}_R(X) \equiv \mathcal{I}'_L \frac{(\mathcal{N}_L, X)}{(\ell_1, X)} \otimes \frac{(X, \mathcal{N}_R)}{(X, \ell_2)} \mathcal{I}'_R$$

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Constructing Local Integrands for Two-Loop Amplitudes

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$$|||$$

$$\frac{(b-1,b+1)(b,X)}{(\ell_1,b-1)(\ell_1,b)(\ell_1,b+1)(\ell_1,X)}\bigotimes\frac{(X,a)(a-1,a+1)}{(X,\ell_2)(\ell_2,a-1)(\ell_2,a)(\ell_2,a+1)}$$

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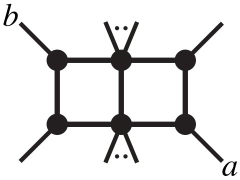

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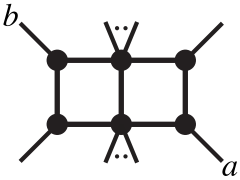

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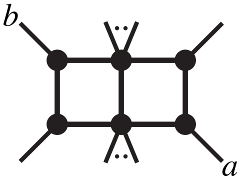

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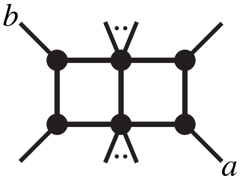

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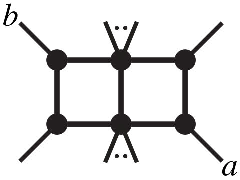

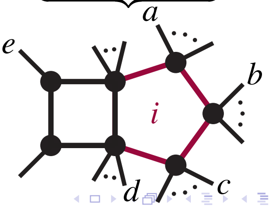

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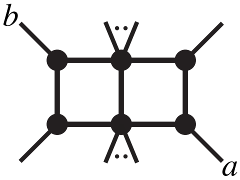

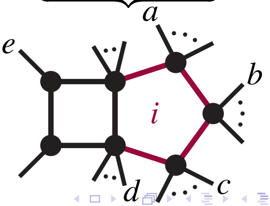

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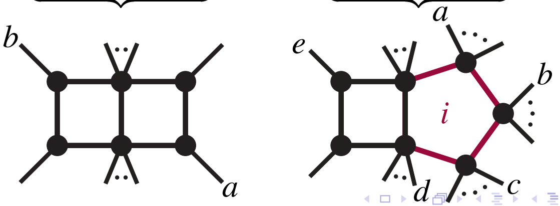

Finite Integrand Contributions to Two-Loop Amplitudes

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1. "Kissing" Boxes:

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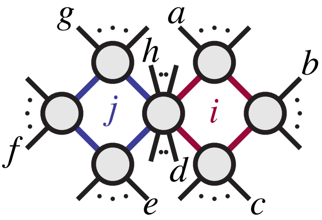

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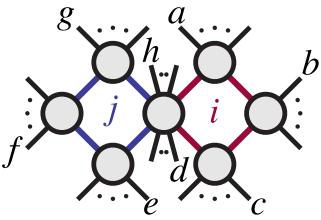

$$\times \left(\mathcal{I}^{j}_{e,f,g,h} \otimes \mathcal{I}^{i}_{a,b,c,d}\right)$$

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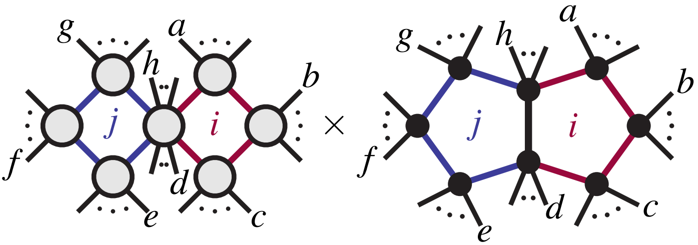

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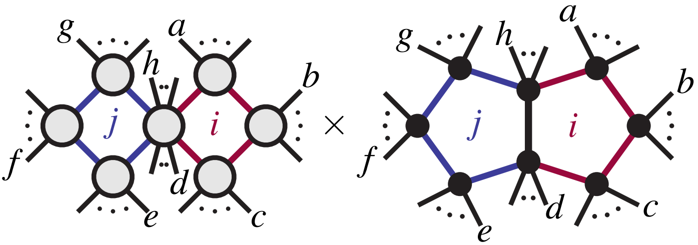

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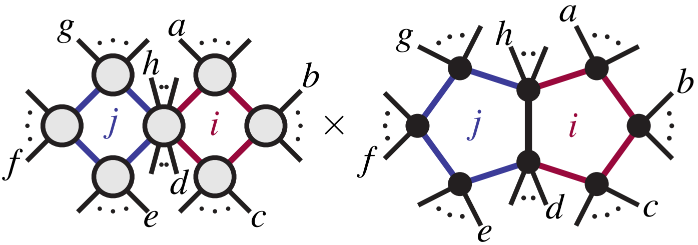

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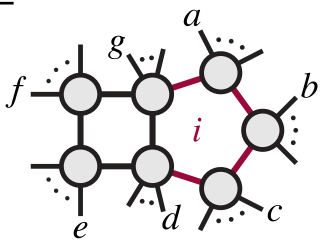

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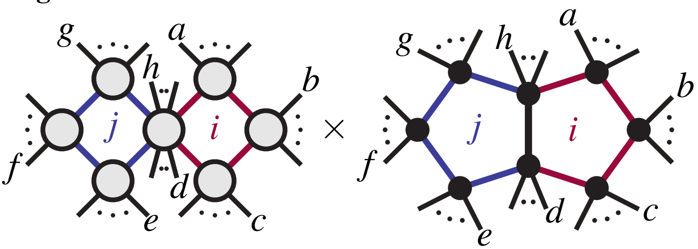

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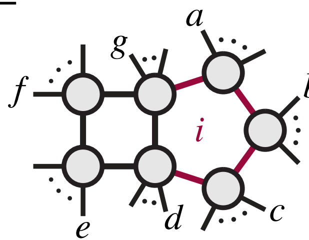

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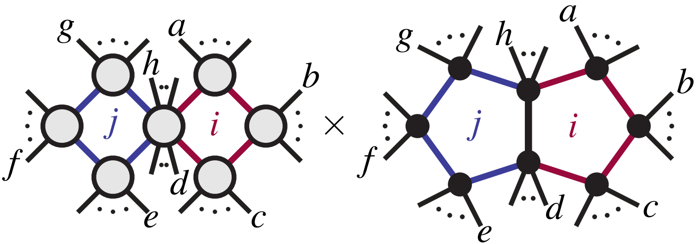

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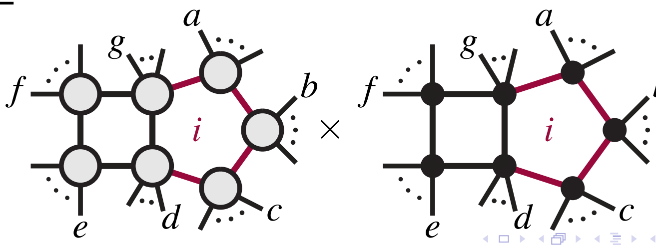

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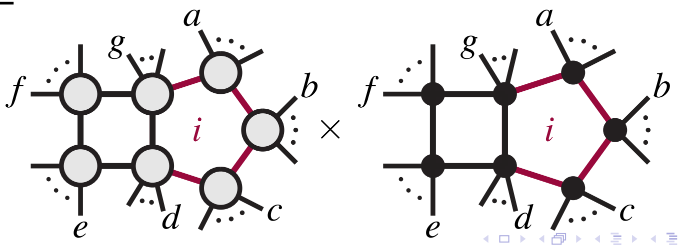

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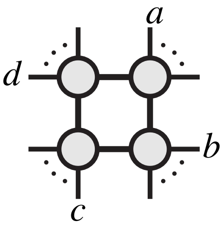

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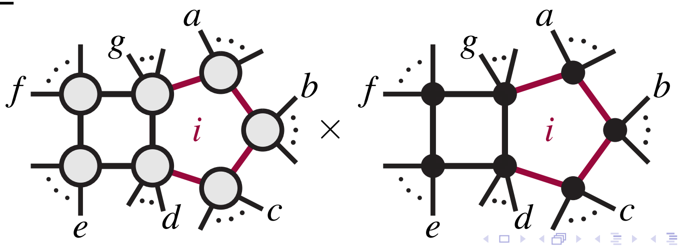

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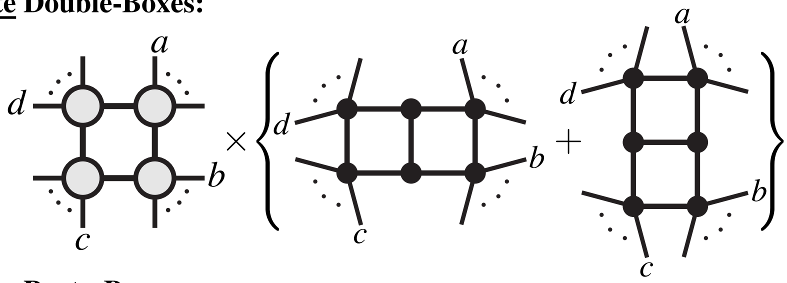

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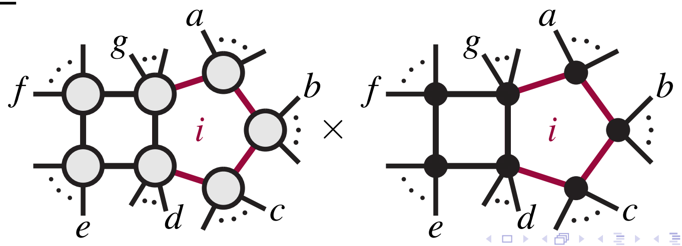

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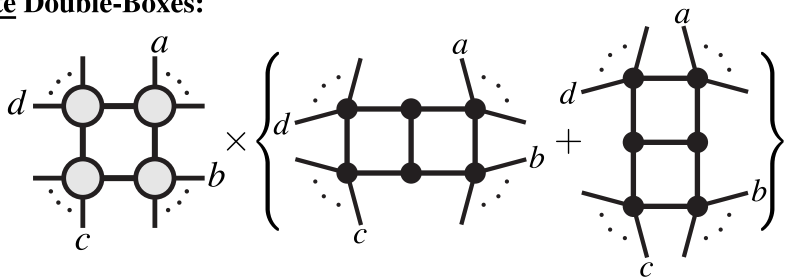

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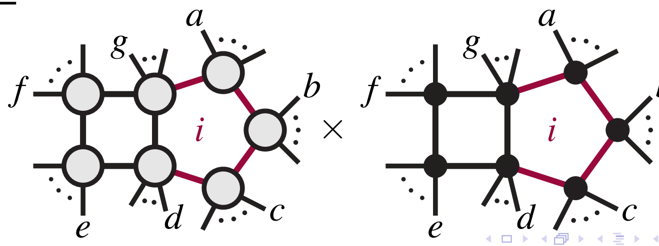

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$$\propto(\widetilde{\eta}_1^1\widetilde{\eta}_1^2)(\widetilde{\eta}_2^1\widetilde{\eta}_2^2)(\widetilde{\eta}_3^1\widetilde{\eta}_3^2)(\widetilde{\eta}_4^2\widetilde{\eta}_4^3)(\widetilde{\eta}_5^2\widetilde{\eta}_5^3)(\widetilde{\eta}_6^3\widetilde{\eta}_6^4)(\widetilde{\eta}_7^3\widetilde{\eta}_7^4)(\widetilde{\eta}_8^3\widetilde{\eta}_8^4)(\widetilde{\eta}_9^4\widetilde{\eta}_9^1)(\widetilde{\eta}_{10}^4\widetilde{\eta}_{10}^1)$$

Novel Contributions Required

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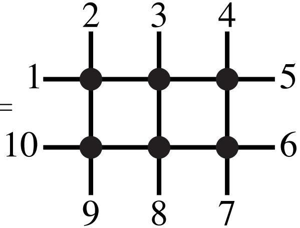
$=$ 

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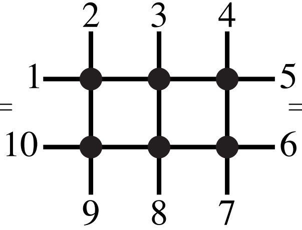
= 

$$=\int\frac{d^4\ell_1 d^4\ell_2}{(\ell_1,9)(\ell_1,1)(\ell_1,3)(\ell_1,\ell_2)(\ell_2,4)(\ell_2,6)(\ell_2,8)}$$

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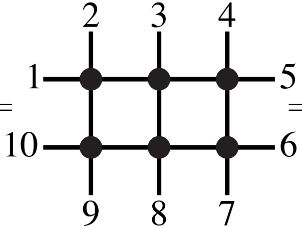
$=$ 

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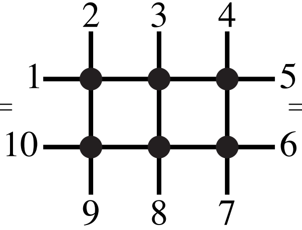

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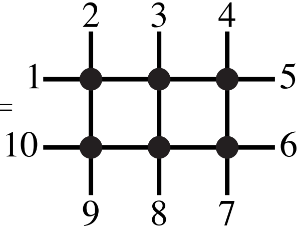

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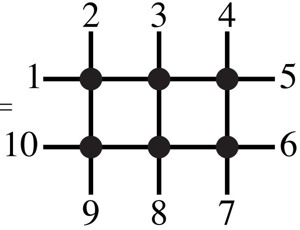
=  $\times\,?$

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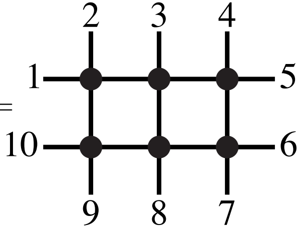
$=$ 

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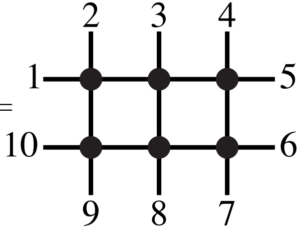
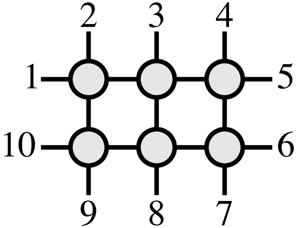
$=$ 

$\times$ 

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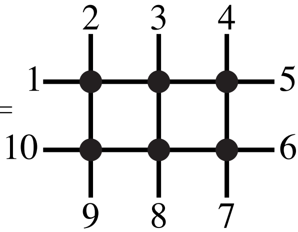
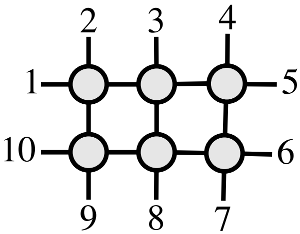
$=$ 

$\times$ 

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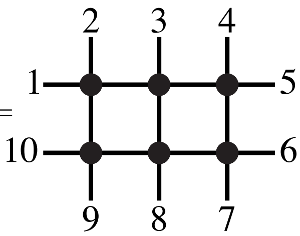
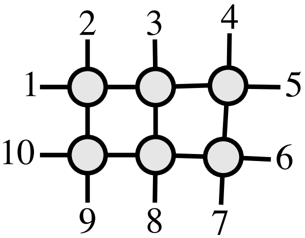
$=$ 

$\times$ 

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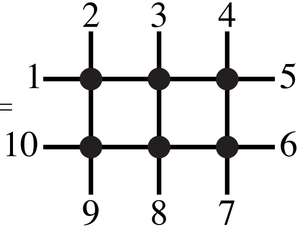
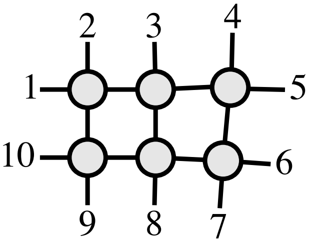
$=$ 

$\times$ 

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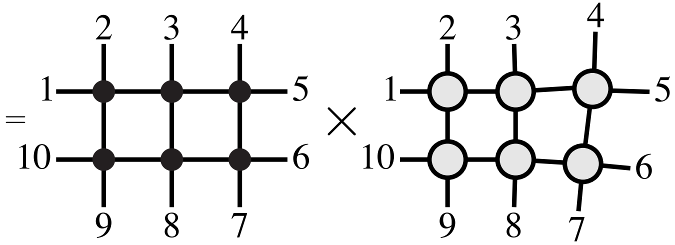

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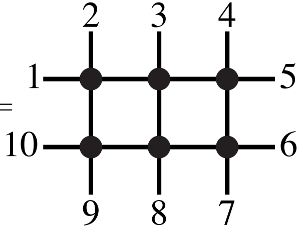
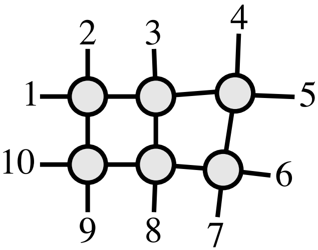
$=$  $\times$ 

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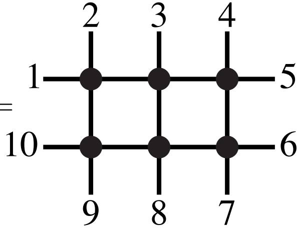
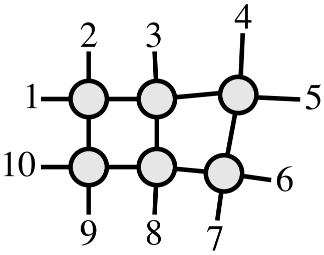
$=$  $\times$ 

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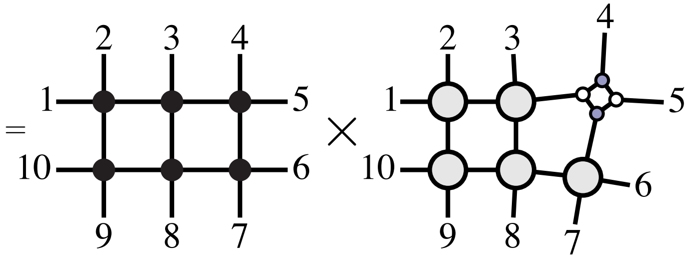

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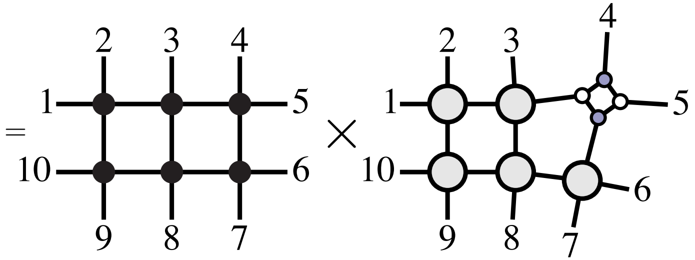

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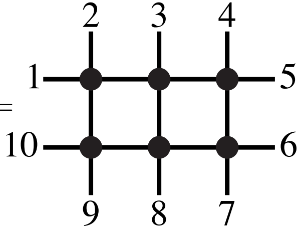
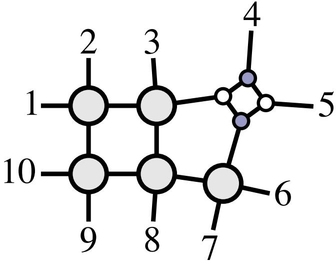
$=$ 

$\times$ 

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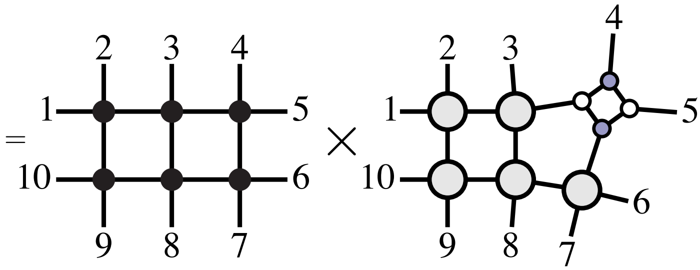

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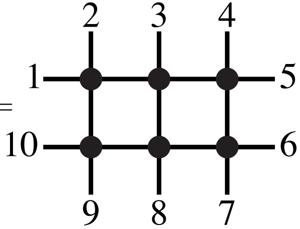
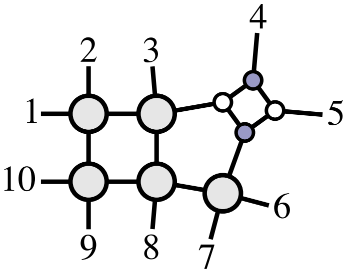
$=$ 

$\times$ 

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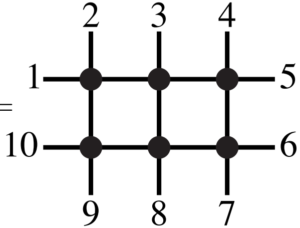
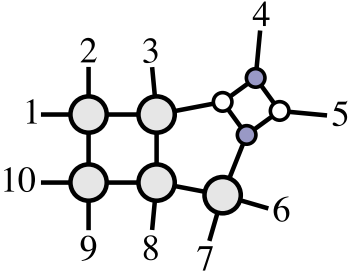
$=$  $\times$ 

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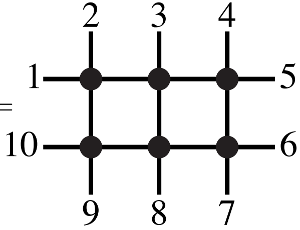
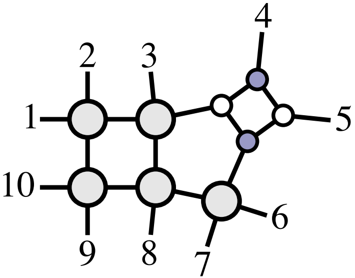
$=$ 

$\times$ 

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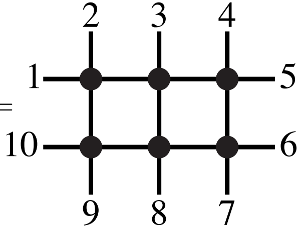
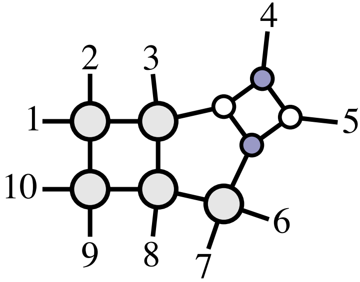
$=$  $\times$ 

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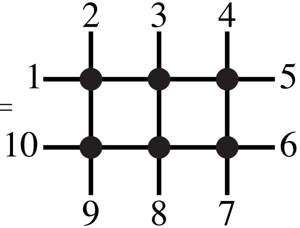
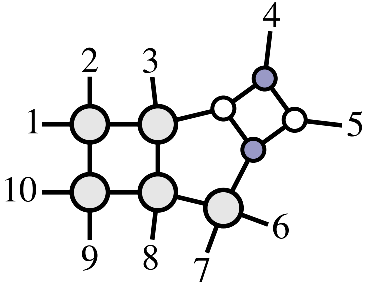
$=$ 

$\times$ 

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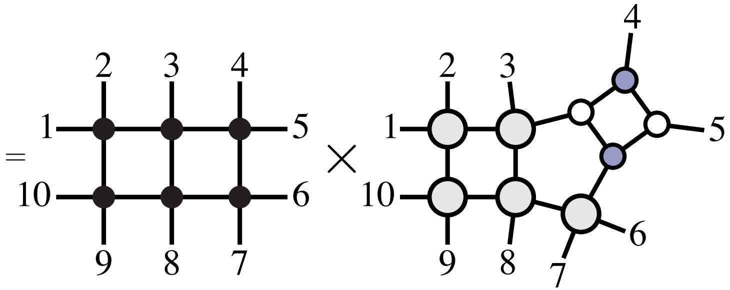

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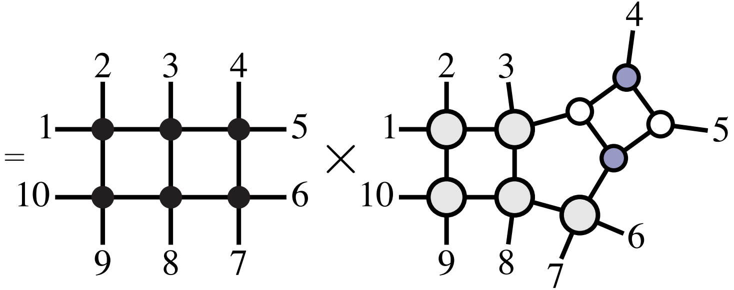

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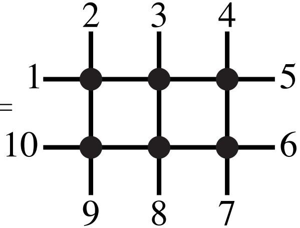
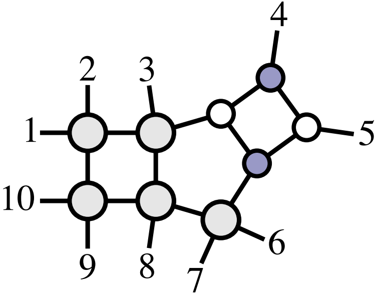
$=$ 

$\times$ 

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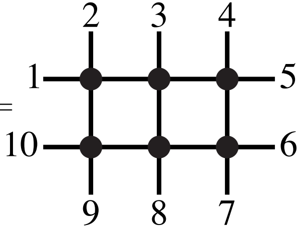
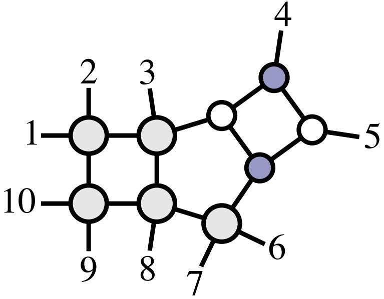
$=$ 

$\times$ 

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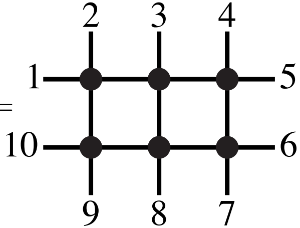
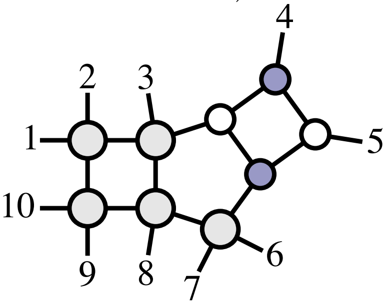
$=$ 

$\times$ 

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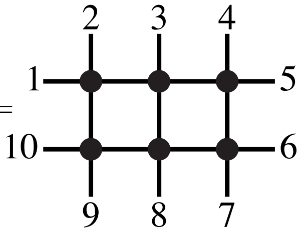
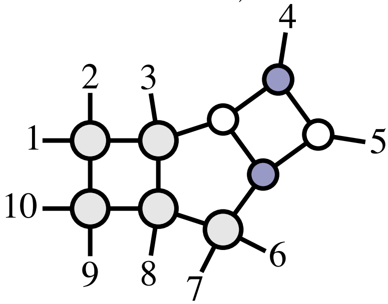
$=$ 

$\times$ 

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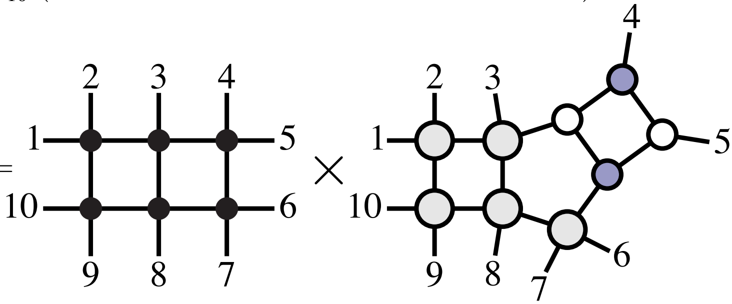
$=$ 

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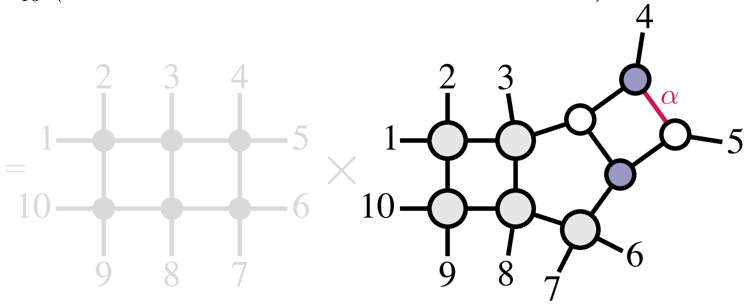

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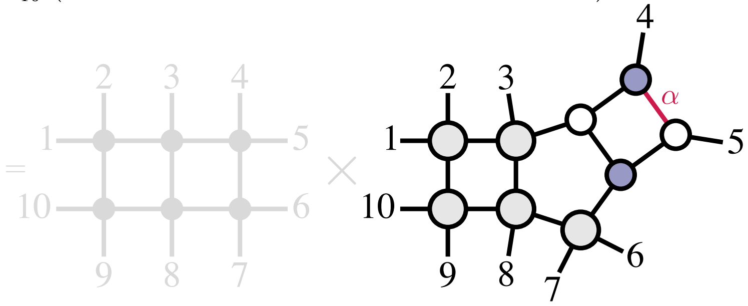

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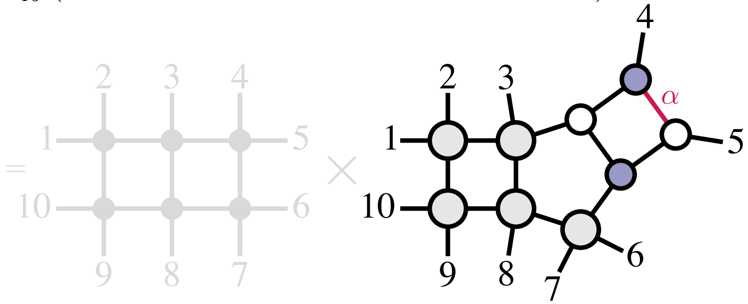

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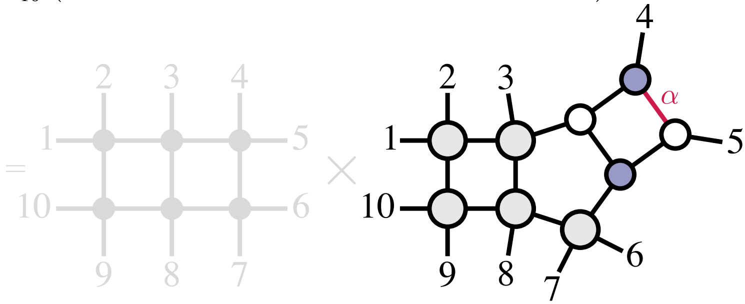

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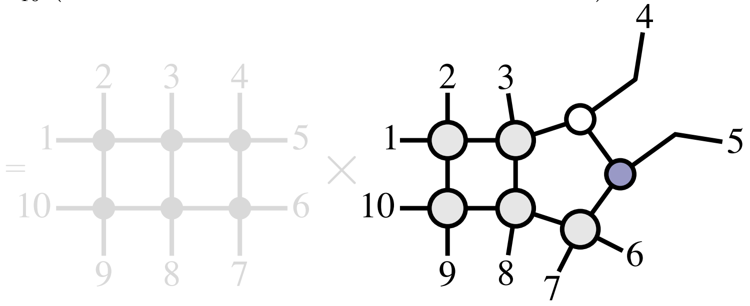

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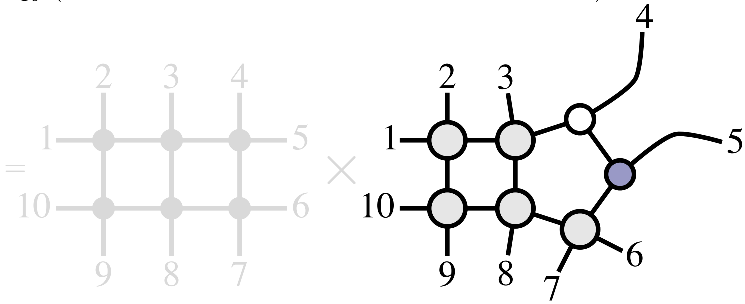

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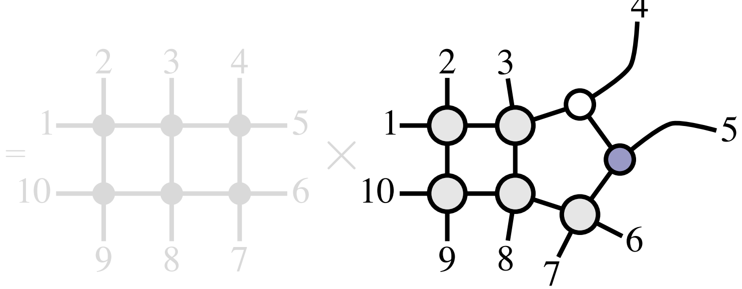

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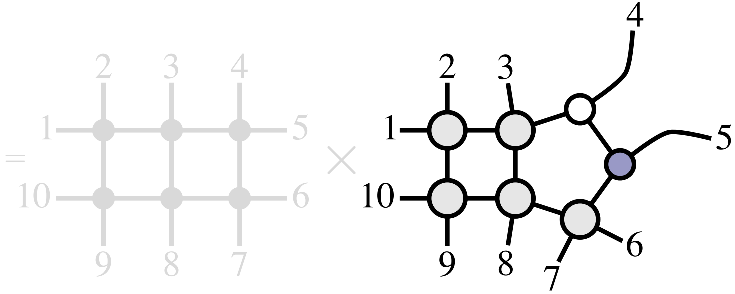

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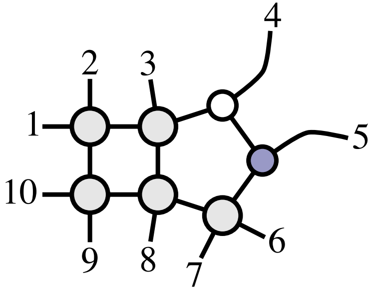
$\times$ 

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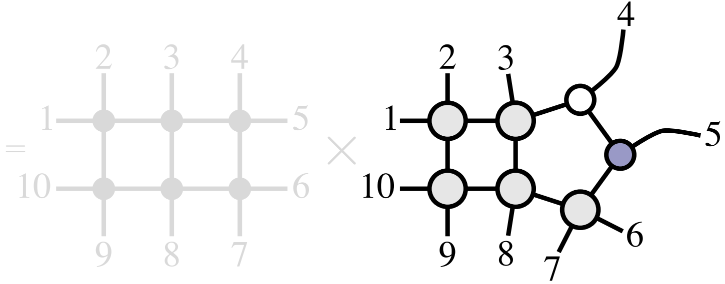

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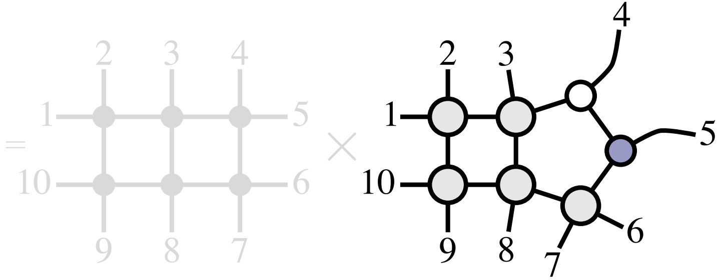

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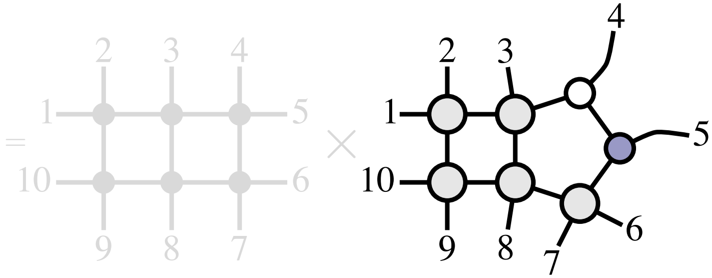

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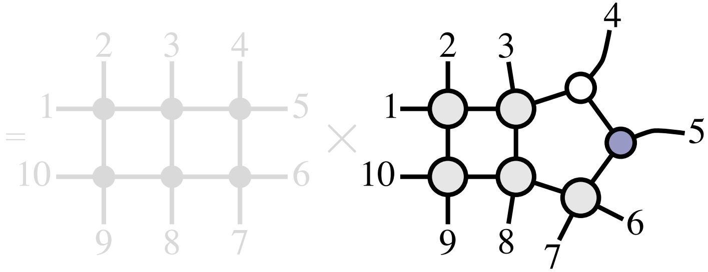

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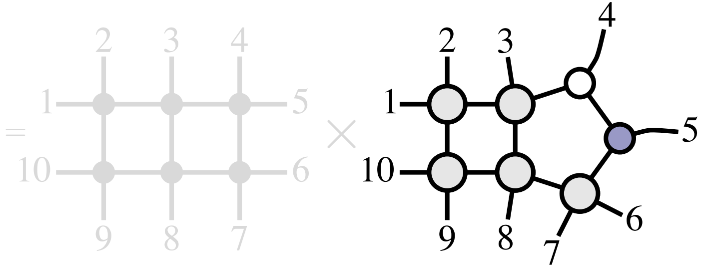

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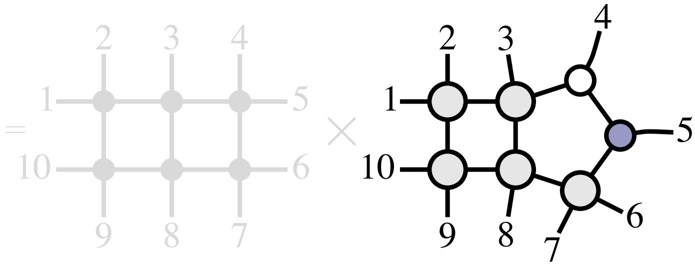

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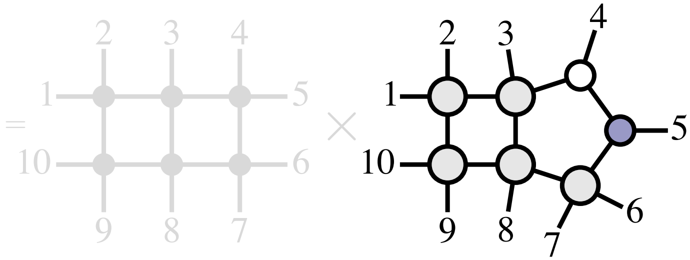

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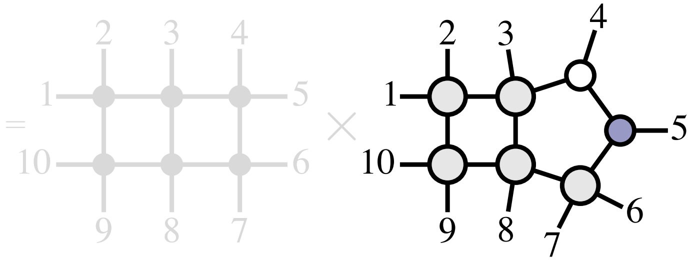

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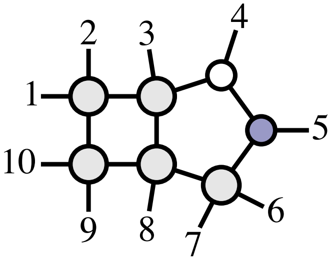
$\times$ 

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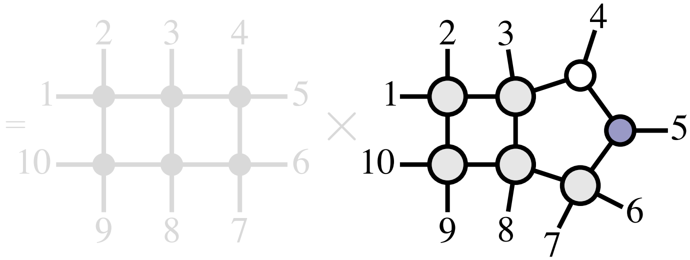

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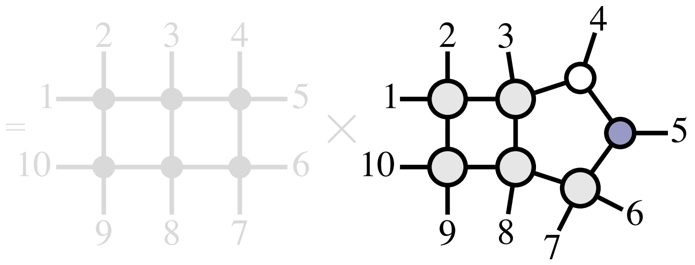

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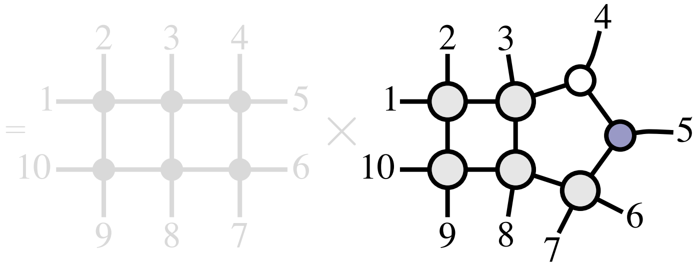

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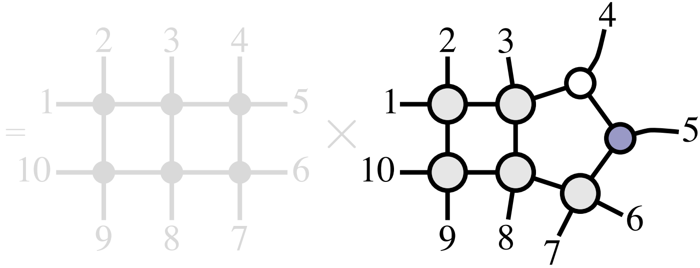

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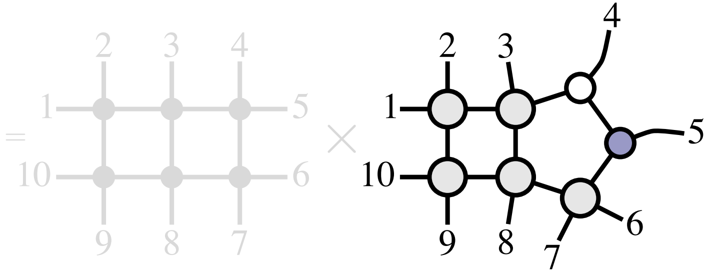

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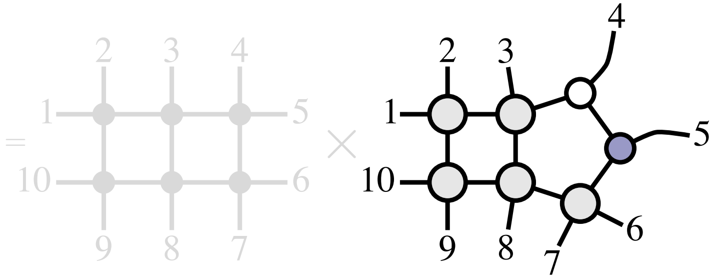

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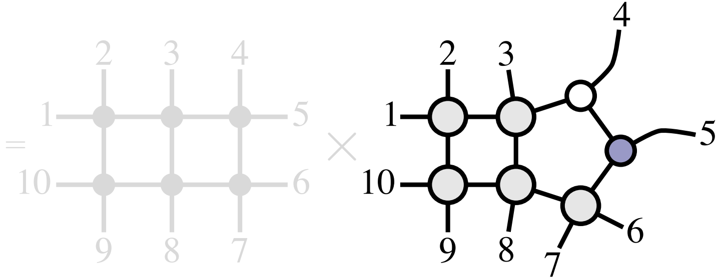

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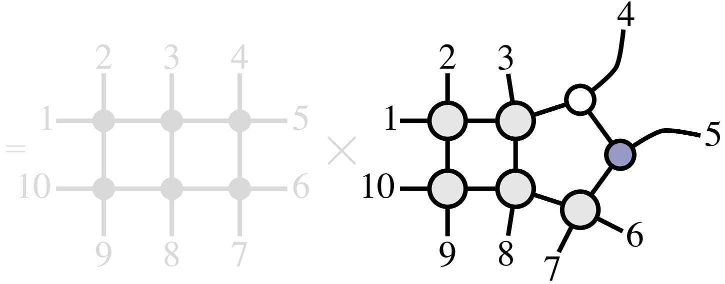

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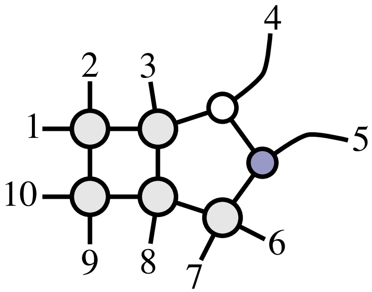
$\times$ 

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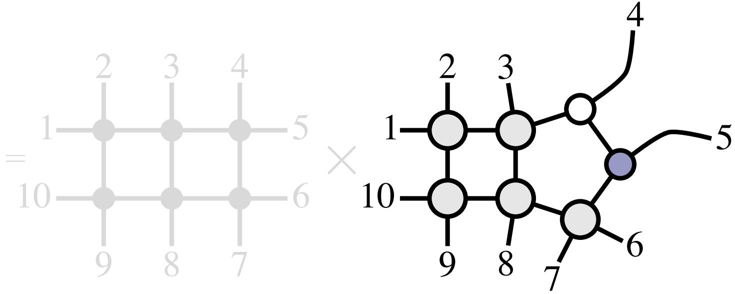

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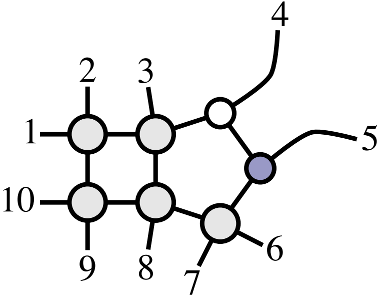

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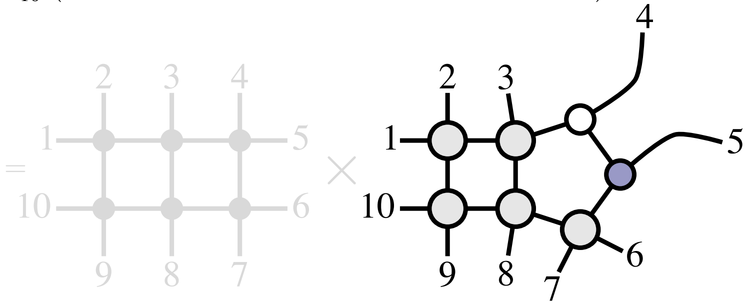

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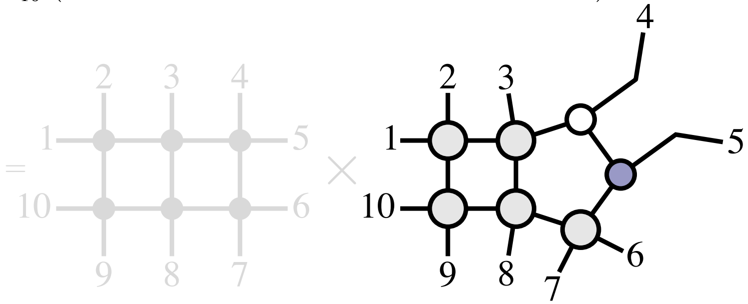

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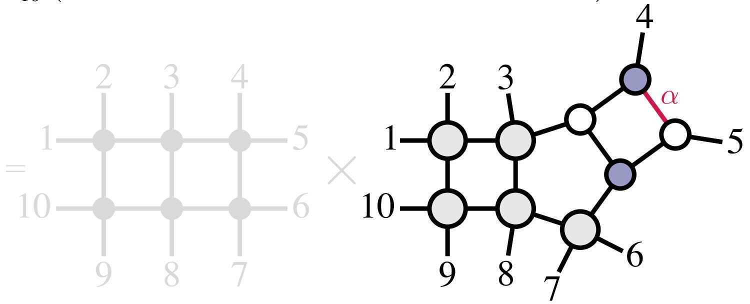

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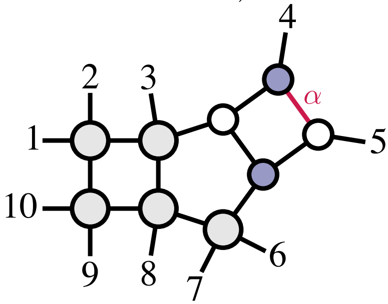
$\times$ 

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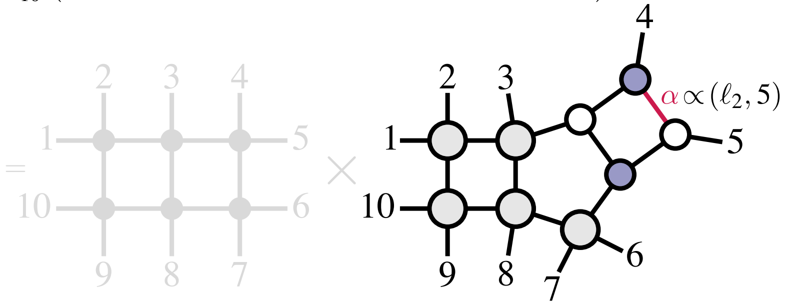

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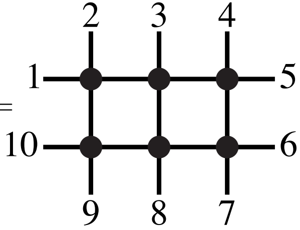
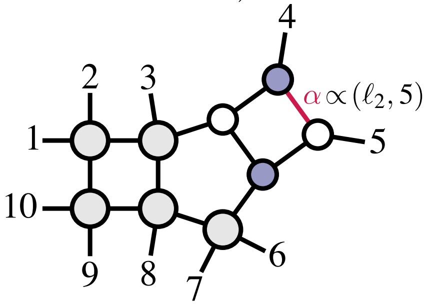
$=$ 

$\times$ 

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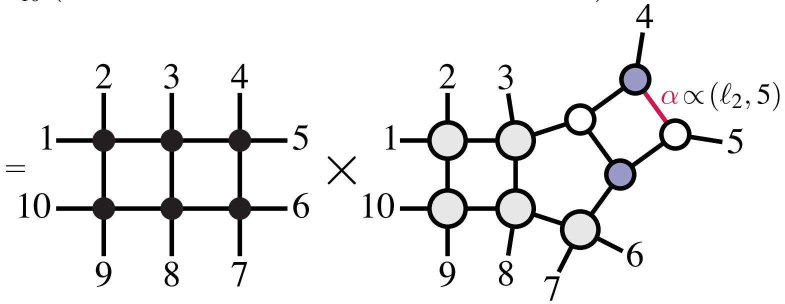

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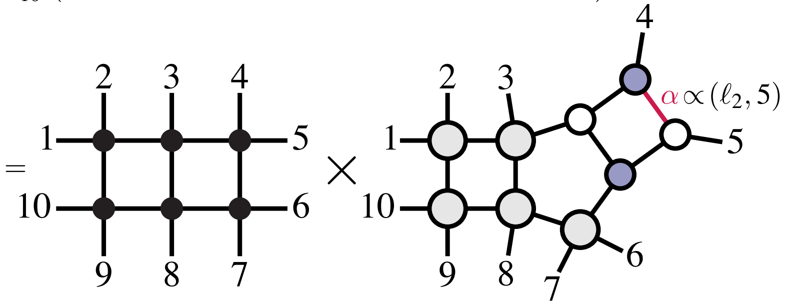

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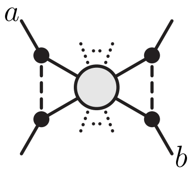

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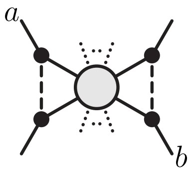

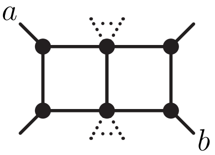

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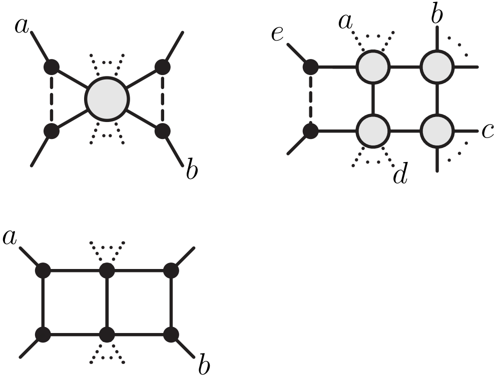

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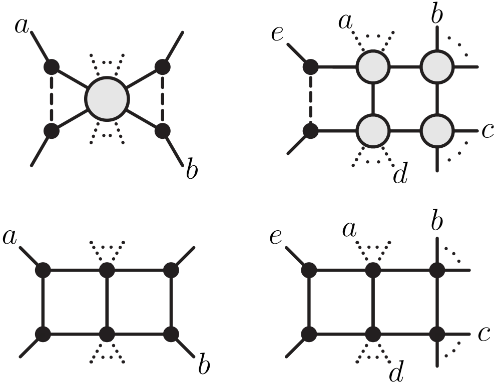

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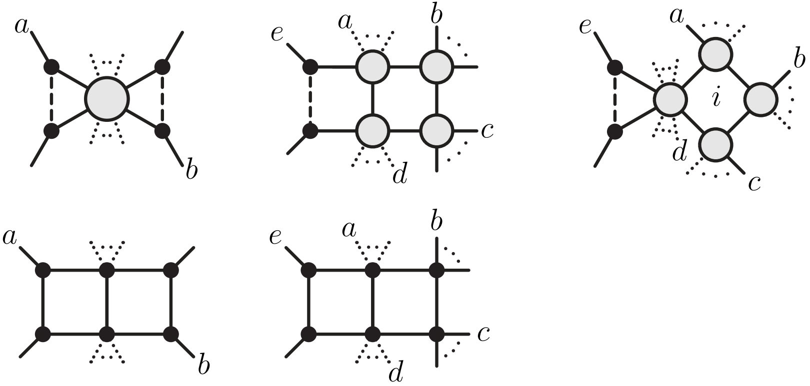

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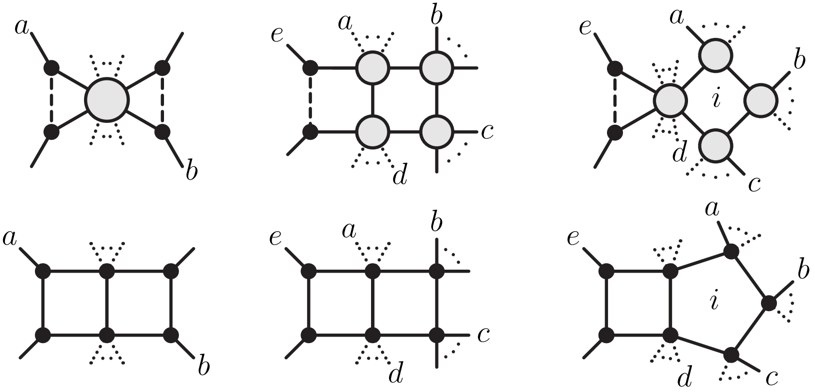

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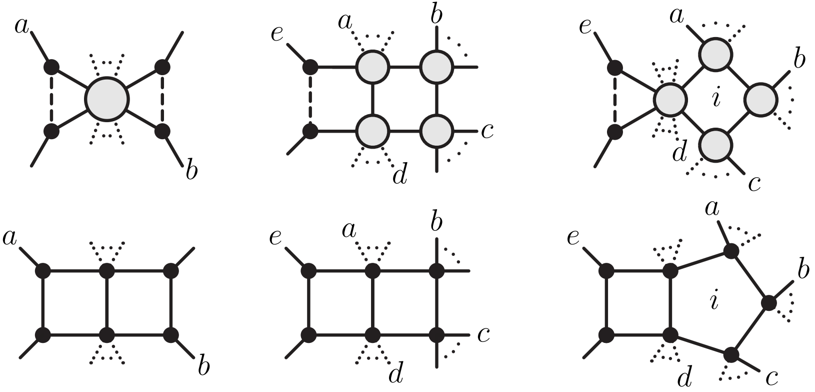

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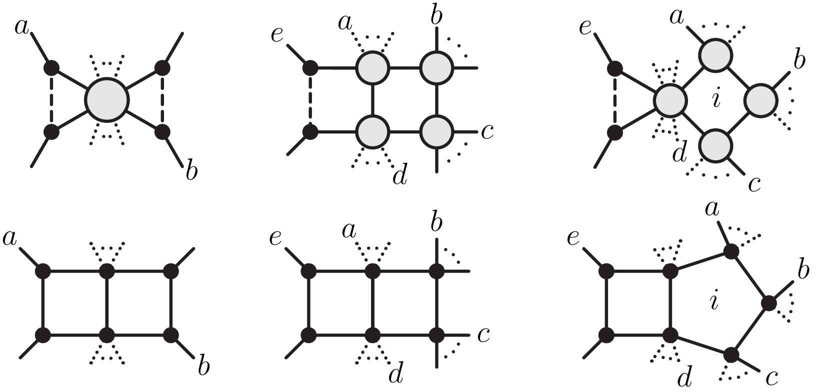

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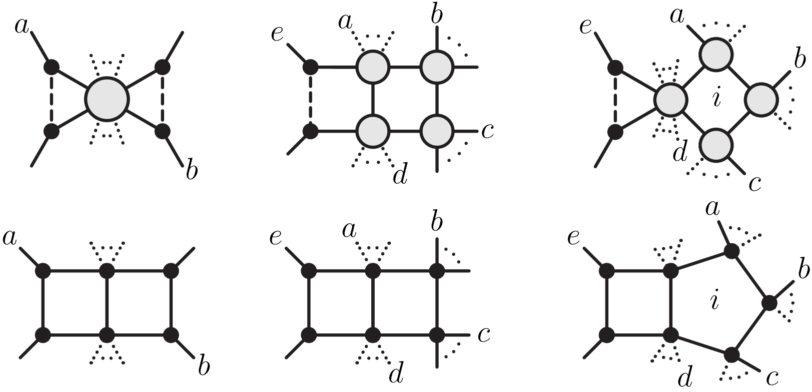

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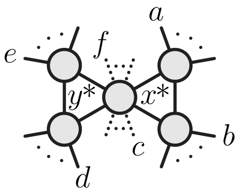

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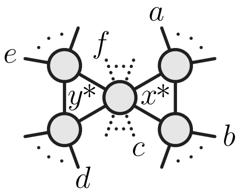

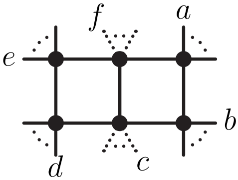

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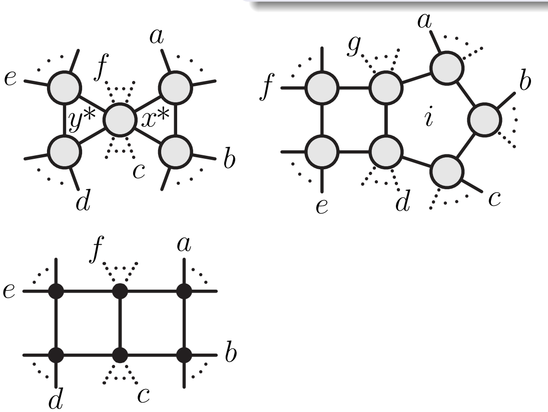

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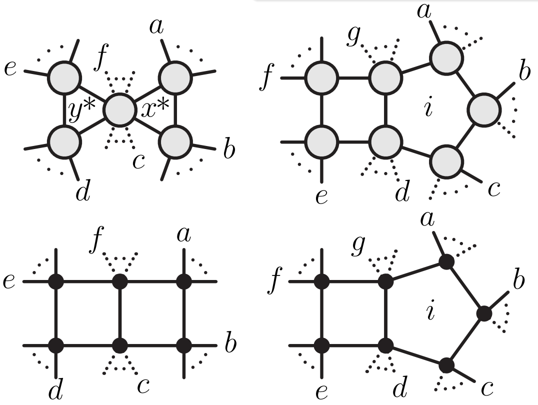

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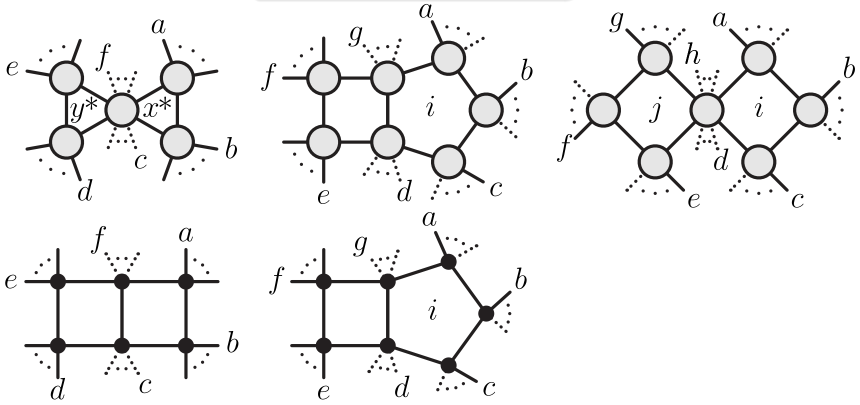

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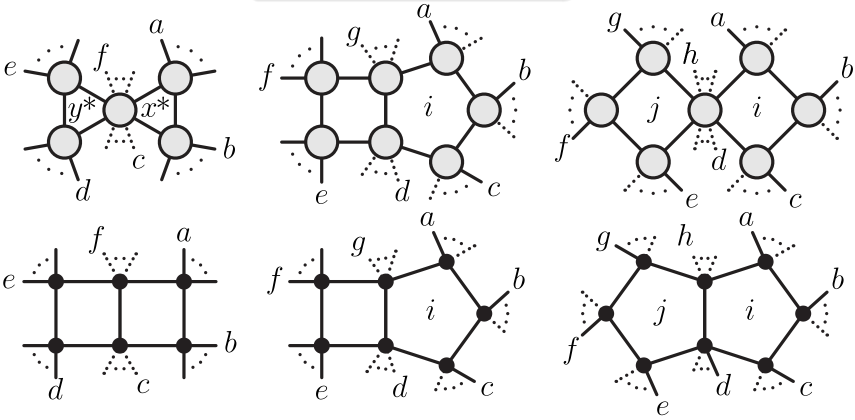

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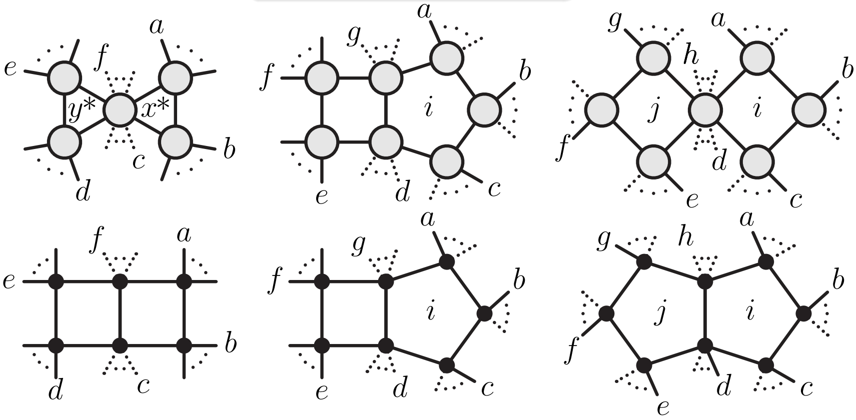

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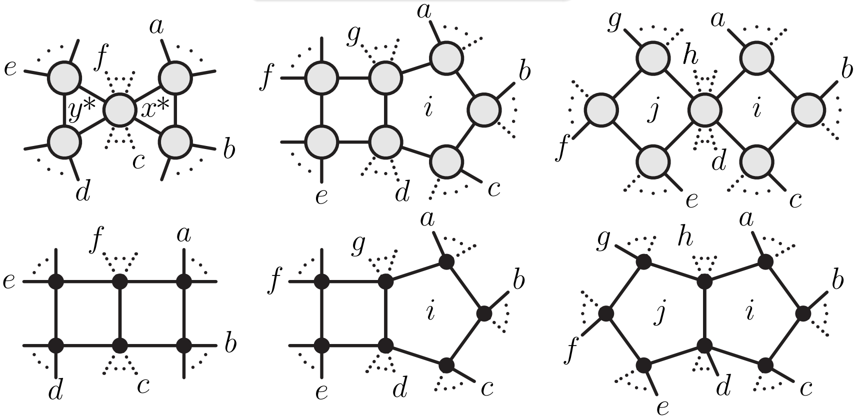

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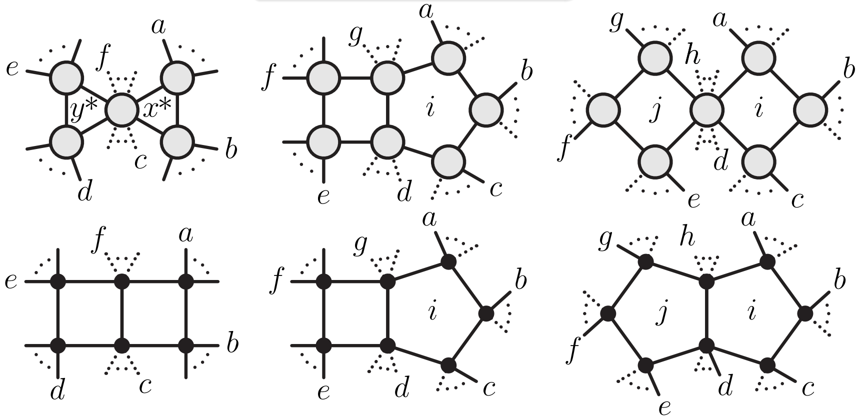

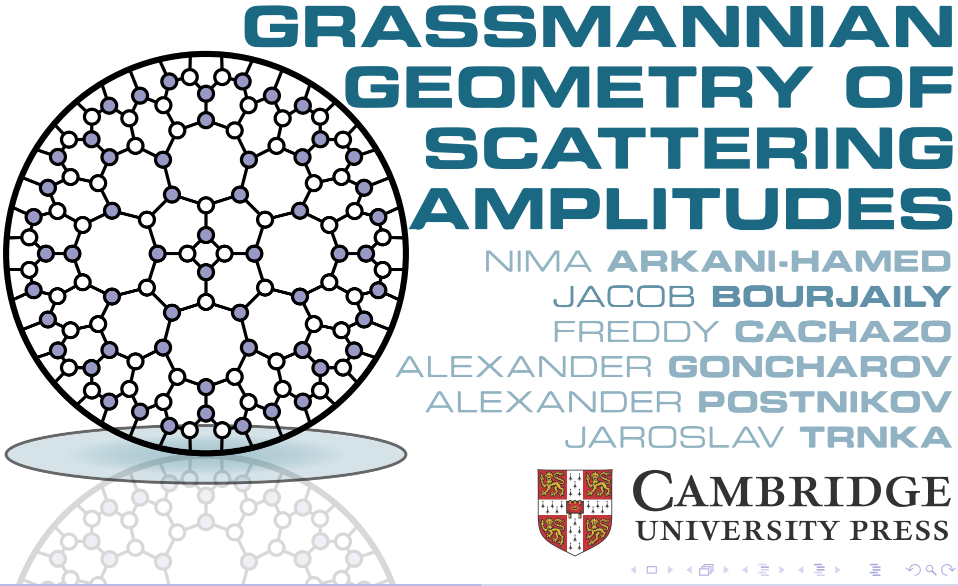
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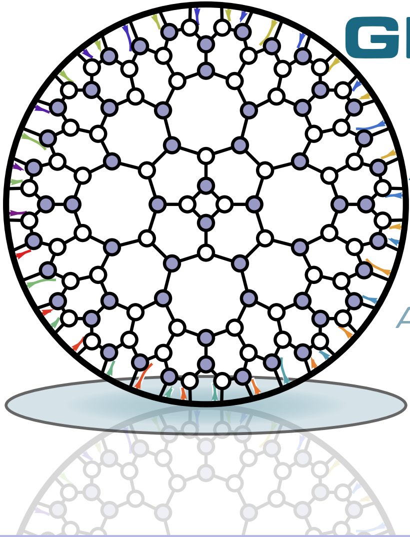

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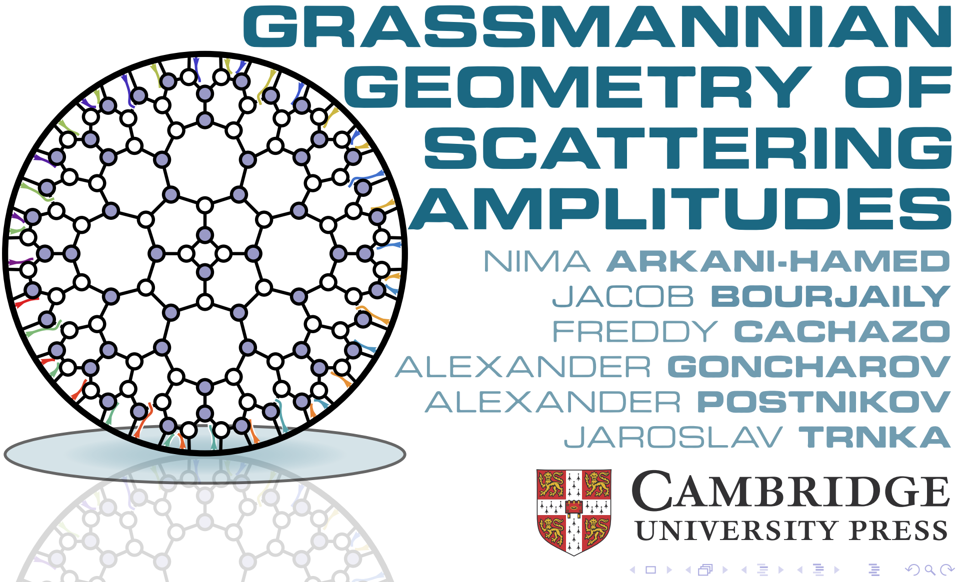
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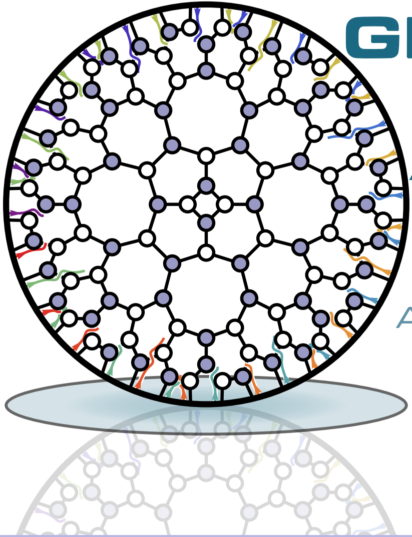

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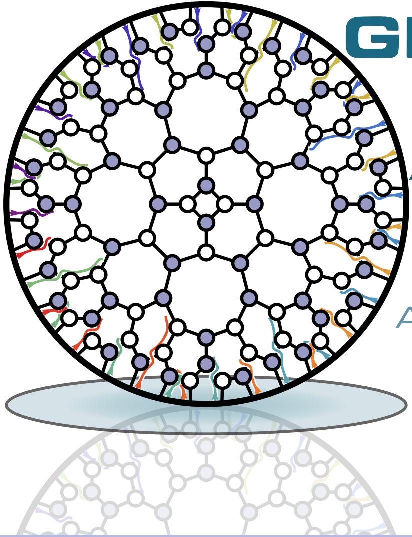

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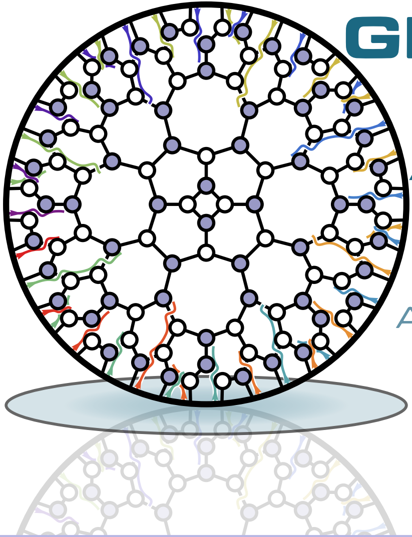

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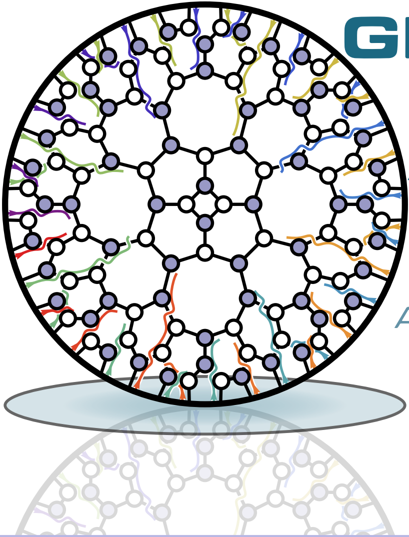

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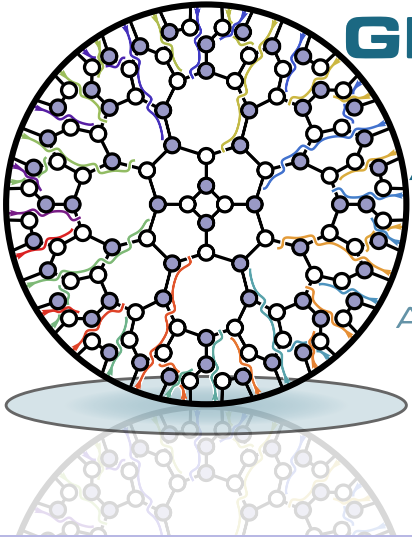

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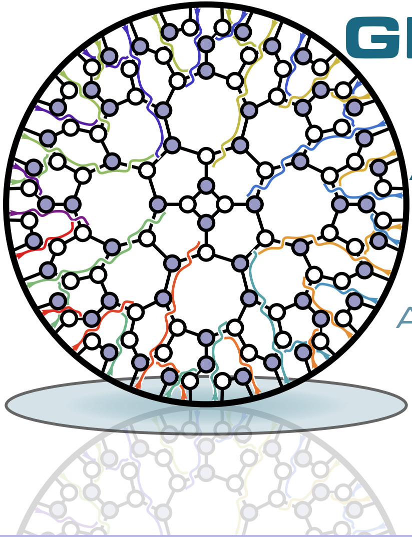

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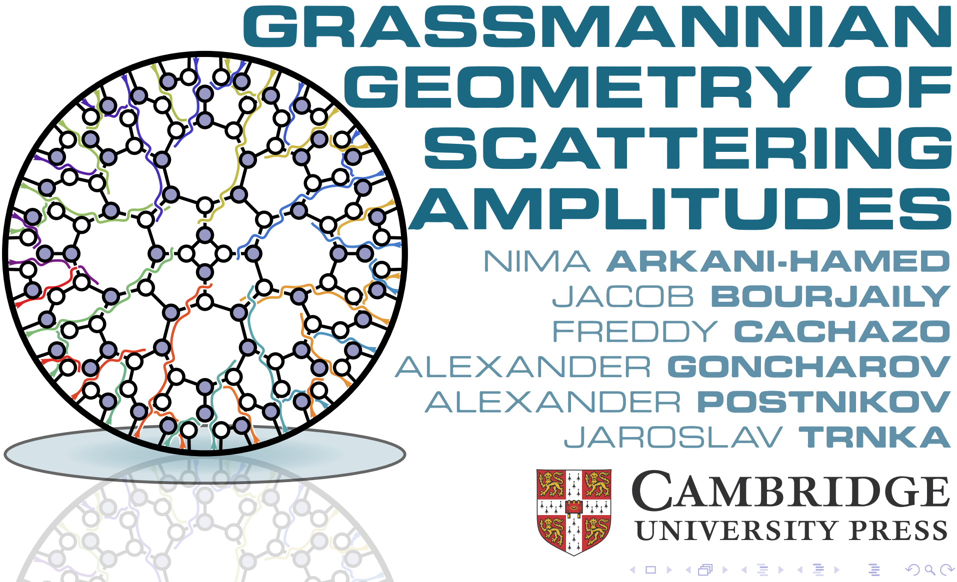
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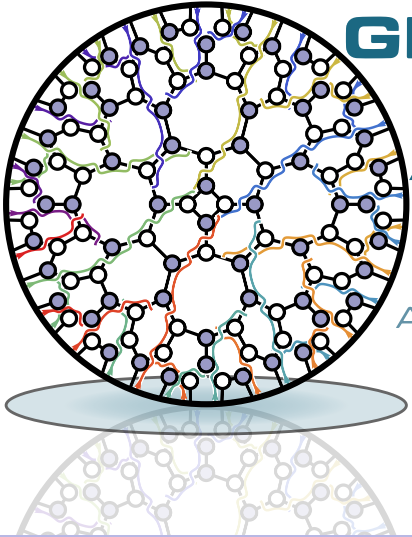

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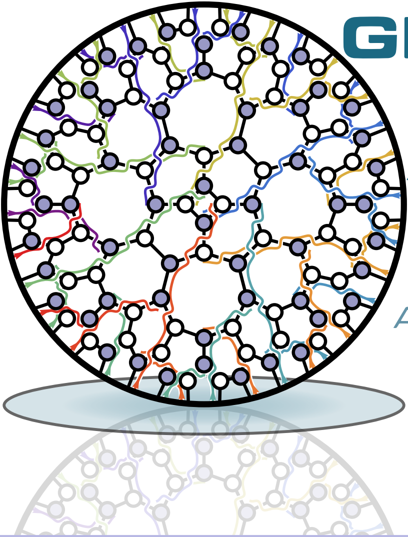

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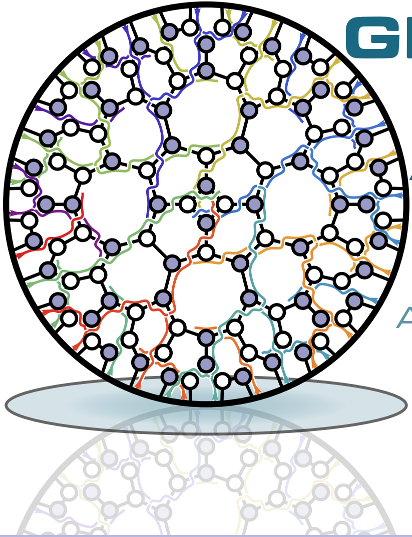

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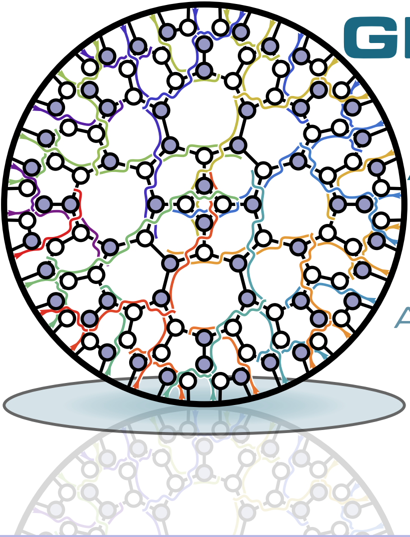

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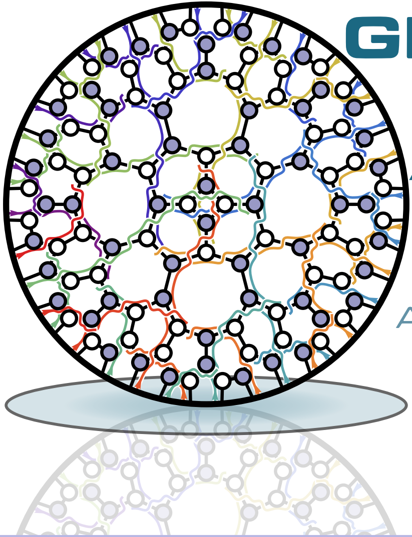

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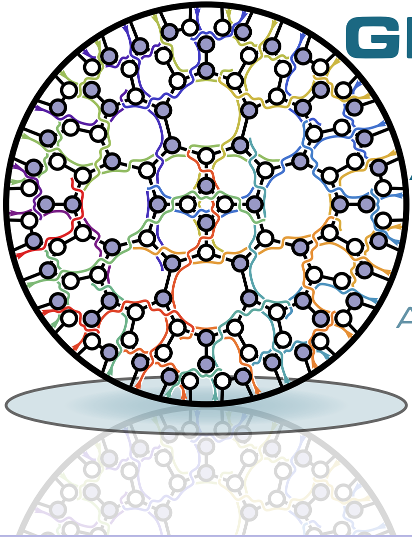

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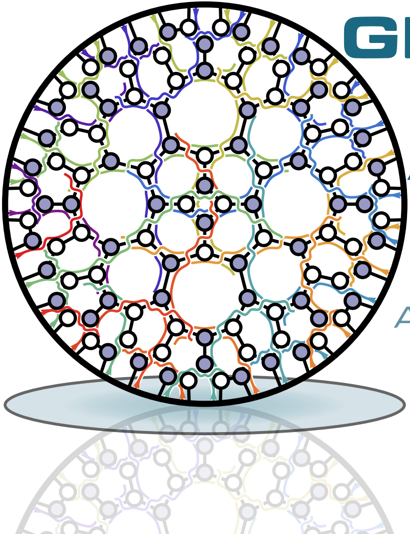

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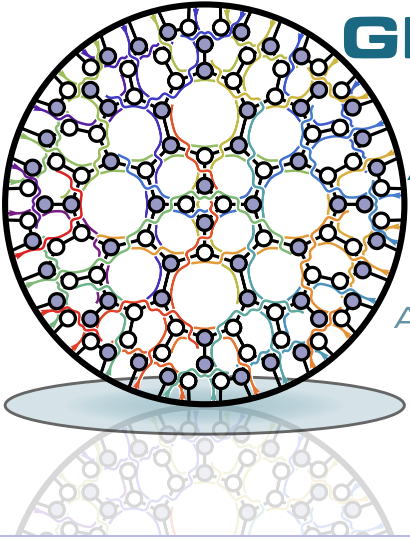

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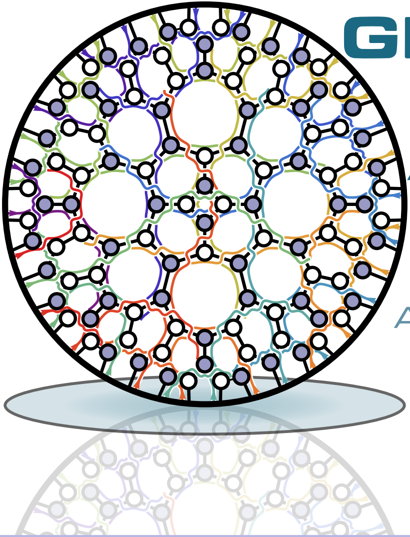

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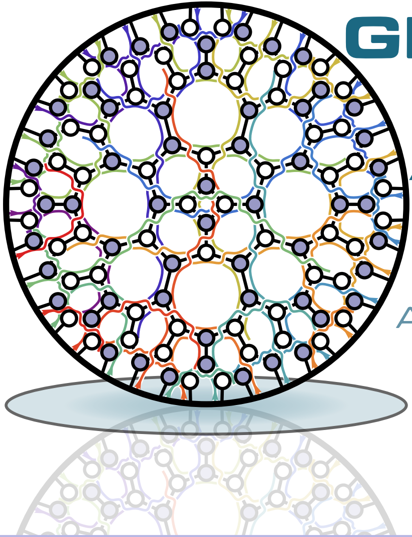

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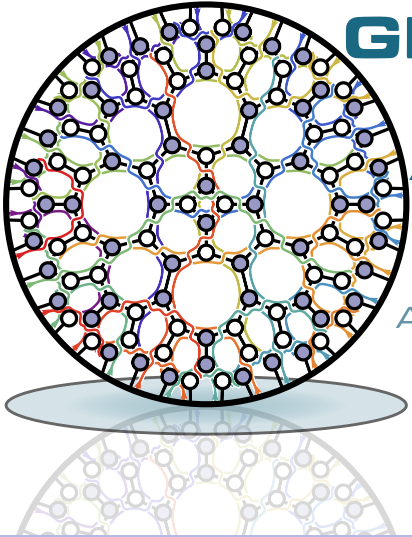

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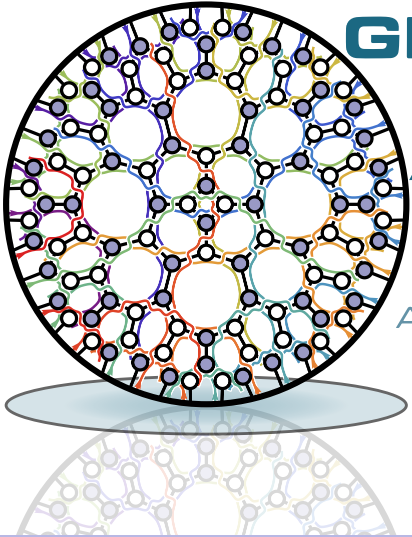

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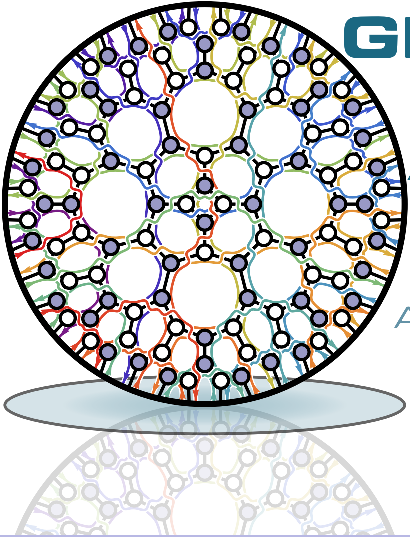

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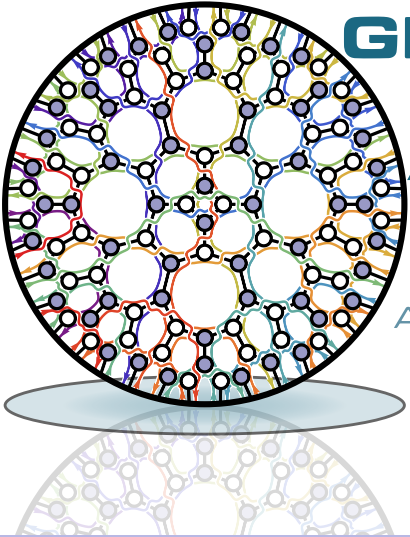

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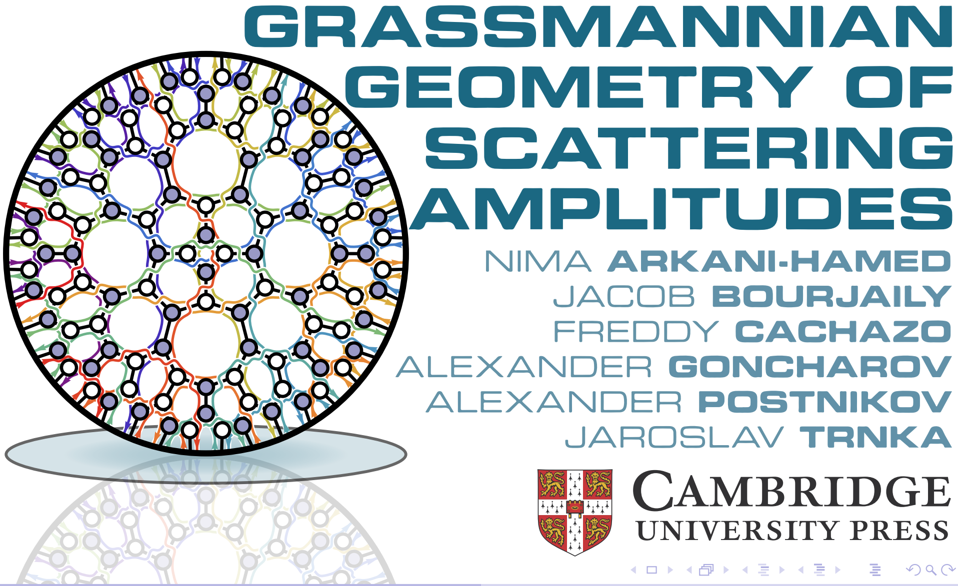
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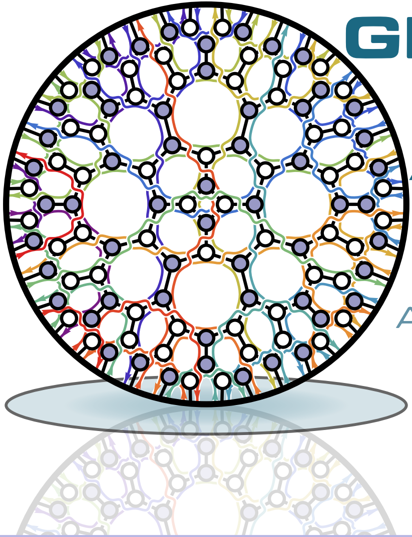

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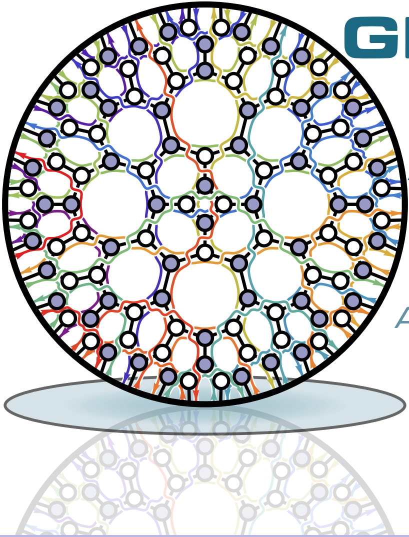

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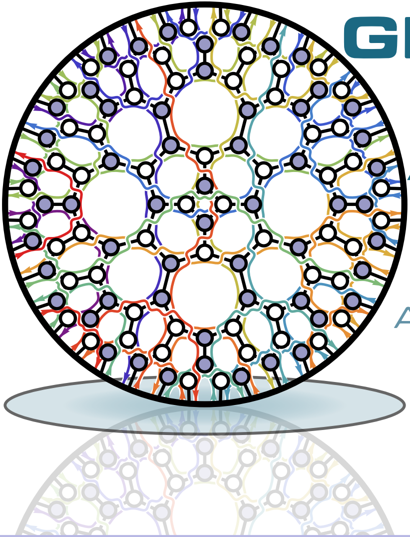

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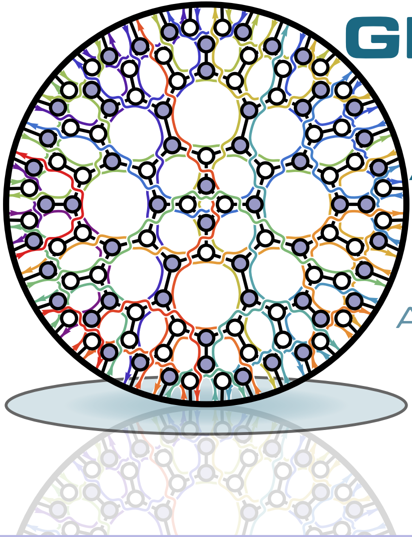

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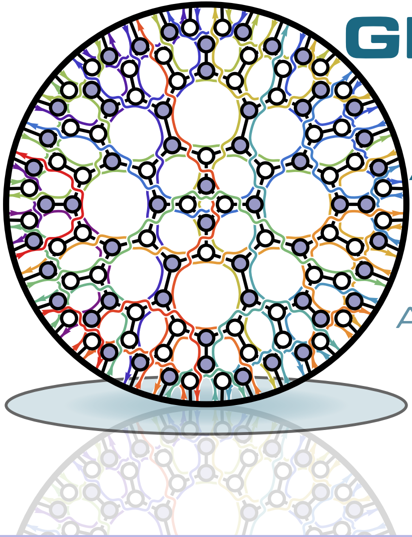

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