Membrane Paradigm at Large Dimension

Sayantani Bhattacharyya

June 25, 2015

References

- This talk is based on arXiv: 1504.06613
  S.B., A. De, S. Minwalla, R. Mohan, A. Saha:
- It builds on earlier work by Emparan, Tanabe, Suzuki and collaborators
  - R. Emparan, R. Suzuki and K. Tanabe : arXiv 1302.6382
  - R. Emparan, D. Grumiller and K. Tanabe: arXiv 1303.1995
  - R. Emparan and K. Tanabe: arXiv 1401.1957
  - R. Emparan, R. Suzuki and K. Tanabe : arXiv 1402.6215
  - R. Emparan, R. Suzuki and K. Tanabe : arXiv 1406.1258

Aim of the talk

- Evolution of the space-time is governed by Einstein equations

$$R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} = 0$$

- These equations are very difficult to solve in general and it is always useful to have new solution-generating techniques.
- In this talk, our aim is to find new perturbative solutions to Einstein equations.
- We shall use the number of dimensions as a perturbation parameter.
  Hence the new solutions would be a series in $\frac{1}{D}$.

Our final result

We found a class of such perturbative solutions with the following properties.

- Solutions have a space-time singularity, but always behind a dynamical event horizon.
- Event horizon could be viewed as a fluctuating membrane embedded in flat-space-time with a 'velocity' field on it.
- Space-time would be regular everywhere outside the event horizon only if
  the dynamics of the membrane and the 'velocity' field satisfy some particular set of equations.

In large dimensions effectively there is one-to-one map between the dynamics of a membrane and solutions to Einstein equations.

Plan of the talk

- First we shall explain how large dimension simplifies the solution.
  To understand this point we shall study Schwarszchild Black Hole at $d \to \infty$ limit.
- Next we describe our set-up
- Then we explain the algorithm we used to generate the perturbative solutions
- Finally we present our results.
  The explicit form of the metric and the membrane-equations at first non-trivial order
- Ongoing works and Future direction

Schwarszchild Black Hole as $D \to \infty$

- The metric of the Schwarszchild BH:

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2_{D-2} + \left(\frac{r_0}{r}\right)^{D-3} (dt + dr)^2,$$

- Naive large D limit:
  at fixed $r > r_0$, $\lim_{D\to\infty} \left(\frac{r_0}{r}\right)^{D-3} = 0$
- But suppose $r = r_0 \left(1 + \frac{R}{D}\right)$
- That is we tune r nearer to r_0 as we take $D \to \infty$
- Now at fixed R

$$\lim_{D\to\infty} \left(\frac{r_0}{r}\right)^{D-3} = \lim_{D\to\infty} \left(1 + \frac{R}{D}\right)^{D-3} = e^{-R}$$
$$\Rightarrow ds^2 = ds^2_{flat} + e^{-R}(dt + dr)^2$$

- Metric departs from flat space-time only within a thin region of thickness $\left(\frac{r_0}{D}\right)$ around the horizon.

QNM analysis around Schwarszchild BH as $D \to \infty$

At large D there are two sets of QNM at every angular momentum.

Emparan, Suzuki, Tanabe arXiv:1502.02820

1. An infinite tower of non-decoupled modes:
   These modes have frequency of order $\mathcal{O}\left(\frac{D}{r_0}\right)$ and behaves as radiation at large r.
2. A few decoupled modes:
   These modes have frequency of order $\mathcal{O}\left(\frac{1}{r_0}\right)$ and fall off exponentially at large r.

From this analysis we learned that

- Decoupled modes are parametrically separated from the others.
- Therefore it is possible to complete their dynamics to full non-linear level without exciting the non-decoupled ones

Our construction essentially captures the non-linear dynamics of these decoupled modes in an expansion in $\frac{1}{D}$

- As $D \to \infty$ both the number of equations and the functions (to be solved) becomes infinite.
- No perturbation would work if the number of functions to be solved also changes with the perturbation parameter.

Therefore

- We shall restrict ourselves to solutions with $SO(d+1)$ isometry, such that

  Total dimension $= D = d{+}p{+}3$ and as $d \to \infty$, p remains finite

  .

- Only a finite $(p+3)$ dimensional subspace of the full space-time will be dynamical.

The set-up

- $D = d + p + 3, \quad ds^2 = ds^2_{p+3} + e^{2\phi} d\Omega^2_d$

  where $d\Omega^2_d$ = Metric of a unit d dimensional sphere
- $ds^2_{p+3} = g_{\mu\nu} dx^\mu dx^\nu, \quad \{\mu, \nu\} = \{0, 1, 2, \cdots, p+2\}$
- Here $g_{\mu\nu}(x^\alpha)$ is the $(p+3)$ dimensional metric (the dynamical part).
- In the remaining d dimensions, $SO(d+1)$ isometry fixes the metric to $(d\Omega^2_d\ )$ upto an overall radius $e^{\phi(x^\mu)}$.
- From the point of view of $(p+3)$ dimensions, the basic variables:
  1. $(p+3)$ dimensional metric: $g_{\mu\nu}(x)$
  2. Scalar field: $\phi(x)$

Strategy

- From the study of Schwarszchild BH we learned,
  - The decoupled modes are effectively confined within a thin region of order $\mathcal{O}\left(\frac{1}{d}\right)$ around the horizon.
  - And therefore if we excite only the decoupled modes, the space-time outside this region will remain flat.
    (We shall call this region 'membrane region'.)
- Now our strategy:
  1. Start with a global ansatz for $g_{\mu\nu}$ that has a nontrivial 'membrane-region'.
  2. Zoom into the membrane region by scaling the coordinates and the distance appropriately.
  3. Ensure that the ansatz solves the Einstein equations at leading order in $\mathcal{O}\left(\frac{1}{d}\right)$ expansion inside this membrane region.
  4. Correct the ansatz order by order so that it continues to solve the equations.

Step-1: Global ansatz

- Start with a global ansatz

$$g_{\mu\nu} = \eta_{\mu\nu} + \psi^{-(d+p)} O_\mu O_\nu$$

Here

$\{\mu, \nu\} = \{0, 1, 2, \cdots, p+2\}$ $\qquad \eta_{\mu\nu} = \text{Diagonal}(-1, 1, 1, \cdots)$

- The ansatz is parametrized by a free function ψ and a one-form $O_\mu dx^\mu$ (both functions of global coordinates)
- Next we have to identify the membrane region for this ansatz metric.

Identifying the membrane region

Ansatz: $g_{\mu\nu} = \eta_{\mu\nu} + \psi^{-(d+p)} O_\mu O_\nu$

- The metric has a singularity at $\psi = 0$ for every value of $(d+p)$
- As $d \to \infty$ the metric blows up for $\psi << 1$ and it approaches Minkowski metric $\eta_{\mu\nu}$ for $\psi >> 1$
- Only when $|\psi - 1| \sim \mathcal{O}\left(\frac{1}{d}\right)$, the deviation of the metric from flat space-time is non-trivial yet finite.

$$\psi = 1 + \frac{\Phi}{d}, \quad \Phi \sim \mathcal{O}(1), \quad \Rightarrow \lim_{d\to\infty} \psi^{-d} = e^{-\Phi}$$

- Clearly this ansatz has a 'membrane region' of thickness $\mathcal{O}\left(\frac{1}{d}\right)$ around $\psi = 1$

Zooming into the membrane region

We want to blow up the membrane region

We shall do this in the following way.

1. Decompose the membrane into many small patches, each of size $\sim \mathcal{O}\left(\frac{1}{d}\right)$
2. In every patch, scale the coordinates by d
3. Then scale the distance by d^2 and the one-form $d\phi$ by d so that they remain finite as we take $d \to \infty$

   We shall refer to these coordinates as 'patch coordinates', whereas the original x^μ coordinates shall be called 'global coordinates'

In terms of equations:

$$x^\mu = x_0^\mu + \frac{X^\mu}{d}, \;\; ds^2 = \frac{1}{d^2} ds^2_{patch}, \;\; d\phi = \frac{\chi}{d}$$

- Here x_0^μ is some arbitrary point on the membrane.
- $ds^2_{patch} = G_{\mu\nu} dX^\mu dX^\nu$ with $G_{\mu\nu}$ being the finite metric inside the membrane region.

Equations in 'patch coordinates'

Our strategy is as follows:

- Solve Einstein equations in each patch as a series in $\left(\frac{1}{d}\right)$
- Join them smoothly to get the global solution.

The equations in 'patch coordinates' take following form:

$$\frac{1}{2}\nabla_\mu \chi^\mu = \left(\frac{d-1}{d}\right) e^{-\phi} - \frac{\chi^\mu \chi_\mu}{4}$$

$$R_{\mu\nu} = \frac{1}{2}\left(\nabla_\mu \chi_\nu + \nabla_\mu \chi_\nu\right) + \frac{1}{4d}\chi_\mu \chi_\nu$$

$$R_{\mu\nu} = \text{Ricci tensor for scaled metric } G_{\mu\nu}$$

- Note that the equations have non-trivial d dependence.
- We shall choose e^ϕ itself as one coordinate: $S = e^\phi$
- This choice completely fixes the functional form of χ

$$\chi = d\left(\frac{dS}{S}\right)$$

Initial choice of membrane

- We should emphasize that there is no 'real' membrane in the space-time.
- Therefore at this stage we can choose any surface in the membrane region to be our 'membrane'.
  Two choices will differ only at order $\mathcal{O}\left(\frac{1}{d}\right)$
- We shall choose the membrane to be at $\psi = 1$
  This implies that the 'patch coordinates' are defined around some arbitrary point x_0^μ on $\psi = 1$ surface.
- Now we shall describe the method we have used to solve these equations in a series in $\left(\frac{1}{d}\right)$.

- Expand the ansatz in Taylor series around an arbitrary point $\{x_0^\mu\}$ on the membrane
- Scale the coordinate and the metric to zoom inside the membrane region.
- Expansion of ψ and $O_\mu dx^\mu$ in patch coordinates:

Patch coordinates are defined as $x^\mu = x_0^\mu + \frac{X^\mu}{d}$

$$\psi = 1 + \left(\frac{X^\mu}{d}\right)[\partial_\mu \psi(x_0^\mu)] + \mathcal{O}\left(\frac{1}{d}\right), \; O_\mu = O_\mu(x_0^\mu) + \mathcal{O}\left(\frac{1}{d}\right)$$

- Ansatz at leading order:

$$\Rightarrow \psi^{-(d+p)} = e^{-X^\mu[\partial_\mu \psi(x_0^\mu)]} + \mathcal{O}\left(\frac{1}{d}\right)$$

$$\Rightarrow G_{\mu\nu} = \eta_{\mu\nu} + e^{-X^\mu[\partial_\mu \psi(x_0^\mu)]} \; O_\mu(x_0^\mu) O_\nu(x_0^\mu) + \mathcal{O}\left(\frac{1}{d}\right)$$

Step-3: Ansatz as a leading solution

$$G_{\mu\nu} = \eta_{\mu\nu} + e^{-X^\mu[\partial_\mu \psi(x_0^\mu)]}\ O_\mu(x_0^\mu) O_\nu(x_0^\mu) + \mathcal{O}\left(\frac{1}{d}\right)$$

- The leading ansatz is parametrized by two constant one-forms.

$$O_\mu(x_0), \quad d\psi(x_0)$$

- It turns out that the leading ansatz will solve the equations provided

$$O \cdot O|_{x_0^\mu} = 0, \quad O \cdot [d\psi - \chi]|_{x_0^\mu} = 0$$
$$(d\psi) \cdot [d\psi - \chi]|_{x_0^\mu} = 0,$$

Here '$\cdot$' denotes contraction with respect to $\eta_{\mu\nu}$

- Since x_0^μ is any arbitrary point on the membrane, the above three relations must be true everywhere on the $\psi = 1$ surface

Parametrization of O_μ and ψ: The data at zeroth order

- It is convenient to parametrize O and ψ in the following way
- $\psi = 1 + \left(\frac{dB\cdot dS}{SdB\cdot dB}\right) B$ where $B = 0$ is the membrane.
- $O_\mu = e^H \left(-u_\mu + \frac{\partial_\mu \psi}{\sqrt{d\psi\cdot d\psi}}\right), \quad u\cdot u = -1, \quad u\cdot d\psi = 0, \quad u\cdot\chi = 0$
- Advantages of this parametrization:
  1. O and ψ automatically satisfy the three constraints, mentioned in the previous slide.
  2. Since $u \cdot d\psi = 0$, we could identify u_μ as a four-velocity field defined along the surface of the membrane.

Some gauge freedom:

- Two ansatz that differ at order $\mathcal{O}\left(\frac{1}{d}\right)$ will be considered as equivalent starting point.
- Now to specify our ansatz at leading order, only data that are required.
  1. The equation of the membrane (equivalently the one form $d\psi$ along the surface)
  2. Null one-form O_μ defined only on the membrane.
- This does not specify the function ψ or one-form O_μ everywhere in the space-time.

  Away from the surface we could extend them in many different ways.
- But two different extensions will change the ansatz only at order $\mathcal{O}\left(\frac{1}{d}\right)$ and therefore will be equivalent starting point.
- We fix this freedom by imposing that both O_μ and $d\psi$ do not change as we move along the normal to the membrane.

$$\partial^\mu\psi\partial_\mu O_\nu = 0, \quad \partial^\mu\psi\partial_\mu\partial_\nu\psi = 0$$

Step-4: Next to leading order

- Recall that for leading order we expanded ψ upto first order and O_μ at zeroth order.
- Therefore now we have to expand ψ upto 2nd order and O_μ upto 1st order in patch coordinates.
- We shall call the expansion coefficients 'first order data'.
- So the' first order data' consists of 2nd derivatives of ψ and 1st derivatives of O_μ evaluated at x_0^μ.
- Recall that our ansatz has a $SO(p)$ symmetry in the directions perpendicular to $d\psi$, dS and u_μ.
  For explicit computation we use this symmetry to classify the first order data as scalar, vector or tensor according to their transformation property
- At this order all first order data will be treated as constants.

Next to leading order

- Now this expanded ansatz will not solve the equations at order $\mathcal{O}\left(\frac{1}{d}\right)$
- So we have to add corrections to the metric.

$$G_{\mu\nu} = G_{\mu\nu}^{(ansatz)} \left[\text{Expanded till order } \mathcal{O}\left(\frac{1}{d}\right)\right] + \frac{h_{\mu\nu}}{d}$$

- Schematically at order $\mathcal{O}\left(\frac{1}{d}\right)$, the equations (once evaluated on the above metric) will take the following form

$$\text{Differential Operator}[h_{\mu\nu}] = \text{Source}$$

- We have to solve these equations subject to the conditions
  - Regularity inside the membrane region.
  - Exponential fall off as $(\psi - 1) >> \frac{1}{d}$

Event horizon and regularity of the metric

- We could argue that the event horizon of this space-time lies within the membrane region.
- The intuition is as follows.
  - We expect the dynamical space-time will eventually settle down to some stationary solution of Einstein equations.
  - Then event horizon could be determined as the unique null surface that at late time joins the event horizon of this final stationary solution.
  - And we know at large d, the event horizons of these stationary solutions are always within their respective membrane regions.
  - Since dynamics is also always confined within the membrane region, it follows that event horizon will remain inside this region for all time.
  - We could determine its position in a series in $\left(\frac{1}{d}\right)$
- We shall demand that metric is regular everywhere on and outside the event horizon. This condition eventually gives the equation for the membrane.

Solving at next to leading order

Equations in schematic form: Differential Operator$[h_{\mu\nu}] =$ Source

- Naively these are complicated linear PDEs with source, but we know that the leading ansatz varies only along one direction, $\partial_\mu \psi$
- Choose this direction to be one patch coordinate. $R = d \times (\psi - 1)$
- With this choice,
  - Leading order ansatz $\sim$ a 'black-brane' with a translationally invariant horizon at $R = 0$.
  - $\frac{1}{d}$ expansion of the ansatz $\sim$derivative expansion along the directions, tangent to the brane

Very similar to what we had in 'Fluid-Gravity correspondence'.

Equations in schematic form: Differential Operator$[h_{\mu\nu}]$ = Source

- It turns out that in this R coordinate, the Source depends only on R .
- Therefore only R dependent $h_{\mu\nu}$ will be enough to cancel the source.
- This reduces the equations for $h_{\mu\nu}(R)$ from PDE to linear coupled ODEs with source
- Now we do a further coordinate redefinition
  - We choose 3 of the $p+3$ dynamical coordinates in the three special directions associated with the leading black-brane

$$1.\ d\psi(x_0),\quad 2.\ O(x_0)\ \text{ and }\ 3.\ d\psi(x_0)-\chi$$

  - Rest of the p coordinates are chosen in the directions perpendicular to these three special directions.
- It turns out that we could easily decouple the ODEs in these coordinates and solve them explicitly.

Next to leading order: Regularity

- Generically the solution for $h_{\mu\nu}(R)$ has a logarithmic singularity at the point where norm of dR vanishes.

  This is the position of the event horizon at leading order

- It turns out that we could construct a regular solution only if the ‘first order data’ satisfy some equations
- Recall that the first order data consists of the derivatives of $d\psi$ and O_μ in the directions along the membrane.
- These equations (imposed by regularity) involve the extrinsic curvature of the membrane and the velocity field hidden in O_μ. These are the leading ‘equations of motion’ for the dynamic membrane and the velocity field u_μ

The final step

- We solve for $h_{\mu\nu}(R)$ after imposing the equations of motion on the first order data.
- We impose the boundary condition that the solution should have exponential fall-off as $R \to \infty$
- Finally we lift the solution to global form using
  - Inverse transformation from patch coordinate to the global coordinates.
  - Replacing the special point (x_0^μ) by some arbitrary point x^μ (not restricted to the membrane).

    The difference here will be exponentially suppressed outside the membrane region and inside the membrane region it will be of higher order in $\left(\frac{1}{d}\right)$

Final Result: Equation of motion

- The equation of motion takes the following form.

$$U^\mu U^\nu \left(\frac{S}{n_s}\right) K_{\mu\nu} = n_s^2 - 1 \quad \text{Scalar equation}$$

$$U^\mu P^{\nu\alpha} \left(\partial_\mu u_\alpha - \frac{S}{n_s} K_{\mu\alpha}\right) = 0 \quad \text{Vector equation}$$

$S =$ Radius of the large d dimensional sphere

$$n_\mu = \frac{\partial_\mu \psi}{\sqrt{d\psi \cdot d\psi}}, \quad n_s = n \cdot dS, \quad U_\mu dx^\mu = \frac{ds}{n_s} - n_\mu dx^\mu - u_\mu dx^\mu$$

$P_{\mu\nu} =$ Projector perpendicular to ds, n, u

$K_{\mu\nu} =$ extrinsic curvature of the membrane

$u_\mu =$ Velocity along the membrane

- Note the vector eqn has p independent components.
- So we have $p+1$ eqns for total $p+1$ variables, namely the shape of the membrane and p components of the velocity.

Final Result: Metric

$$ds^2 = \left(\eta_{\mu\nu} + \psi^{-(d+p)} O_\mu O_\nu\right) dx^\mu dx^\nu$$

$$+ \left(\frac{\psi^{-d}}{d}\right) (O_\mu dx^\mu) \Big[K_1(x) O_\nu + 2K_2(x)(\partial_\nu \psi - \chi_\nu) + 2K_\alpha(x) P^\alpha_\nu\Big] dx^\nu$$

$$+\mathcal{O}\left(\frac{1}{d^2}\right)$$

- K_1, K_2 and K_α are simple quadratic polynomials in $d(\psi - 1)$
- The coefficients in these polynomials are different linear combination of the first order data

  (i.e., The components of the extrinsic curvature and the derivatives of the velocity field)
- K_1 and K_2 depend on the scalar piece of data whereas K_α depends on the vector data.

  (Scalar and Vector decomposition has been done with respect to the spatial $SO(p)$ symmetry)

Checks

- Matching with known stationary solutions:
  - Schwarzschild solution and Kerr solution are known in arbitrary dimension.
  - We could take large d limit on these solutions and read off ψ and $O_\mu dx^\mu$.
  - We have explicitly checked that they satisfy our equations of motion at leading order.
  - We have also checked that Schwarzschild solution expanded upto order $\mathcal{O}\left(\frac{1}{d}\right)$ match with our answer for the metric.
- Spectrum:
  - We have linearized the equations of motion around ψ_{sch} and O_{sch} and computed the spectrum.
  - We have checked that the spectrum matches with the spectrum of the decoupled modes, computed by Emparan, Suzuki and Tanabe in arXiv:

Ongoing work

- This method could be generalized to Einstein-Maxwell system.
- In this case, apart from ψ and O_μ we have one more scalar function, parametrizing the charge.
- Here also we get the equations of motion from the condition of regularity.
- We have some preliminary results about the equations of motion in this case.
- We found two scalar and one vector equations as expected.

Future Direction

- It would be nice to understand the equations of motion more physically.
  For example, it would be very interesting if we could find some exact equivalence between these equations and the real equations of moving membrane in terms of its extrinsic curvature and surface velocity.
- It would be very interesting to find out how this dynamical horizon radiates gravitational waves, within our set-up of large d expansion.
- It would be interesting to see how horizon area increase theorem leads to entropy production in this this set-up.
- Finally it would be great if we could somehow connect and use this formalism to real astrophysical phenomenon in four dimension.

Thank You