Functional determinants, Index theorems, and Exact quantum black hole entropy

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Black hole entropy $\Longrightarrow$ underlying ensemble of microscopic states

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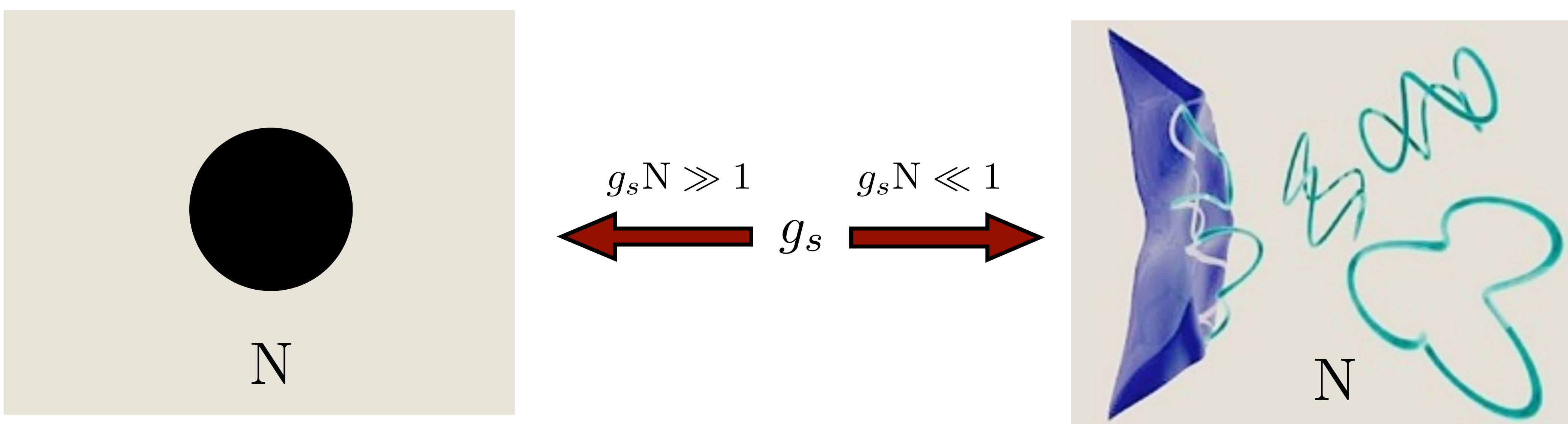
Macroscopic

Microscopic



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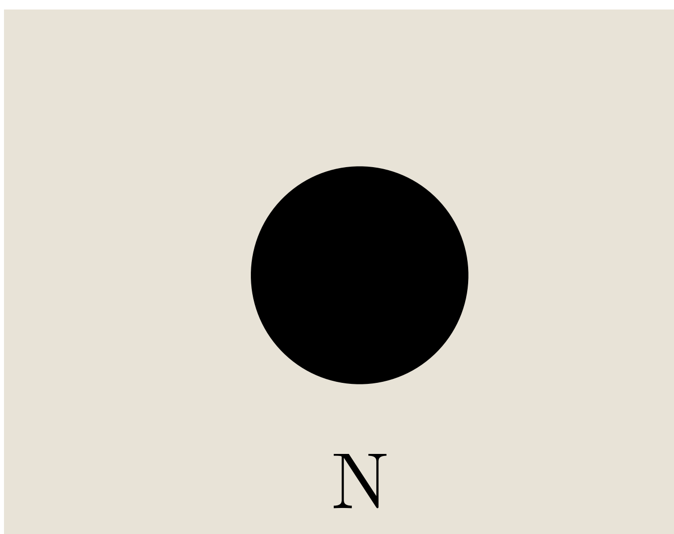
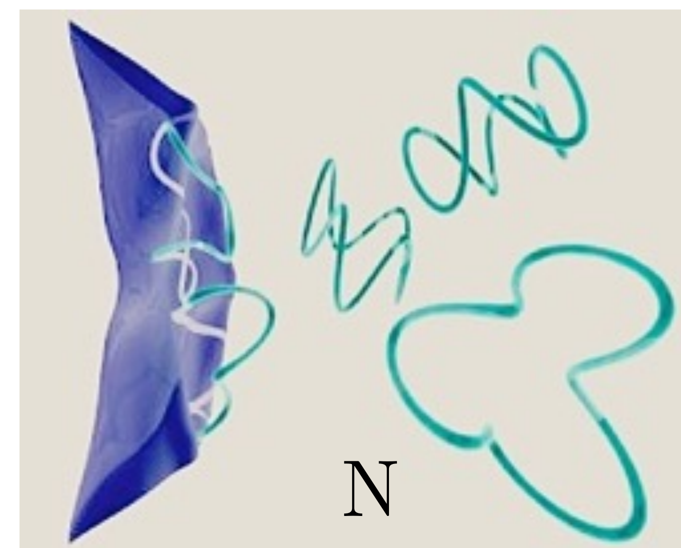


$g_s\mathrm{N} \gg 1$ $\longleftarrow$ g_s $\longrightarrow$ $g_s\mathrm{N} \ll 1$

Microscopic



Q: How many microstates are associated to a BH?

Aim of talk

In a theory of

- 4d N=2 supergravity,
- coupled to vector multiplets and hyper multiplets,
- with generic higher derivative couplings,

compute the perturbatively exact formula for the quantum gravitational entropy of a 1/2 BPS black hole.

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In a theory of

- 4d N=2 supergravity,
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compute the perturbatively exact formula for the quantum gravitational entropy of a 1/2 BPS black hole.

Based on:

S.M., V. Reys, arXiv:1504.01400, B. de Wit, S.M., V. Reys, in progress.

and on the work of:

Sen (Quantum entropy function program), Ooguri,Strominger,Vafa (OSV), Cardoso, de Wit, Mohaupt;

Banerjee, Dabholkar, David, Denef, Gaiotto, Gomes,Gupta, Hama, Hosomichi, Jatkar, Lal, Mahapatra, Mandal, Moore, Pestun, Pioline, Shih, Yin, …

Near-horizon region is an independent quantum system, fixed by charges

4d BPS black hole with charges (q_I, p^I)

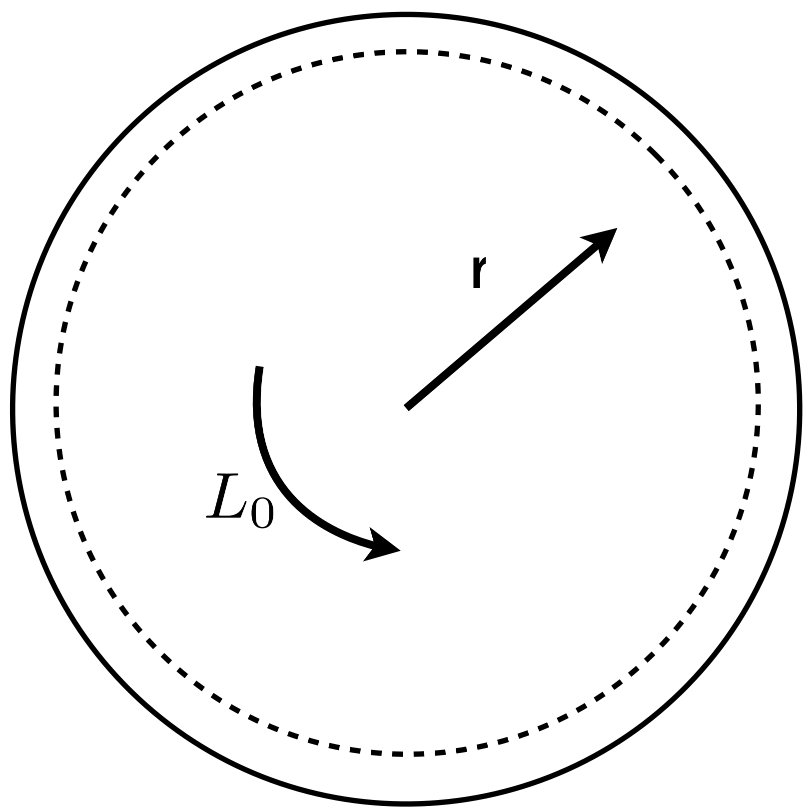
$$ds^2 = v\left((r^2-1)\,d\theta^2 + \frac{dr^2}{r^2-1}\right) + v\left(d\psi^2 + \sin^2\psi\, d\phi^2\right)$$

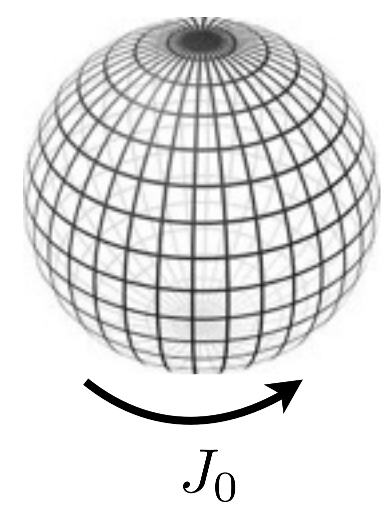
$$F^I_{rt} = e^I_* \,, \qquad F^I_{\psi\phi} = p^I \sin\psi$$

$$X^I = X^I_*$$

Attractor geometry:

(Ferrara, Kallosh, Strominger '95)





$\mathbf{Euclidean\ AdS_2 \times S^2}$

Quantum BPS black hole entropy is an AdS_2 functional integral

(Sen '08)

$$\exp(S^{\rm qu}_{BH}(q_I)) \equiv Z_{AdS_2}(q_I) = \left\langle \exp\left[- i\, q_I \oint A^I \right] \right\rangle^{\rm reg}_{\rm AdS_2}$$

- Saddle point evaluation $\Longrightarrow$ $S^{\rm qu}_{\rm BH} = S^{\rm class}_{\rm BH} + \cdots$

  Attractor entropy including local higher-derivative corrections.

  (Cardoso, de Wit, Mohaupt, '98, …)

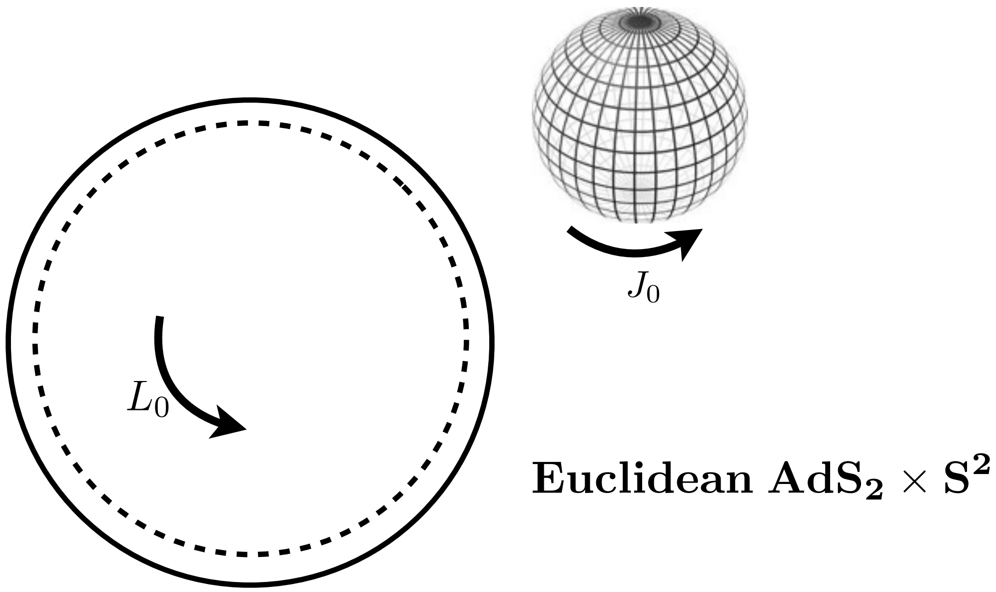
- Leading logarithmic one-loop corrections systematized.

  (Sen + Banerjee^2, Gupta, Mandal, '10-'14, Larsen, Keeler, Lisbão '14, '15).

The functional integral localizes onto the solutions of the off-shell BPS equations

Duistermaat-Heckmann, Atiyah-Singer-Bott, Berligne-Vergne, Witten (1980s), Pestun '07

$$\boxed{Z_{\mathrm{AdS}_2}(q_I) \equiv \int_{\mathcal{M}} d\mu\, \mathcal{O}\, e^{-\mathcal{S}}}$$



Superalgebra $Q^2 = L_0 - J_0 \equiv H$

$$I := \int_{\mathcal{M}} d\mu\, \mathcal{O}\, e^{-\mathcal{S}}$$

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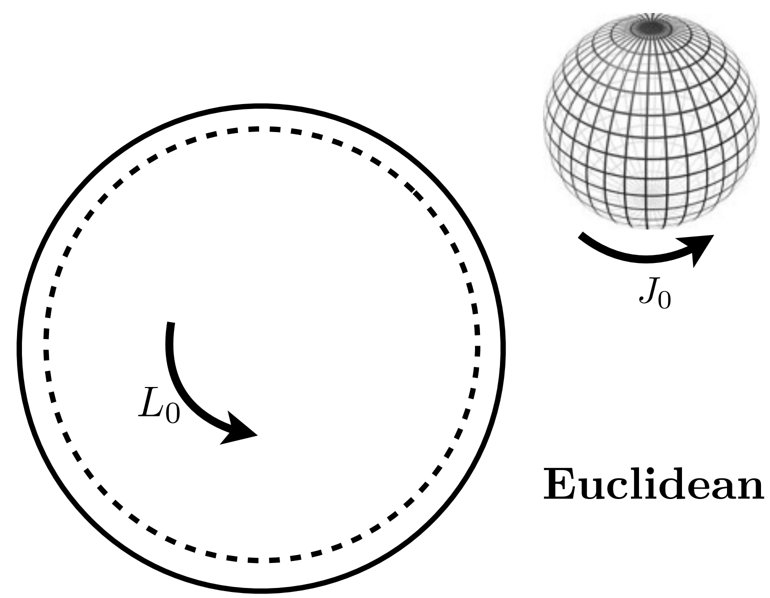

Euclidean $\mathbf{AdS_2 \times S^2}$

Superalgebra $Q^2 = L_0 - J_0 \equiv H$

$$I(\lambda) = \int_{\mathcal{M}} d\mu\, \mathcal{O}\, e^{-\mathcal{S} + \lambda Q \mathcal{V}} \,, \qquad \mathcal{V} = \sum_{\psi} \int d^4x\, \overline{\psi}\, Q\, \psi$$

$$I(0) = I(\infty) = \int_{\mathcal{M}_Q} d\mu_Q\, \mathcal{O}\, e^{-\mathcal{S}}\, Z_{1\text{-loop}}(Q\mathcal{V})$$

$$= \{Q\Psi = 0\}$$

Localization in supergravity

(A.Dabholkar, J.Gomes, S.M. '10, '11, '14)

1. Describe the supergravity theory in a formalism which admits off-shell supersymmetry transformations.

2. Find all solutions of localization equations $Q\,\Psi = 0$, subject to $AdS_2 \times S^2$ boundary conditions.

3. Evaluate full supergravity action on these solutions (including all higher derivative terms).

4. Compute quantum measure on the solution space.

The fields of N=2 conformal supergravity

(de Wit, van Holten, Van Proeyen '80)

- Weyl multiplet coupled to $n_{\rm v}+1$ U(1) vector multiplets

  $$X^I \to \lambda(x)\, X^I,\ (I=0,\cdots,n_{\rm v})\,,\quad g_{\mu\nu} \to \lambda(x)^{-2}\, g_{\mu\nu}$$

- $\boxed{e^{-\mathcal{K}} \equiv -i(X^I \bar{F}_I - \bar{X}^I F_I)}$ appears in the action as

  $\sqrt{-g}\, e^{-\mathcal{K}}\, R + \cdots$ (conformal compensator)

- Can choose gauge $e^{-\mathcal{K}} = 1$ or $\sqrt{g} = 1$

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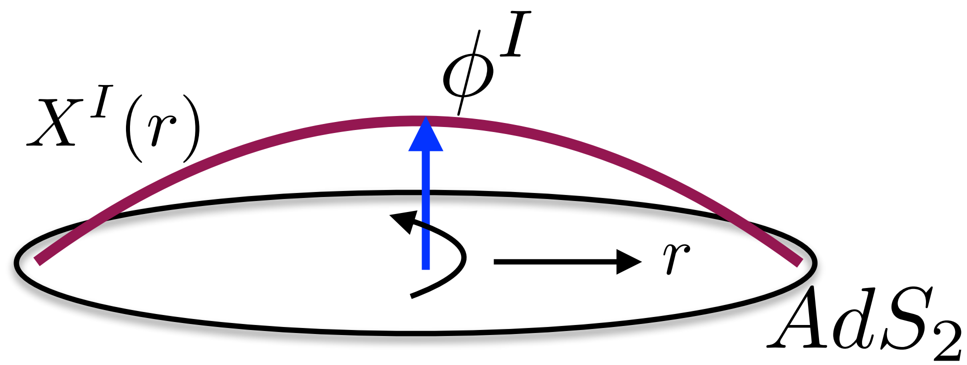
- Can choose gauge $e^{-\mathcal{K}} = 1$ or $\sqrt{g} = 1$

In this gauge, $e^{-\mathcal{K}(X^I)}$ measures the size of AdS_2

The space of off-shell BPS solutions

- In vector multiplet sector, scalar fields go off-shell:



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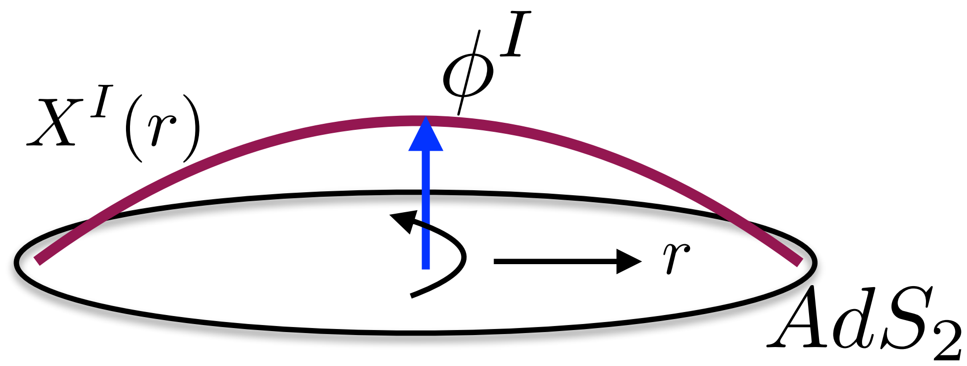

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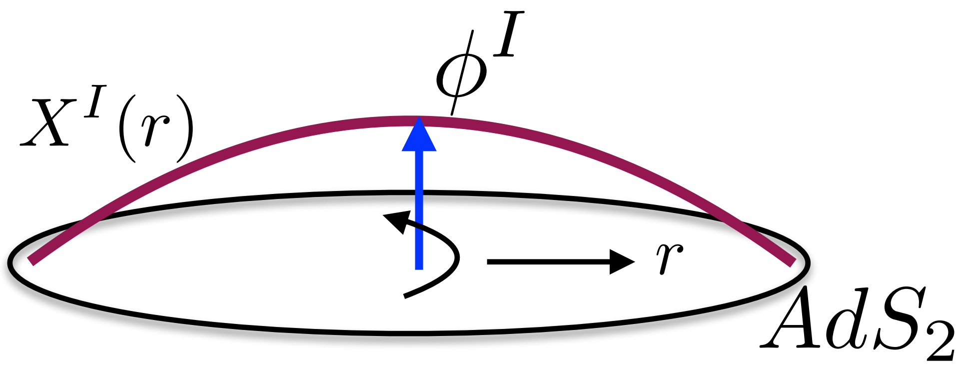

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Localization manifold labelled by one parameter $\phi^I \equiv X^I(r=1)$ for each vector multiplet X^I.

- Classical measure for scalar manifold $\prod_I dX^I \, M(X^I)$ has been computed. Does not scale with charges.

(Cardoso, de Wit, Mahapatra '12)

Action of supergravity theory

Chiral superspace action is governed by holomorphic prepotential $F(X^I)$, computable in string theory.

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$$Z^{\text{pert}}_{AdS_2}(q,p) = \int \prod_{I=0}^{n_{\text{v}}} d\phi^I \, M(\phi^I) \times$$
$$\times \, \exp\Big(- \pi \, q_I \, \phi^I + 4\pi \, \text{Im} \, \text{F}\big(\phi^I + ip^I\big)\Big) \, Z^{Q\mathcal{V}}_{\text{1-loop}}(\phi^I)$$

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1-loop determinant is a function of dynamical scale of quantum supergravity

The length scale at the center of AdS_2 $\ell^2_{\mathrm{AdS}} = e^{-\mathcal{K}(\phi^I)}$ depends on the position in the localization manifold.

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$$Z^{Q\mathcal{V}}_{\text{1-loop}}(\phi^I) = \exp\big(-a_0\,\mathcal{K}(\phi^I)\big), \qquad a_0 = \sum_{\text{multiplets}} a_0^{\text{multiplet}}$$

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Index theorem $\Rightarrow$ $$a_0^{\rm vector} = -a_0^{\rm hyper} = -\frac{1}{12}$$

(Result for vector multiplets also in Ito, Gupta, Jeon 1504.01700)

(Graviton multiplet in progress with B. de Wit, V. Reys)

Sketch of computation

$$X \xrightarrow{Q} QX$$

- Separate fields into doublets of Q

$$\Psi \xrightarrow{Q} Q\Psi$$

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Eigenvalues of H encoded in index

$$\mathrm{ind}(D_{10})(t) = \mathrm{Tr}_X\, e^{-iHt} - \mathrm{Tr}_\Psi\, e^{-iHt}$$

$$= \sum_n a(n)\, e^{-it\lambda_n}$$

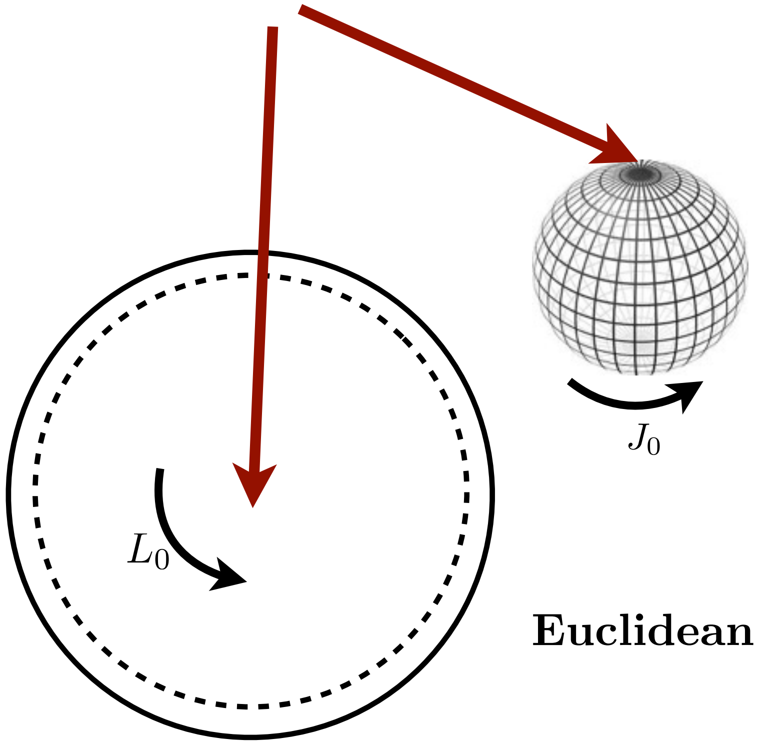
The index is captured by fixed points of H

Atiyah-Bott index theorem

$$\mathrm{ind}(D_{10})(t) = \sum_{\text{x=fixed pts of } H} \frac{\mathrm{Tr}_{X,\Psi}\,(-1)^F\, e^{-iHt}}{\det(1-\partial\widetilde{x}/\partial x)}\,.$$

$(\widetilde{x} = Hx)$

Fixed points of
$H = L_0 - J_0$



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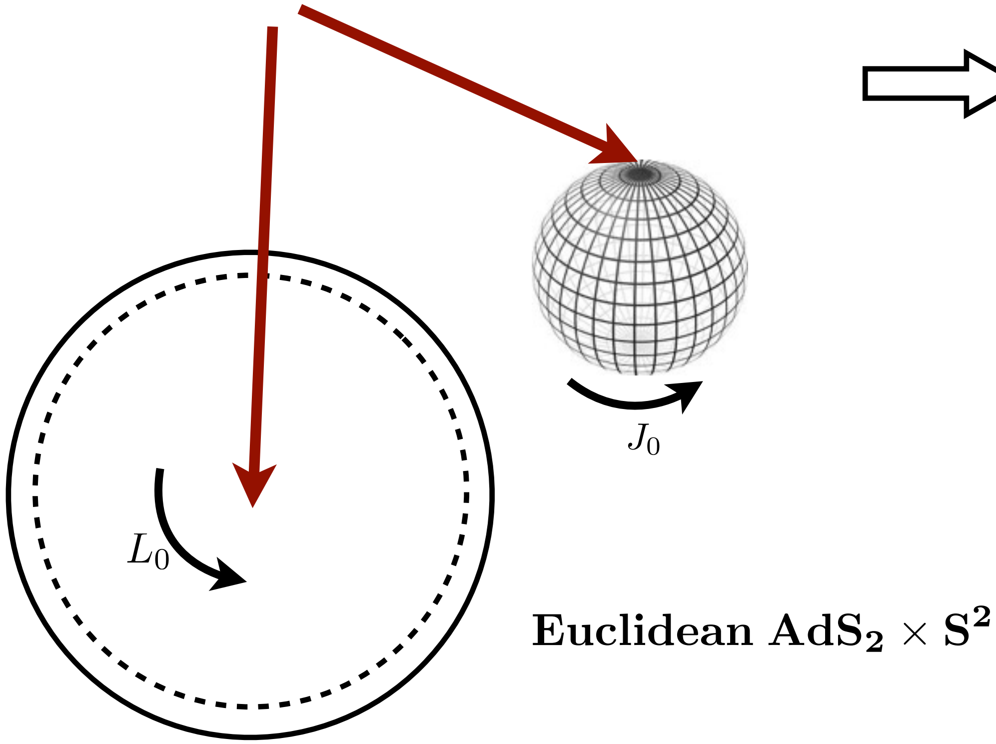

$\Longrightarrow$

$$\mathrm{ind}_{\mathrm{vec}}(D_{10}) = \sum_{n\geq 1} 4n\, e^{-i\,tn/\ell_{\mathrm{AdS}}}$$

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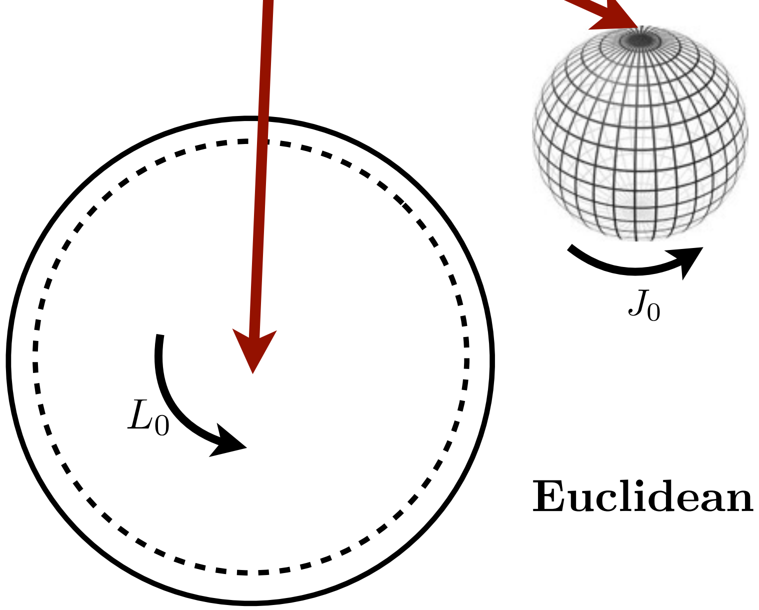

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Degeneracies

Eigenvalues

Regularize determinant using zeta functions.

Technical comments

- In supergravity, $Q^2 = L_0 - J_0 + \delta_{\text{gauge}}$

  Introduce (b,c) ghost system and BRST operator Q_B

  Introduce $\widehat{Q} = Q + Q_B$ with $\widehat{Q}^2 = H = L_0 - J_0$

- Boundary modes ("pure" gauge modes that do not vanish at the boundary) need to be treated separately.

(See e.g. Banerjee, Gupta, Sen '10; Sen '11)

Checks

$$Z^{Q\mathcal{V}}_{\text{1-loop}}(\phi^I) = \exp\big(-a_0\,\mathcal{K}(\phi^I)\big)\,, \qquad a_0 = \sum_{\text{multiplets}} a_0^{\text{multiplet}}$$

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- Microscopics of N=8 string theory:

  Truncation of N=8 to N=2 theory. Quantum BH entropy agrees with microscopic integer degeneracies exactly!

  (A. Dabholkar, J. Gomes, S.M., '11 '14)

  Can check that the 1-loop contributions of fields that are thrown away add up to zero.

$\frac{1}{2}$-BPS N=2 BH entropy: where do we stand?

$$Z_{\mathrm{AdS}_2}^{\mathrm{pert}}(q,p) = \int \prod_{I=0}^{n_{\mathrm{v}}} d\phi^I \, (\phi^0)^{2-\frac{\chi}{12}} \, e^{-\mathcal{K}(\phi)} \times$$

$$\times \exp\Big(- \pi \, q_I \, \phi^I + 4\pi \, \mathrm{Im} F(\phi^I + ip^I) \Big)$$

$$\chi = 2(n_{\mathrm{v}} + 1 - n_{\mathrm{h}})$$

(*cf*. Ooguri-Strominger-Vafa '04, Denef-Moore '07, Sen, 1108.3842)

- Note: Still need to complete graviton calculation.
- This assumes a classical measure that has been derived for two-derivative theories.

  (Cardoso, de Wit, Mahapatra '12)
- This formula implies a surprising cancellation of states in N=2 string theory.

Resolution of topological string puzzle

Consider type II string theory on a CY3. The (perturbative) topological string partition function at strong topological string coupling $\lambda \to \infty$ is:

$$F_{\mathrm{top}} = -i \frac{(2\pi)^3}{6\lambda^2} C_{ABC}\, t^A\, t^B\, t^C - \frac{i\pi}{12} c_{2A}\, t^A + F_{GW}(\lambda, t^A)$$

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Upon analytic continuation to weak coupling (supergravity regime), the correct expression is: (Pioline '06, Denef-Moore '07)

$$\widetilde{F}_{\rm top} = -i\frac{(2\pi)^3}{6\lambda^2}\, C_{ABC}\, t^A\, t^B\, t^C - \frac{\chi}{24}\log\lambda - \frac{i\pi}{12}\, c_{2A}\, t^A + F_{GW}(\lambda, t^A)$$

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Resolution: arises in 1PI effective action of supergravity.

(See Dedushenko,Witten '14)

The exact N=2 BH entropy formula

$$Z^{\text{pert}}_{\text{AdS}_2}(q,p) = \int_{\mathcal{M}_Q} \prod_{I=0}^{n_{\text{v}}} d\phi^I \, M_{\text{class}}(\phi^I) \times$$

$$\times \, \exp\Big(- \pi \, q_I \, \phi^I + 4\pi \, \mathrm{Im}\, \mathrm{F}\big(\phi^I + i p^I\big)\Big) \, Z_{\text{1-loop}}(\phi^I)$$

$$Z_{\text{1-loop}} = \exp\Big(-\mathcal{K}(\phi^I)\big(2 - \frac{\chi}{24}\big)\Big) \,, \qquad \chi = 2(n_{\text{v}} + 1 - n_{\text{h}}) \,.$$

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$$M_{\text{class}} = \big(\phi_0^{-2} \, \exp(-\mathcal{K}^0(\phi^I))\big)^{\frac{\chi}{24} - 1}$$

at first order in large charges. (Cardoso, de Wit, Mahapatra '12)

The exact N=2 BH entropy formula

$$\boxed{e^{S^{\mathrm{qu}}(q,p)} = \int_{\mathcal{M}_Q} \prod_{I=0}^{n_{\mathrm{v}}} d\phi^I \, e^{-\pi\, q_I\, \phi^I} \, |\lambda_{\mathrm{top}}|^{-2} \, e^{-\mathcal{K}^0} \, |Z_{\mathrm{top}}|^2}$$

$$e^{-\mathcal{K}^0} = -i(X^I \bar{F}^0_I - \bar{X}^I F^0_I) \ , \qquad X^I = \phi^I + ip^I \ ,$$

$$Z_{\mathrm{top}} = \left(\frac{\lambda_{\mathrm{top}}}{2\pi}\right)^{\chi/24} e^{-2\pi i F(X^I)} \ , \qquad \lambda_{\mathrm{top}} = 4\pi/X^0 \ ,$$

$$\chi = 2(n_{\mathrm{v}} + 1 - n_{\mathrm{h}}) \ .$$

Why does the truncation work?

We kept: zero modes of 8 vector multiplets $\qquad -\frac{1}{2} \times 8$

We threw away:

- Non-zero modes of 8 vector multiplets $\qquad +\frac{5}{12} \times 8$
- (15-8) vector multiplets $\qquad -\frac{1}{12} \times (15-8)$
- 10 hyper multiplets $\qquad +\frac{1}{12} \times 10$
- 6 gravitini multiplets $\qquad -\frac{11}{12} \times 6$
- 1 gravity multiplet $\qquad +2$

Total 0