Holographic Entanglement entropy and second law of Black holes

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Plan of the talk:

1. Higher curvature Corrections to GR.

2. Black hole entropy beyond GR. Wald’s formula.

3. Lineaized second law for generic curvature square gravity.

4. Holographic entanglement entropy and second law.

5. Open issues.

General Relativity:

The action is,
$$A = \frac{1}{16\pi}\int_M d^D X \sqrt{-g}\; R\left(g_{ab}, \partial_a g_{bc}, \partial_a \partial_b g_{cd}\right)$$

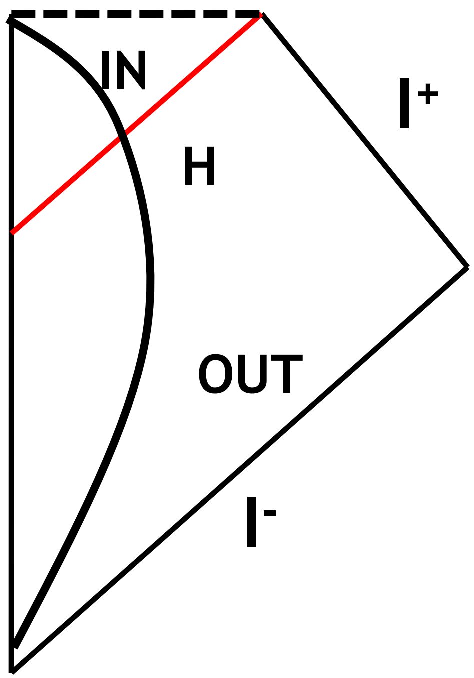
Non-renormalizable, may make sense as an effective theory working perturbatively in the powers of a dimensionless small parameter *G (Energy)*$^{D-2}$

String theory : Cancellation of Weyl anomaly in one loop requires the background to obey Einstein's equation. Higher loops introduce higher curvature terms.

$$A = \frac{1}{16\pi}\int_M d^D X \sqrt{-g}\; \left[R + \alpha\; O(R^2) + \beta\; O(R^3) + \ldots\right]$$

Black hole mechanics with higher curvature terms:

Black hole: Compliment of the past of future null infinity.
Event Horizon: Boundary of the black hole region.



$$B = M - J^{-}\left(I^{+}\right)$$

$$H = \partial B$$

IN and OUT regions are causally disconnected.

Event horizon is a global notion.

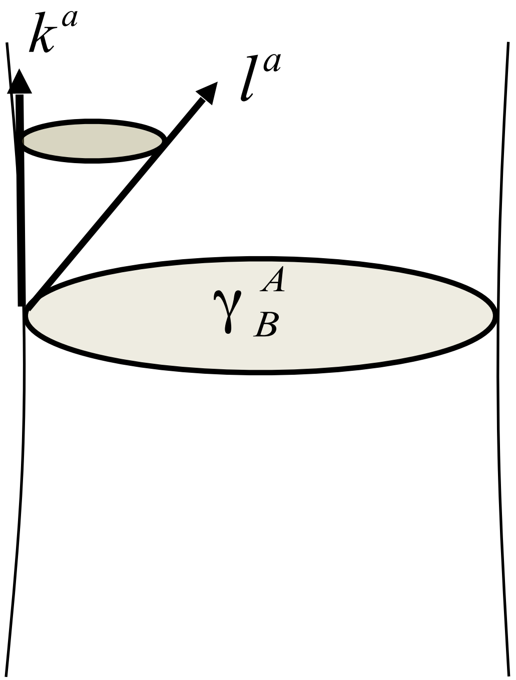
Event horizon is also a null hypersurface generated by null geodesics which is future complete.
Penrose

Horizon basis:

$$\{k^a, l^a, e_A^a\}$$

$$k^a k_a = l^a l_a = 0 \; ; \; k^a l_a = -1$$

$$g_{ab} = -2k_{(a} l_{b)} + \gamma_{ab}$$



Expansion: $$\theta = \frac{1}{\delta A} \frac{d(\delta A)}{d\lambda}$$

Raychaudhuri eq.

$$\frac{d\theta}{d\lambda} = -\frac{\theta^2}{(D-2)} - \sigma_{ab}\sigma^{ab} - R_{ab} k^a k^b$$

Laws of black hole mechanics:

First Law: (Equilibrium State Version)

Wald, Wald and Iyer.

For any diff. invariant Lagrangian, it is possible to show that a stationary black hole obeys.

$$\frac{\kappa}{2\pi}\delta S = \delta M - \Omega_H \delta J$$

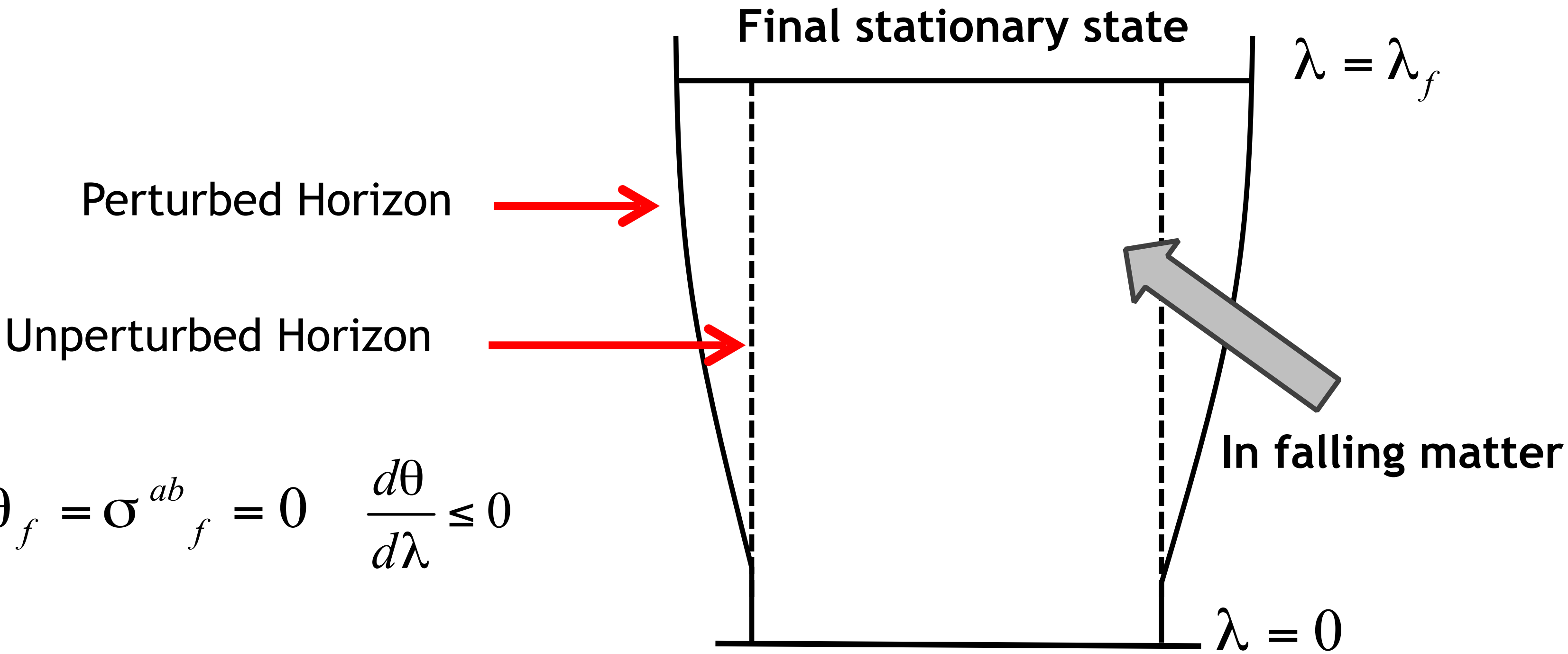
$$S = 2\pi \int_C X^{abcd} \varepsilon_{ab} \varepsilon_{cd} \; ; X^{abcd} = \frac{\partial L}{\partial R_{abcd}};$$

The Wald entropy

In GR: $S = \frac{1}{4}\int_C \sqrt{\sigma}\, d^{D-2}x = \frac{A}{4}$

In general, *entropy is no longer proportional to area* beyond GR.

Linearized **second Law** in GR: $$\frac{d\theta}{d\lambda} \approx -R_{ab}k^a k^b = -T_{ab}k^a k^b \leq 0$$



$$\theta_f = \sigma^{ab}{}_f = 0 \quad \frac{d\theta}{d\lambda} \leq 0$$

Which implies, for every slice prior to future,

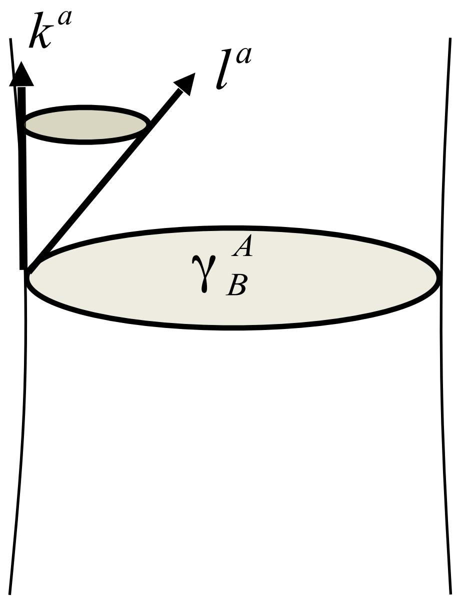
$$\theta(\lambda) \geq 0 \Rightarrow \frac{dA}{d\lambda} \geq 0$$

Extension of second law beyond GR:

1. The derivation of Wald entropy requires stationary Killing Horizon with regular bifurcation surface.

2. Wald entropy expression have several ambiguities which are important when we consider a dynamical black hole.

3. Any proof of second law for generic gravity theories needs the resolution of these ambiguities.



These ambiguities can be expressed in the form:

$$S = S_W + a\ \theta_{(k)}\theta_{(l)} + b\ \sigma^{ab}_{(k)}\sigma_{(l)ab}$$

Find the constant coefficients such that the second law holds.

Consider the most general quadratic curvature theory:

Theory: $$L = \frac{1}{16\pi}\left(R + \alpha R^2 + \beta R_{ab}R^{ab} + \gamma R_{abcd}R^{abcd}\right)$$

Entropy candidate: **?**

Prove that, the entropy always increases for small perturbations as long as NEC holds.

Second law has been proven for f(R) types of theories with Wald entropy:
Kang, Jacobson, Myers

For Lovelock class of gravity: $A = \int d^D X \sqrt{-g} \left[\frac{1}{16\pi} \left(R + \alpha L_{GB} \right) \right];$

$$S_W = \frac{1}{4} \int_C \left[1 + 2\alpha \left(R + 4 R_{ab} k^a l^b + R_{abcd} k^a l^b l^c k^d \right) \right] dA$$

The correct expression of entropy which obeys a local increase law is:

$$S_{JM} = \frac{1}{4} \int_C \left(1 + 2\alpha \ {}^{D-2}R \right) dA$$

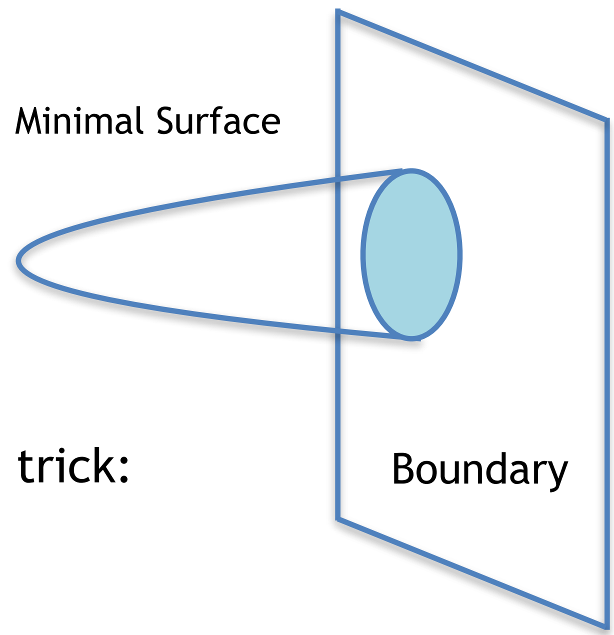
$$S_W - S_{JM} \sim \theta_{(k)} \theta_{(l)} + \sigma_{(k)} \sigma_{(l)}$$

Both Wald and JM entropy obey the first law, but only JM entropy obeys a local increase theorem (at least for linearised perturbation to a Killing horizon)
SS, Wall

A similar proof can be obtained for f(lovelock) gravity:

Holographic entanglement entropy conjecture:
Ryu-Takayanagi

$$S_{CFT} = \frac{A}{4G_{d+2}}$$



This formula now can be proven using the replica trick:
Lewkowycz and Maldacena

When the bulk is the solution of higher curvature gravity:
There are several generalizations of this results….

Fursaev, Patrushev and Solodukhin, Camps,A. Bhattacharyya, A. Kaviraj and A. Sinha, Hung, Myers, Dong,….

Note: Except an apparent similarity with Bekenstein entropy, this formula has a priory nothing to do with black holes!

If there is a stationary black hole in the bulk, then the holographic entropy is related to the black hole entropy.

For time dependent case, additional assumptions are necessary.
Hubeny, Rangamani, Takayanagi, Wall,

In case of a quadratic curvature theory, we obtain:
(Dong)

$$L = \frac{1}{16\pi}\left(R + \alpha R^2 + \beta R_{ab}R^{ab} + \gamma R_{abcd}R^{abcd}\right)$$

$$L = \frac{1}{16\pi}\left(R + \alpha R_{ab}R^{ab}\right) \qquad S = \frac{1}{4}\int_C (1+\rho)\, dA$$

$$\rho_{HEE} = \rho_W + \alpha\ \theta_{(k)}\theta_{(l)} \quad \rho_W = -2\alpha\ R_{ab}k^a l^b$$

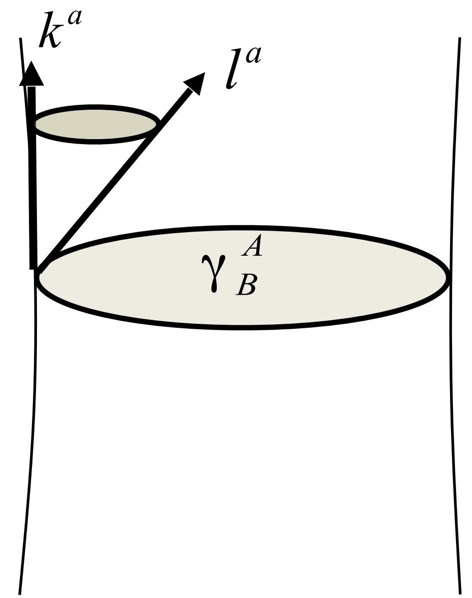
Note that all these expressions coincide in the stationary limit.

Q. Which one obeys the second law?

But all these happening at the boundary:

One could translate the same formula for black hole event horizon.

$$\theta_k = \frac{1}{2}\gamma^{ab} L_k \gamma_{ab}$$



$$\theta_l = \frac{1}{2}\gamma^{ab} L_l \gamma_{ab}$$

For stationary black hole: $\theta_k = 0$

$$\rho = \rho_W$$

Change in entropy: $$\Delta S = \frac{1}{4}\int_C \left(\theta_k + \rho\ \theta_k + \frac{d\rho}{dt} \right)$$

Define: $$\Theta = \theta_k + \rho\ \theta_k + \frac{d\rho}{dt}$$ Note: $\Theta_f = 0$

To calculate the evolution of this generalized expansion make the following assumptions:

1. The perturbation is small. So all changes are of first order in some small parameter.

2. The final state is stationary, so all Lie derivatives of the dynamical fields vanish in the future.

We intent to prove an equation like:

$$\frac{d\Theta}{dt} = -8\pi\ T_{ab}k^a k^b + \mathrm{O}\left(\varepsilon^2\right)$$

The evolution of the generalised expansion is given by:

$$\Theta = \theta_k + \rho \ \theta_k + \frac{d\rho}{dt}$$

$$\frac{d\Theta}{dt} = -8\pi \ T_{ab}k^a k^b + E_{ab}k^a k^b + O\left(\varepsilon^2\right)$$

$$E_{ab}k^a k^b = \nabla_k \nabla_k \rho - \rho R_{kk} + \alpha \ H_{kk}$$

For the correct choice of the entropy expression, all linear order parts of this extra terms should cancel out each other.

We use a dynamical black hole solution.

Assume that the theory has a Vaidya-like solution of the form:

$$ds^2 = -f(r,v)dv^2 + 2dv\,dr + r(v)^2 d\Omega^2$$

This is a non stationary black hole solution and the location of the event horizon is determined by the equation:

$$\frac{dr(v)}{dv} = \frac{f(r,v)}{2}$$

The null generators of the horizon are:

$$k^a = \{2,\, f(r,v),\; 0,\; 0,\ldots\}$$

Case I: Use Wald Entropy $\rho = \rho_W$

$$E_{kk} = \frac{2\alpha\ (D-2)^2}{r^2}\frac{\partial^2 f(r,v)}{\partial v^2} + O\left(\varepsilon^2\right)$$

Coase II: Use HEE $\rho_{HEE} = \rho_W + \alpha\ \theta_{(k)}\theta_{(l)}$

For this Vaidya-like solution:

$$\theta_k = \frac{(D-2)f(r,v)}{r(v)} \qquad \theta_l = -\frac{(D-2)}{2r(v)}$$

$$E_{kk} = O\left(\varepsilon^2\right)$$

And we obtain:

$$\Theta_{HEE} = \theta_k + \rho_{HEE}\ \theta_k + \frac{d\rho_{HEE}}{dt}$$

$$\frac{d\Theta_{HEE}}{dt} = -8\pi\ T_{ab}k^a k^b + O\left(\varepsilon^2\right)$$

Assuming that the final state is again stationary, this gives (with NEC):

$$\Theta_{HEE} > 0$$

HEE obeys a linearized (classical) second law:

HEE obeys a linearized (semi classical) second law:
Bhattacharjee, SS, Wall

Next Q: Is it possible that HEE for this theory obeys an *increase theorem* even beyond small perturbation assumption?

It depends on the signs of the higher order terms.

$$\frac{d\Theta}{d\lambda} = -T_{ab}k^a k^b + \mathrm{O}\left(\varepsilon^2\right)$$

GR result: $$\frac{d\theta}{d\lambda} = -\frac{\theta^2}{(D-2)} - \sigma_{ab}\sigma^{ab} - T_{ab}k^a k^b \leq 0$$

We need to calculate higher order terms:
(SS, in progress)

Conclusions:

1. The naive application of Wald formula does not give correct expression of entropy beyond GR. Except for simple f(R) gravity, validity of second law requires corrections to the Wald's expression in the dynamical regime.

2. For a general quadratic curvature theory, these corrections matches exactly with HEE !

Somehow the holographic principle already contains the validity of the basic laws of black hole thermodynamics.

How?

As of now, I leave this to the competent guys (string theorists) to answer………..

Thanks