Knot polynomials, homological invariants & topological strings

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Talk at STRINGS 2015

Plan:

(i) Knot polynomials from Chern-Simons gauge theory

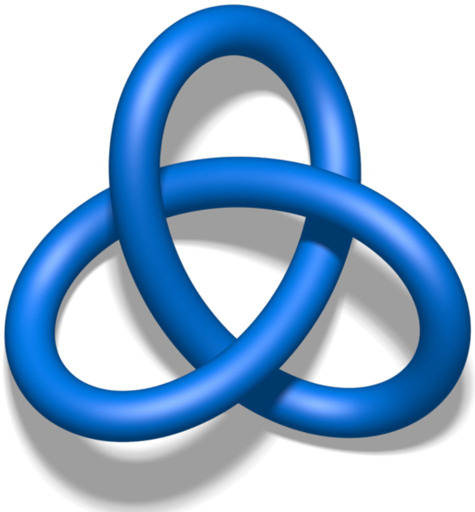
(ii) Our results on mutant knots

(iii) Homological invariants

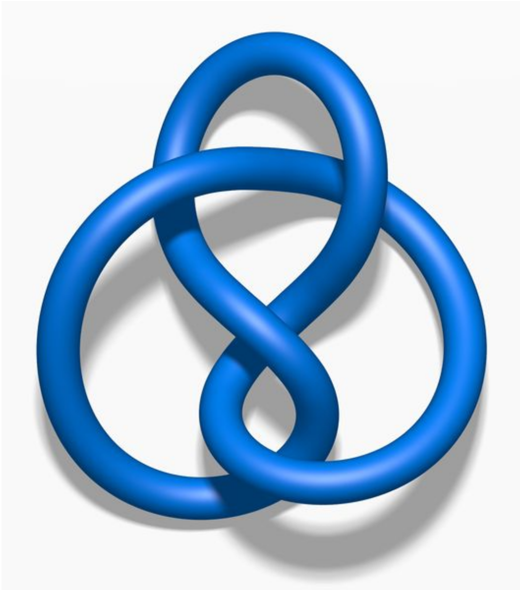
(iv) Large N Chern-Simons & their closed topological string duals

(v) Conclusions & challenging problems .

Knot polynomials from Chern-Simons theory



Torus Knot :Trefoil 3_1



Hyperbolic Knot: Figure-Eight knot 4_1

Chern-Simons theory provides a natural framework for the study of knots

• **Chern-Simons action on S^3 based on gauge group G:**

$$S_{CS}[A] \;=\; \frac{k}{4\pi}\int_{S^3} Tr\left(A\wedge dA+\frac{2}{3}A^3\right)$$

k is the coupling constant, A's are the gauge connections.

• Any knot K carrying representation R are described by **expectation value of Wilson loop operators** $W_R(K)=Tr[Pexp\oint A]$:

$$V_R^G[K] \;=\; \langle W_R(K)\rangle \;=\; \frac{\int_{S^3}[\mathcal{D}A]\; W_R(K)\; exp(iS_{CS}[A])}{Z[S^3]}$$

$$\text{where}\;\; Z[S^3]=\int_{S^3}[\mathcal{D}A]\; exp(iS_{CS}[A])\;\;\text{(partition function)}$$

$V_R^G[K]$ are the **knot invariants.**

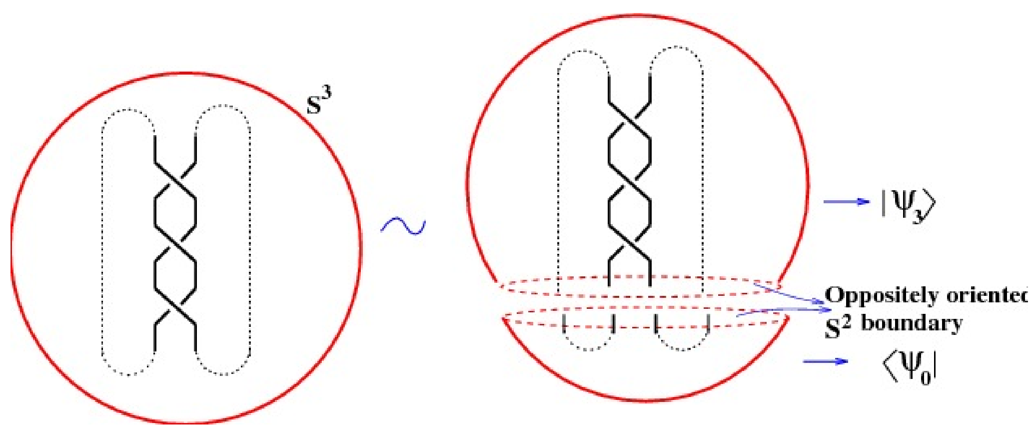
Knot invariant computations

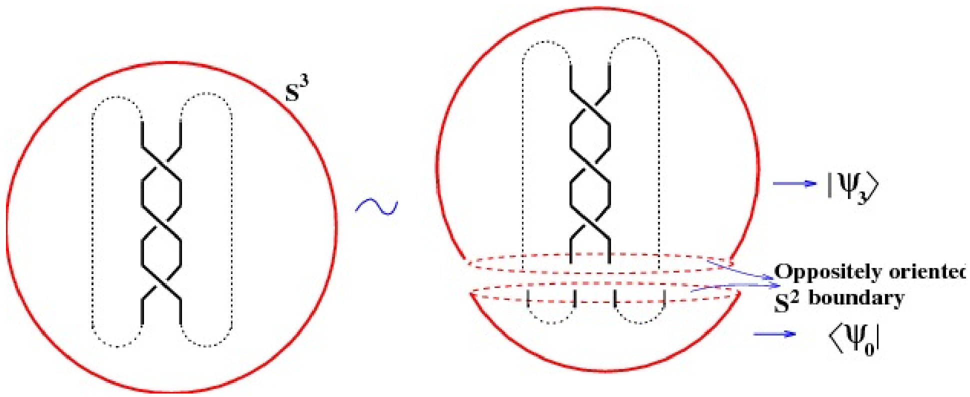
•These knot invariants $(V_R^G[K])$ can be directly evaluated using two inputs:(*Kaul , Govindarajan,PR (1992)*)

1) Relation between Chern-Simons theory to G_k Wess-Zumino conformal field theory (Witten 1989).

2) Any knot can be obtained as a closure or platting of braid(Alexander, Birman)

For example, the trefoil can be redrawn as





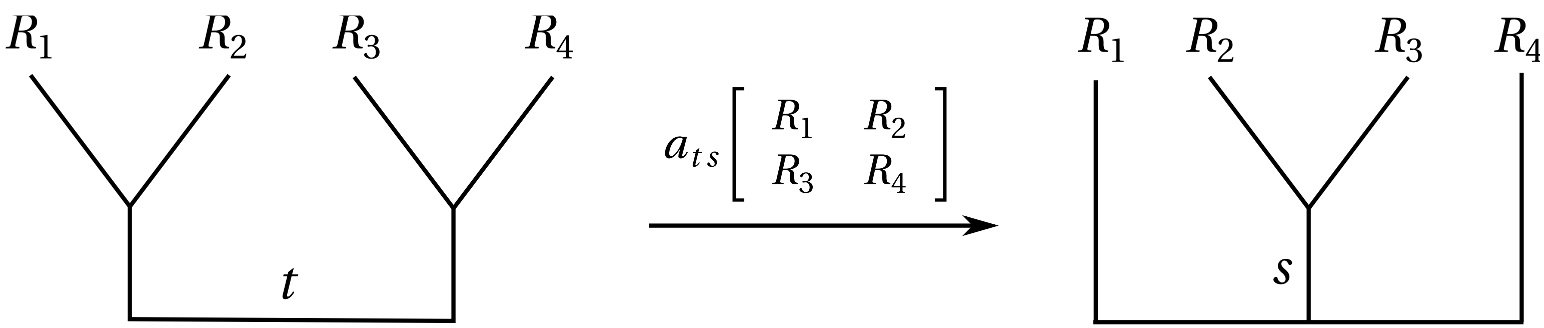
$$V_R[3_1] = \langle\psi_0|\psi_3\rangle = \langle\psi_0|\mathcal{B}^3|\psi_0\rangle$$

where $\mathcal{B}$ is the braiding operator.

To write the polynomial form of the knot invariant:

Expand the state $|\psi_0\rangle$ in a suitable basis in which $\mathcal{B}$ is diagonal.

For the four-punctured S^2 boundary, the conformal block bases are:



where $t \in R_1 \otimes R_2 \cap \bar{R}_3 \otimes \bar{R}_4$ and $s \in R_2 \otimes R_3 \cap \bar{R}_1 \otimes \bar{R}_4$.

$a_{st}\begin{bmatrix} R_1 & R_2 \\ R_3 & R_4 \end{bmatrix}$ is the **duality matrix** (fusion matrix) relating these two bases which is proportional to the quantum Wigner 6j symbols:

$$a_{ts}\begin{bmatrix} R_1 & R_2 \\ R_3 & R_4 \end{bmatrix} \propto \begin{Bmatrix} R_1 & R_2 & t \\ R_3 & R_4 & s \end{Bmatrix}$$

For knots, two of the R_i's will be R and the other two will be conjugate $\bar{R}$ depending on the orientation.

In the braid diagram for trefoil, middle two strands are parallely oriented and they are braided.

$$|\Psi_0\rangle = \sum_{s \in R \otimes R} \mu_s \, |\hat{\Phi}_s(\bar{R}, R, R, \bar{R})\rangle$$

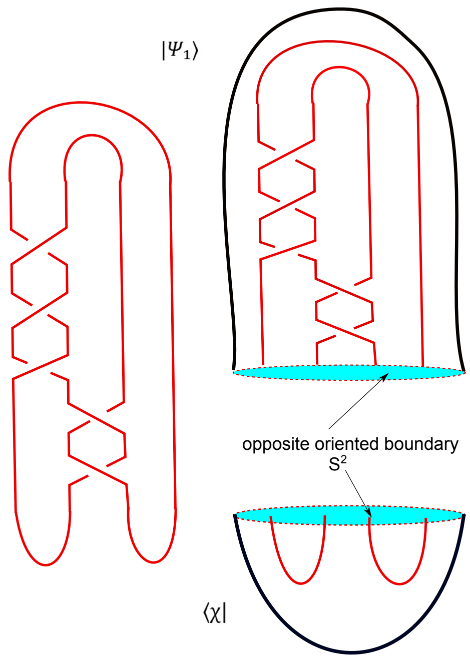
where $\mu_s = \sqrt{S_{0s}/S_{00}} \equiv \sqrt{dim_q t}$ (**unknot** normalisation)

$$V_R[3_1] = \langle \Psi_0 | \mathcal{B}^3 | \Psi_0 \rangle = \sum_s dim_q s \, (\lambda_s(R, R))^3$$

where braiding eigenvalue for parallelly oriented right-handed half-twists is

$$\lambda_t^{(+)}(R, R) = (-1)^{\epsilon_t} q^{2C_R - C_t/2}, \; q = e^{\frac{2\pi i}{k + C_v}} \text{ where } \epsilon_t = \pm 1$$

Antiparallel braiding eigenvalue will be $\lambda_s^{(-)}(R, \bar{R}) = (-1)^{\epsilon_s} q^{C_s/2}$. We require them for figure-eight drawn as quasi-plat.



$$V_R[4_1] = \sum_{t,s \in R \otimes \bar{R}} \sqrt{dim_q t \; dim_q s} \, a_{ts} \begin{bmatrix} \bar{R} & R \\ \bar{R} & R \end{bmatrix} (\lambda_t^{(-)})^2 (\lambda_s^{(-)})^{-2}$$

- **Knot invariants involves braiding eigenvalues & fusion matrices**

- **Fusion matrices proportional to quantum Wigner 6j**
(completely known for $SU(2)$ (Kirillov, Reshetikhin) but not for other groups)

- **For few R's, we determined using knot equivalence and properties of Wigner 6j**-Kaul, Govindarajan,PR (1992), Zodinmawia,PR(2012)

- **Hence the knot invariants $V_R^G[K]$ can be written in variables dependent on k and rank of the group**

Well-known knot polynomials match with knot invariants (normalized by unknot) when R is fundamental representation

Gauge Group	$\square$	[n]-colored
$SU(2)$	Jones' J[K;q]	$J_n[K, q]$
$SU(N)$	HOMFLY $P[K; a = q^N, q]$	$P_n[K; a, q]$
$SO(N)$	Kauffman $F[K; \lambda = q^{N-1}, q]$	$F_n[K; \lambda, q]$

Jones' polynomial for trefoil:

$$J[T; q] = V_{\square}^{SU(2)}[T]/V_{\square}^{SU(2)}[U] = q + q^3 - q^4 \ .$$

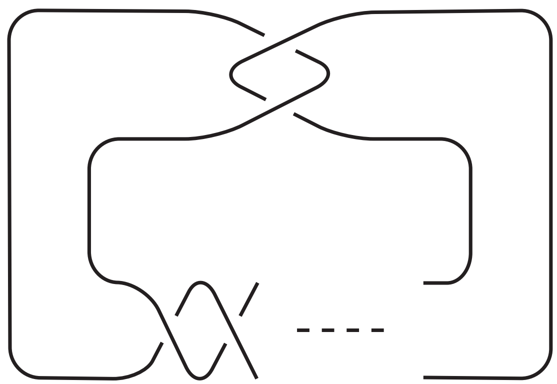
We can place any representation R (higher spins) of $SU(2)$ on the knot and obtain colored Jones' polynomials $J_n[K; q]$
where subscript n means spin $(n-1)/2$ or Young diagram single row with $n-1$ boxes. **The colored Jones for $3_1\&4_1$ are**

$$J_n[3_1; q] = \sum_{k=0}^{n-1} (-1)^k q^{k(k+3)/2+nk} (q^{-n-1}, q)_k (q^{-n+1}, q)_k$$

where $(z;q)_k = \prod_{j=0}^{k-1}(1-zq^j)$ is called q-Pochhammer symbol.

$$J_n[4_1;q] = \sum_{k=0}^{n-1} (-1)^k q^{nk} (q^{-n-1}, q^{-1})_k (q^{-n+1}, q)_k$$

These two colored Jones polynomial is sufficient to determine $J_n[K_p;q]$ for twist knots K_p

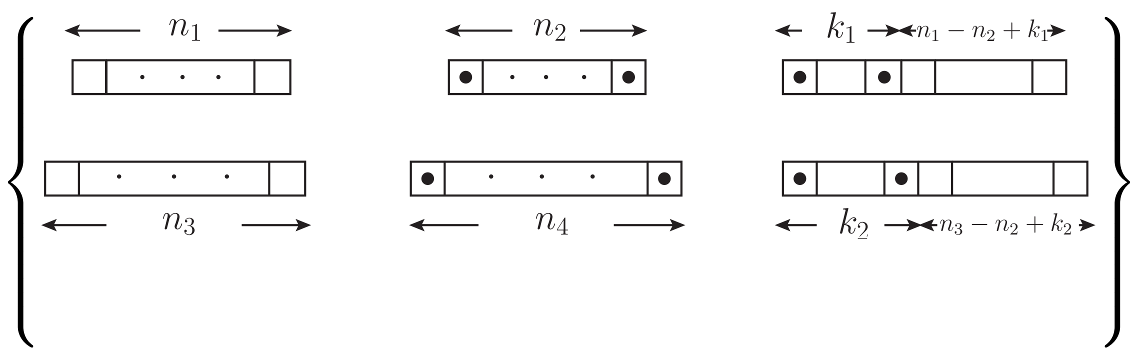


p	-4	-3	-2	-1	0	1	2	3	4
knots	$\mathbf{10_1}$	$\mathbf{8_1}$	$\mathbf{6_1}$	$\mathbf{4_1}$		$\mathbf{3_1}$	$\mathbf{5_2}$	$\mathbf{7_2}$	$\mathbf{9_2}$

• $[n]$-colored HOMFLY $P_n[K_p; a, q]$ (Nawata,Zodin, PR),2012 gave more data leading to conjecture a closed form expression for $U_q(sl_N)$ quantum Wigner 6j symbols for a class of R_i's (Nawata,Zodin,PR),2013

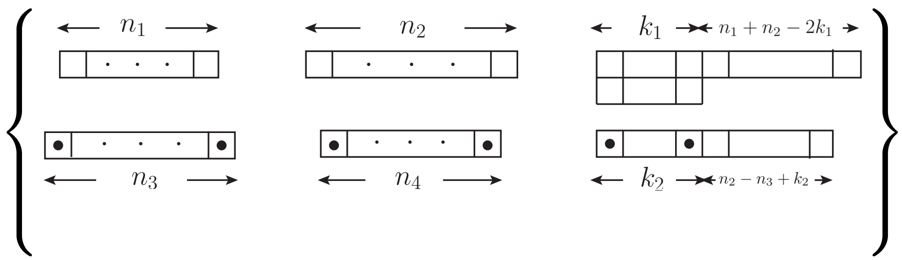
There are two types of Wigner 6j for $SU(N)$:

Type I:



where $n_2 \leq n_1 \leq n_3$, $k_1 \leq n_2$ and $k_2 \leq n_1$.

Type II



where $n_1 \leq n_2$, $k_2 \leq \min(n_1, n_3)$ and $k_1 \leq \min(n_1, n_3, n_4)$. The fusion rule requires $n_1 + n_2 = n_3 + n_4$.

$$
\begin{aligned}
\left\{\begin{matrix} \lambda_1 & \lambda_2 & \lambda_{12} \\ \lambda_3 & \lambda_4 & \lambda_{23} \end{matrix}\right\} \;=\;& \Delta_{(1,2,12)}\Delta_{(3,4,12)}\Delta_{(1,4,23)}\Delta_{(2,3,23)}[N-1]!\sum_{z\geq 0}(-)^z \\
& [z+N-1]!C_z\{[z-\frac{1}{2}\langle\lambda_1+\lambda_2+\lambda_{12},\alpha_1^\vee+\alpha_{N-1}^\vee\rangle]! \\
& \left[z-\frac{1}{2}\langle\lambda_3+\lambda_4+\lambda_{12},\alpha_1^\vee+\alpha_{N-1}^\vee\rangle\right]! \\
& \left[z-\frac{1}{2}\langle\lambda_1+\lambda_4+\lambda_{23},\alpha_1^\vee+\alpha_{N-1}^\vee\rangle\right]! \\
& \left[z-\frac{1}{2}\langle\lambda_2+\lambda_3+\lambda_{23},\alpha_1^\vee+\alpha_{N-1}^\vee\rangle\right]! \\
& \left[\frac{1}{2}\langle\lambda_1+\lambda_2+\lambda_3+\lambda_4,\alpha_1^\vee+\alpha_{N-1}^\vee\rangle-z\right]! \\
& \left[\frac{1}{2}\langle\lambda_1+\lambda_3+\lambda_{12}+\lambda_{23},\alpha_1^\vee+\alpha_{N-1}^\vee\rangle-z\right]! \\
& \left[\frac{1}{2}\langle\lambda_2+\lambda_4+\lambda_{12}+\lambda_{23},\alpha_1^\vee+\alpha_{N-1}^\vee\rangle-z\right]!\}^{-1},
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta_{(1,2,3)} \equiv \Delta(\lambda_1, \lambda_2, \lambda_3) \;\; = \;\; & \Big\{ [\left[\frac{1}{2} \langle -\lambda_1 + \lambda_2 + \lambda_3, \alpha_1^\vee + \alpha_{N-1}^\vee \rangle \right]! \\
& \times \left[\frac{1}{2} \langle \lambda_1 - \lambda_2 + \lambda_3, \alpha_1^\vee + \alpha_{N-1}^\vee \rangle \right]! \\
& \times \left[\frac{1}{2} \langle \lambda_1 + \lambda_2 - \lambda_3, \alpha_1^\vee + \alpha_{N-1}^\vee \rangle \right]! \Big\}^{1/2} \\
& \times \left\{ \left[\frac{1}{2} \langle \lambda_1 + \lambda_2 + \lambda_3, \alpha_1^\vee + \alpha_{N-1}^\vee \rangle + N - 1 \right]! \right\}
\end{aligned}
$$

$$C_z^{(I)} = \begin{cases} \delta_{z,z_{\mathsf{min}}+i} \begin{bmatrix} N-2+k_2-i \\ k_2-i \end{bmatrix}^{-1} & \text{for } k_1 > k_2\,, \\ \delta_{z,z_{\mathsf{min}}+i} \begin{bmatrix} N-2+k_1-i \\ k_1-i \end{bmatrix}^{-1} & \text{for } k_1 \leq k_2\,, \end{cases},$$

$$C_z^{(II)} = \begin{cases} \delta_{z,z_{\mathsf{max}}-i} \begin{bmatrix} N-2+k_2-i \\ k_2-i \end{bmatrix}^{-1} & \text{for } k_1 > k_2\,, \\ \delta_{z,z_{\mathsf{max}}-i} \begin{bmatrix} N-2+k_1-i \\ k_1-i \end{bmatrix}^{-1} & \text{for } k_1 \leq k_2\,, \end{cases}$$

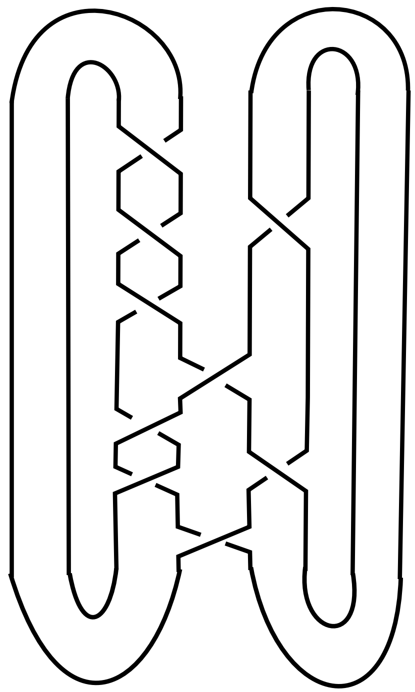
OBSERVATIONS

- [n]-colored HOMFLY for knots drawn as quasi-plat of 4-strand braids can be obtained using this data.

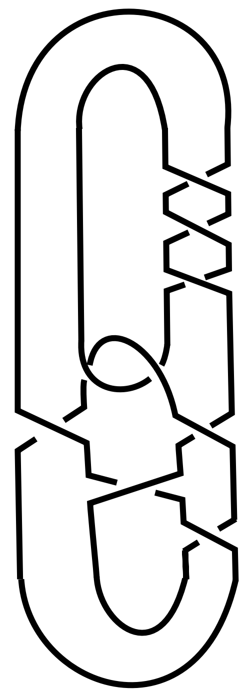
- Our data is not sufficient to write the polynomial for knots obtained from quasi-plat of braids with more than 4-strands.

(dual basis of 6-point or higher point conformal blocks requires Wigner 6-j beyond our conjectured class)

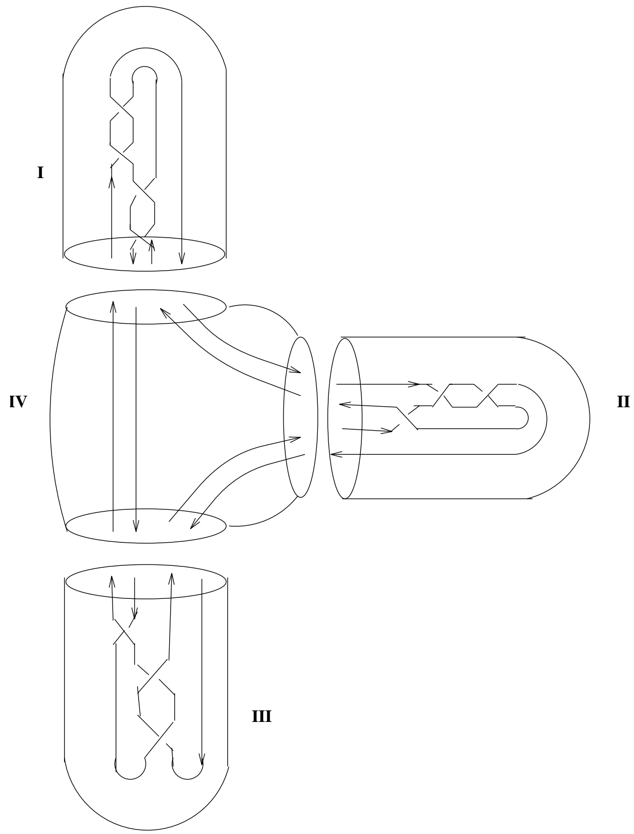
How do we obtain [n]-colored HOMFLY for 9_{42} knot involving six braids?

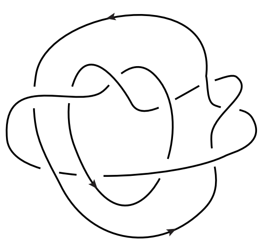


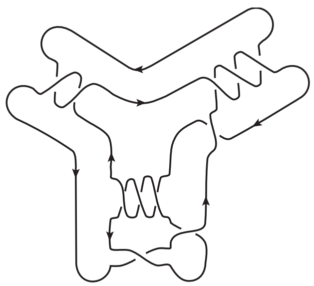
9_42 knot



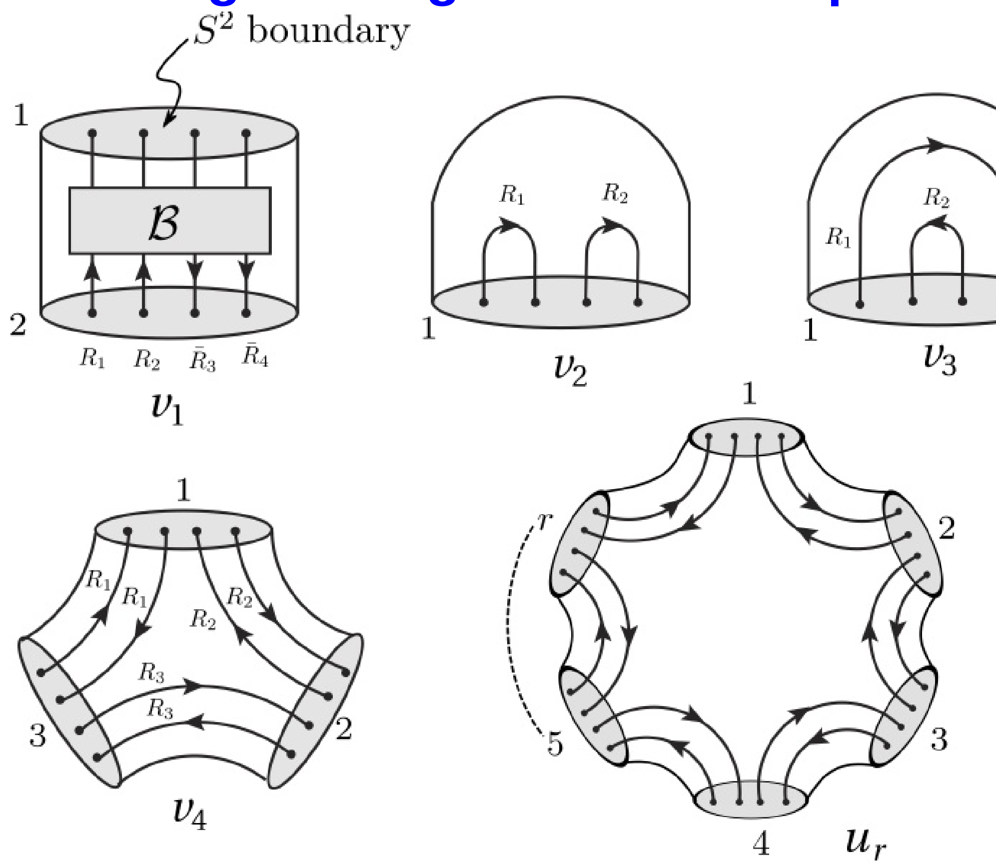
Knots $10_{71}, 10_{152}$ can be drawn gluing 3-manifolds involving three S^2 boundaries each with four-punctures:



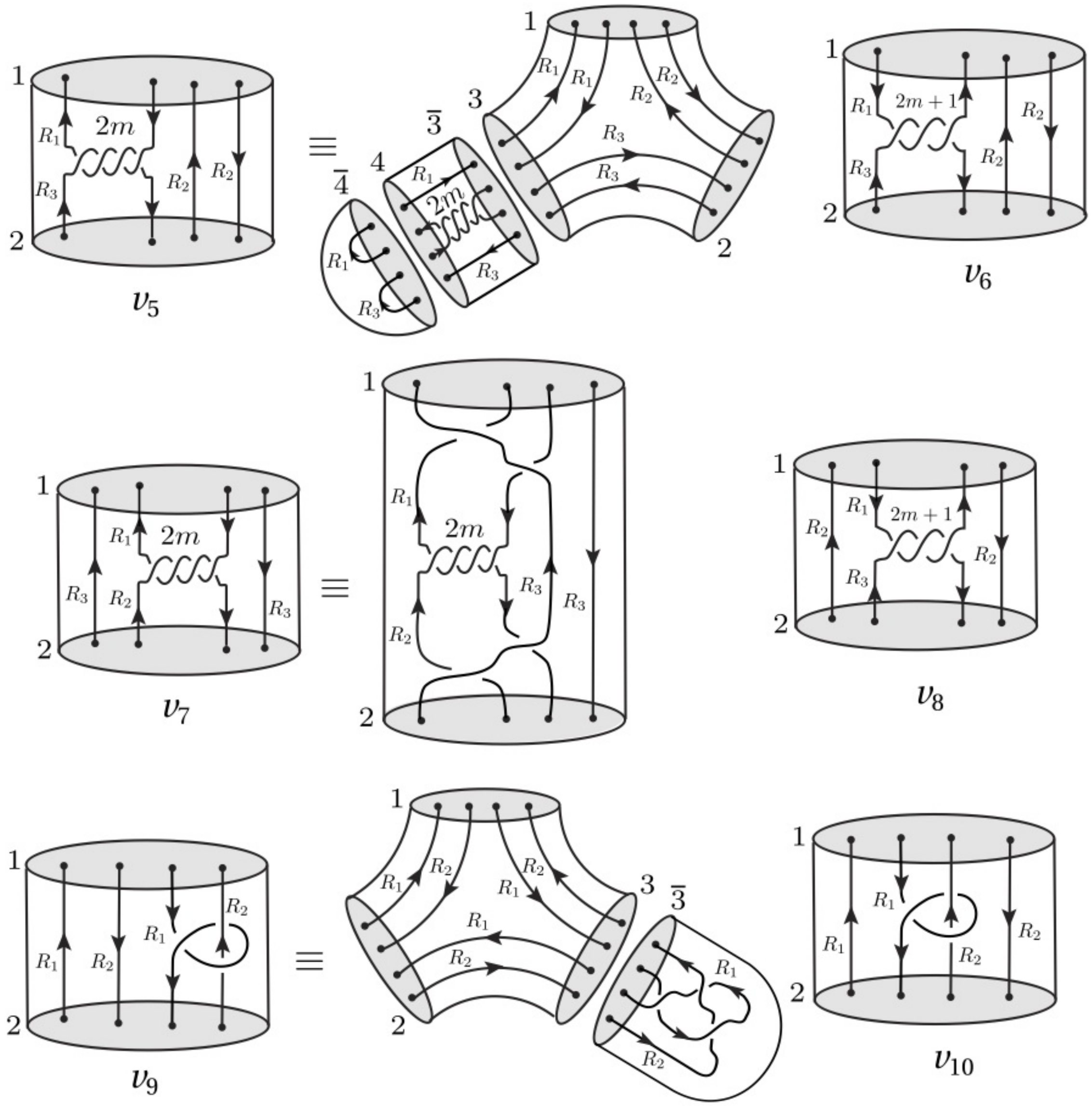




Requires the following building blocks to compute knot polynomials

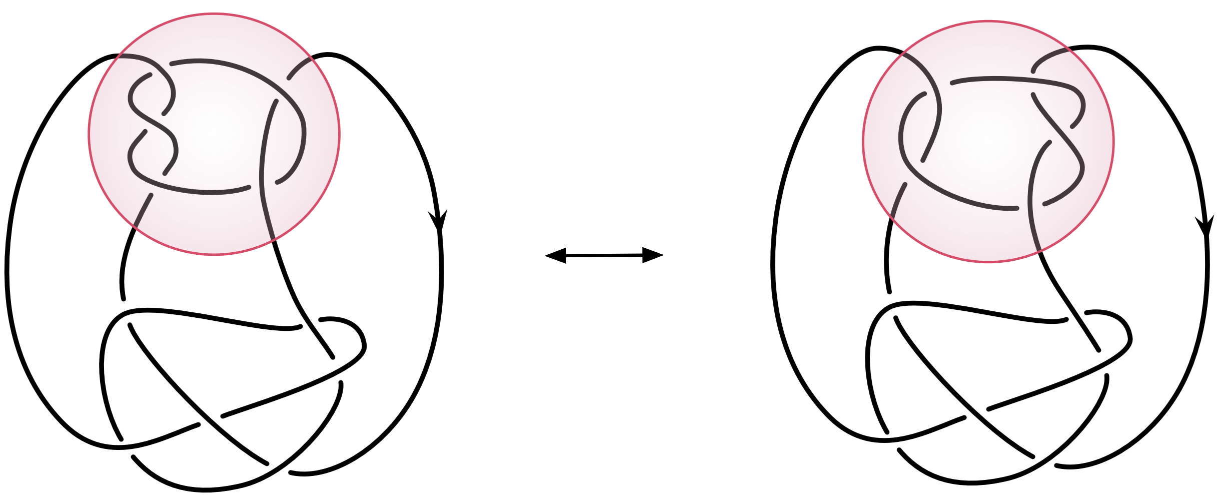


$$u_r = \sum_t (dim_q t)^{(1-r/2)} |\phi_t^{(1)}\rangle \ldots |\phi_t^{(r)}\rangle$$



v_5
v_6
v_7
v_8
v_9
v_{10}

• **We have redrawn many knots using these building blocks enabling evaluation of $[n]$-colored HOMFLY polynomials. These polynomials do not distinguish mutants.**



Kinoshita-Terasaka & Conway mutants

- **Need to work out polynomials for mixed representations.**

- **The two types of $U_q(sl_N)$ Wigner 6j has been recently determined for** $[2,1]$ **(Gu,Jockers),2014**-first mixed representation

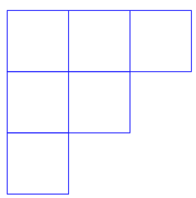
- **Wanted to check the power [2,1] colored HOMFLY for the mutant pair**

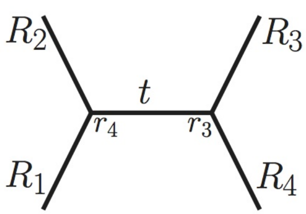
- **The two knots (mutant pairs) are indeed distinguished (Nawata,Singh,PR,2015)**

- **Enumerated class of mutants which can be distinguished but some pretzel mutants with antiparallel odd-braids cannot be distinguished (Mironov, Morozov,Morozov, Singh, PR),2015**

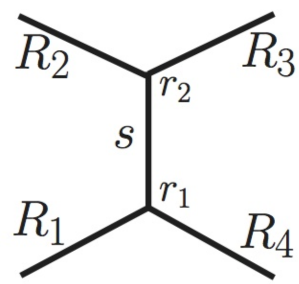
• **Crucial input in the context of mixed representation: multiplicity**

$$
\begin{aligned}
(21;0)\otimes(21;0) \;=\;& (42;0)_0 \oplus (2^3;0)_0 \oplus (31^3;0)_0 \oplus (321;0)_0 \\
&\oplus (321;0)_1 \oplus (41^2;0)_0 \oplus (3^2;0)_0 \oplus (2^21^2;0)_0
\end{aligned}
$$

We see  **appears twice. Multiplicity incorporated four-point conformal blocks:**

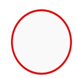
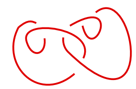
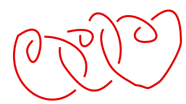
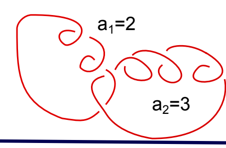
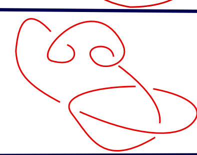


$= |\phi^{(1)}_{t,r_3r_4}(R_1,\ldots,R_4)\rangle$,

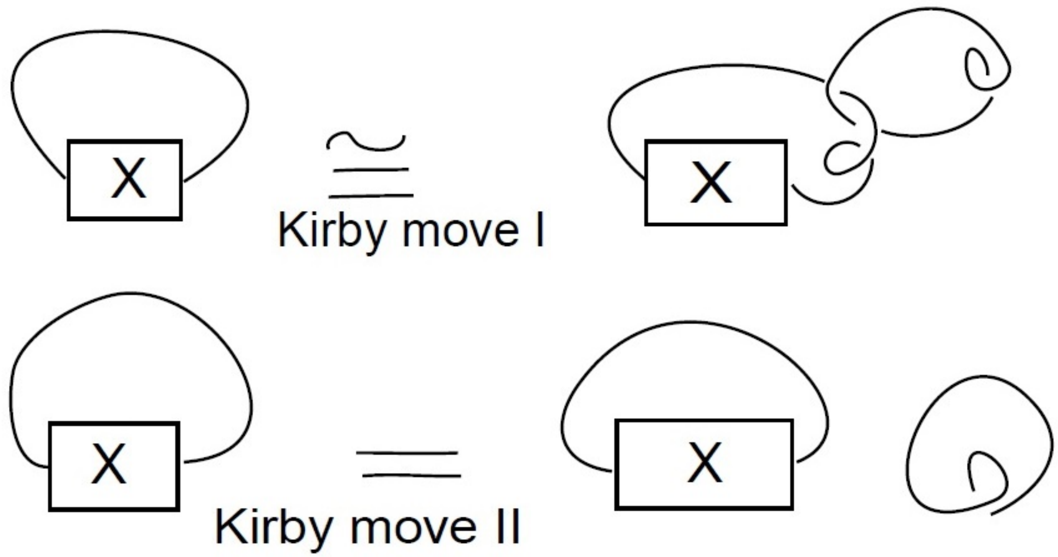


$= |\phi^{(2)}_{s,r_1r_2}(R_1,..,R_4)\rangle$

Table gives the three-manifolds obtained from surgery of framed knots

Frame link knot	3-Manifold
 	$S^2 \times S^1$
 	S^3
	RP^3
	L(P,1)
 $\frac{p}{q} = \frac{a_1}{a_2 - \frac{1}{a_3} \cdots}$	L(5,3)
	P^3

Framed links related by Kirby moves gives the same three-manifold



Three-manifold invariant proportional to Chern-Simons partition function $Z[M]$ respecting Kirby moves is(Kaul,PR),2000

$$Z[M] \propto \sum_R dim_q R \, V_R^{SU(N)}[K]$$

Questions

- **large N expansion $\log Z[M]$?**
- **For any knot, we observe Laurent series:**

$$P[K;a,q]=\sum_{i,j}c_{i,j}a^iq^j$$

where $c_{i,j}$ are integers. Topological meaning or reason?

Why integers?- two parallel developments

- **From physics (topological strings,BPS states counting)**
  Ooguri,Vafa (1999)

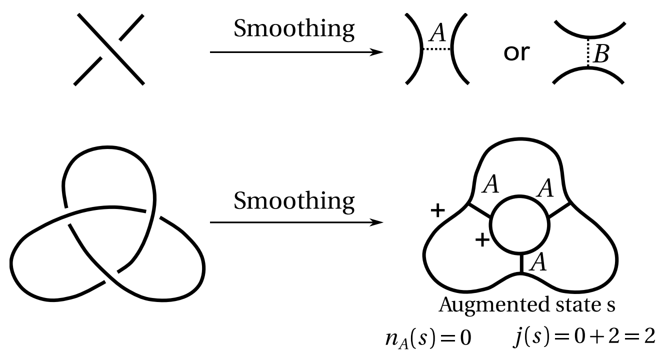
- **From mathematics(homological chain complex)**
  Khovanov(1999), Khovanov-Rosansky(2004)

$$c_{i,j}=\sum_k(-1)^k dim H_{i,j,k}$$

Homological Invariants

Introduce A and B slicing as shown in the diagram.

Define $n(s) = n_B$, $j(s) = n_B + n_+ - n_-$

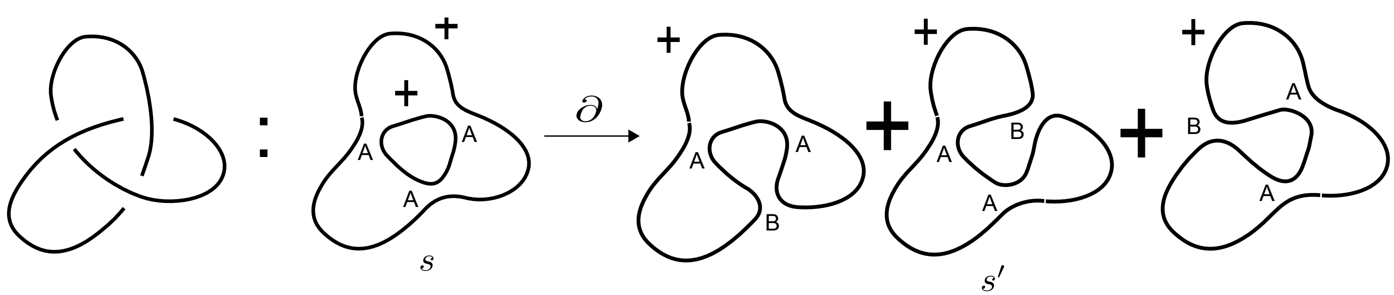


C_{nj} is the vector space with basis as states with $n(s) = n$ and $j(s) = j$.

$$J[K;q] = \sum_{n,j} (-1)^n q^j \mathrm{dim}(C_{nj})$$

where the homology chain

$$\partial : C_{n,j} \longrightarrow C_{n+1,j}, \quad \partial^2 = 0$$



$n_B(s) = 0$
$j(s) = 0 + 2 = 2$

$n_B(s') = 1$
$j(s') = 1 + 1 = 2$

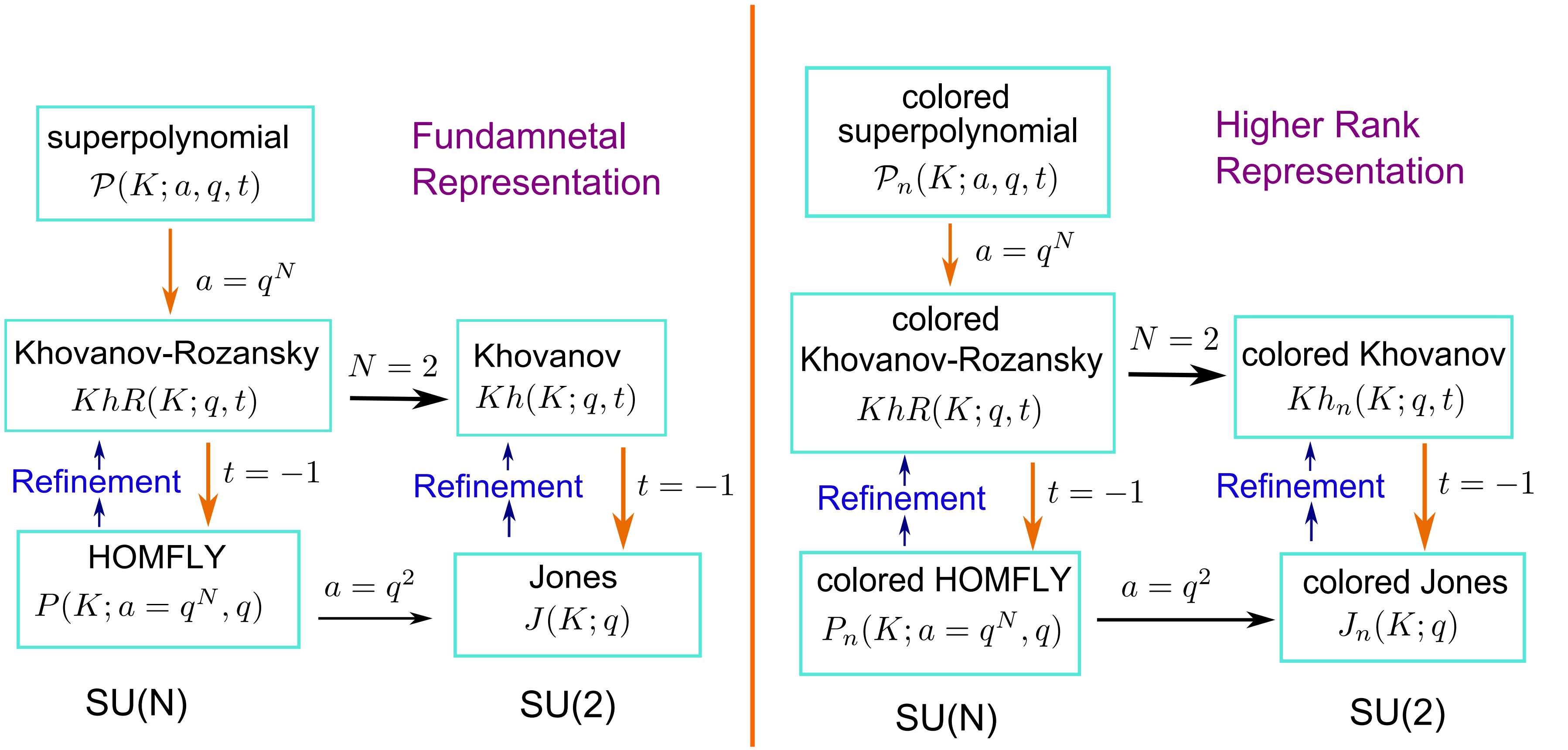
$$C_{*j}:\ C_{0j} \longrightarrow C_{1j} \longrightarrow C_{2j} \longrightarrow \ldots$$

The vector space

$$H_n(C_{*j}) = \frac{\mathrm{ker}(\partial : C_{n,j} \longrightarrow C_{n+1}, j)}{\mathrm{Image}(\partial : C_{n-1,j} \longrightarrow C_{n,j})}$$

$$Kh(K;q,t) = \sum_{n,j} t^n q^j \mathrm{dim}(H_{nj}).$$

Taking $t = -1$ gives the Jones polynomial $J[K;q]$



$$P_n(a,q)=\sum_{i,j} c_{i,j}q^i a^j$$

integer

$$Kh(K;q,t)=\sum_{i,j} t^i q^j \dim\mathcal{H}_{i,j}$$

homology of bigraded vector space

$$\boxed{\mathcal{P}_n(K;\,a,q,t)=\sum_{i,j,k} t^k q^j a^i \dim\mathcal{H}_{i,j,k}}$$

Large N Chern-Simons &topological strings

• Gopakumar-Vafa duality gives A-model closed topological string on a resolved conifold from large N expansion of $\log Z[S^3]$.

• Ooguri-Vafa conjecture a form for reformulated knot invariants $f_R(q, \lambda)$.

• We verified Ooguri-Vafa conjecture for non-torus knots (Sarkar, PR 2000) and higher crossing knots(Nawata,Zodin,PR 2013) using our $[n]$-colored HOMFLY polynomials.

• Making use of these two duality conjectures, we attempted the N expansion of $\log Z[M]$ for some three-manifolds M giving(P. Borhade, T. Sarkar,PR(2003)

$$
\begin{aligned}
\log Z_0[M] &= \sum_{n=1}^{\infty} \sum_g \frac{1}{n} \left(\sinh \frac{d g_s}{2} \right)^{2g-2} \times \\
& \{ \sum_Q \sum_{\{\ell_\alpha\}} \sum_{\{s_\alpha\}} \hat{N}_{(R_{\ell_1,s_1},\ldots R_{\ell_r,s_r}),g,Q} (-1)^{\sum_\alpha s_\alpha} \\
& (-1)^{n \sum_\alpha \ell_\alpha p_\alpha} \lambda^{\frac{1}{2} n \sum_\alpha \ell_\alpha p_\alpha} \Big(\lambda^{n\{Q + \sum_\alpha (\frac{-\ell_\alpha}{2} + s_\alpha)\}} \\
& - \lambda^{n\{Q+1+\sum_\alpha (\frac{-\ell_\alpha}{2} + s_\alpha)\}} \Big) \} \\
&= \sum_{g,n,m} \frac{1}{n} (2 \sinh \frac{n g_s}{2})^{2g-2} n_{g,m} e^{-dmt}
\end{aligned}
$$

Subtle Issues

In the large k limit

$$Z[M] = \sum_c Z_c[M]$$

t' Hooft proposal requires

$$\ln Z_c[M] = \text{Closed String expansion}$$

whereas we find

$$\ln \left(\sum_c Z_c[M] \right) = \text{Closed String partition function}$$

Hence we cannot predict duality between Chern-Simons gauge theory on M with the A-model string theory with the $n_{g,m}$'s we have determined

Generalisation of the duality to SO gauge groups

A-model closed strings on an orientifold of the resolved conifold is dual to SO/Sp Chern-Simons theory (Sinha and Vafa)

$$ln Z^{(SO)}_{(CS)}[S^3] = \frac{1}{2}\mathcal{Z}^{(or)} + \mathcal{Z}^{(unor)}$$

- Incorporating Wilson loop observables

$$\ln\langle Z(\{U_\alpha\},\{V_\alpha\})\rangle = \mathcal{F}_{\mathcal{G}}(V) = \frac{1}{2}\mathcal{F}^{(or)}_{\mathcal{R}}(V) + \mathcal{F}^{(unor)}(V)$$

Not clear how to seperate, we showed LHS (Pravina Borhade and PR)

$$\langle Z(\{U_\alpha\},\{V_\alpha\})\rangle = \exp\left[\sum_{n=1}^{\infty}\sum_{\{R_\alpha\}} g_{R_1,R_2,\ldots R_r}(q^n,\lambda^n)\,\frac{1}{n}\prod_{\alpha=1}^{r} Tr_{R_\alpha}V_\alpha^n\right]$$

where $g_{R_1,\ldots R_r}(q,\lambda) = \sum_{Q,s} \frac{1}{(q^{1/2}-q^{-1/2})} N_{(R_1,R_2\ldots R_r),Q,s} q^s \lambda^Q$

$N_{(R_1,\ldots R_r),Q,s}$ are integers-how to find oriented contribution?

•Composite representation invariants to extract oriented contribution (Marino)

$$\mathcal{F}_{\mathcal{R}}^{(or)}(V) = \sum_{R,S} \langle W_{(R,S)}[C]\rangle (Tr_R V)(Tr_S V) = \sum_t \mathcal{R}_t[C] Tr_t V$$
$$= \sum_{d=1}^{\infty} \sum_R h_R(q^d,\lambda^d) Tr_R V^d$$

where R and S are representation on the knot C in two Lagrangian submanifolds $\mathcal{N}_\epsilon$ and $\mathcal{N}_{-\epsilon}$ related by orientifolding action. The composite representation (R,S) has highest weight $\Lambda_R + \Lambda_{\bar{S}}$.

• Obtained invariants of torus knots & links for some composite representations and verified the form of $h_R(q,\lambda)$(2010, C. Paul, P. Borhade,PR)

• Using [r]-colored Kauffman polynomials for figure-eight knot from the study of structural properties of colored Kauffman homologies of knots (Nawata,Zodin,PR,2013) and Morton's result for ($\square$, $\square$) composite invariant for figure-eight, we verified the integrality structure for $h_{\square\square}$ and $g_{\square\square}$.

Summary

• Elegant computation of [n]-colored HOMFLY for many knots from Chern-Simons approach

• [2,1]-colored HOMFLY distinguishes many mutants including Kinoshita-Terasaka & Conway mutant pair.

• These invariants are definitely useful to verify integrality structures predicted by $U(N)$ and SO topological string duality conjectures

Challenging problems

(i) Obtaining closed form expression for $SU(N)_q$ fusion matrices for general R's which could be rectangular or mixed representation.

(ii) Can we get a better understanding of polynomial variable t in homological invariants?

(iii) The results on refined CS for $[2, 2p + 1]$-torus knots suggests that there could be a combinatorial definition of colored Khovanov invariants for non-torus knots. So far no success!

(iv) Any idea on getting # of solutions to elliptic Nahm equations accounting for integer coefficients in Khovanov invariant for trefoil? Gaitto,Witten (2011)

(v) Appears Kevin Costello work may be applicable to the determine spectral parameter dependent R-matrix for N-vertex models using 4-d twisted supersymmetric field theory.

Due to time constraint, I have not discussed volume conjecture, A-polynomials and their relations to (i) $SL(2, C)$ 3-d gravity(Gukov,2003) and (ii) their appearance in mirror Calabi-Yau geometry (Aganagic,Vafa(2012)-interesting developments.

Thank You