Anomalies and Entanglement Entropy

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based on a work with A. Yarom (Technion)
(to appear)

Motivation

Does entanglement entropy observe anomalies in QFT?

- Conformal anomalies in CFT$_d$:

$$S_A = \frac{c_{d-2}}{\epsilon^{d-2}} \cdots + c_0 \log \epsilon + \cdots \ , \qquad c_0 \sim \text{central charges}$$

- Gravitational anomaly in CFT$_2$ with $c_L \neq c_R$
  [Wall 11, Castro-Detournay-Iqbal-Perlmutter 14]

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How about chiral and (mixed-)gravitational anomalies in other dimensions?

Strategy

- Use a thermodynamic partition function W reproducing flavor and (mixed-)gravitational anomalies

- Evaluate W on the n-fold cover $\mathcal{M}_n$ that is an S^1 fibration over an entangling region A

- Calculate the variation of the entanglement entropy with W

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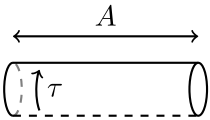

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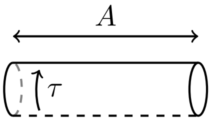

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Outline

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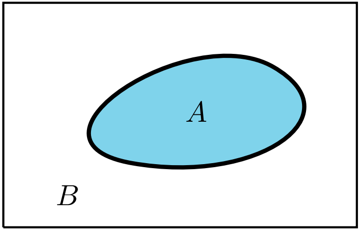
Definition of entanglement entropy

- Divide a system to A and $B = \bar{A}$: $\mathcal{H}_{tot} = \mathcal{H}_A \otimes \mathcal{H}_B$



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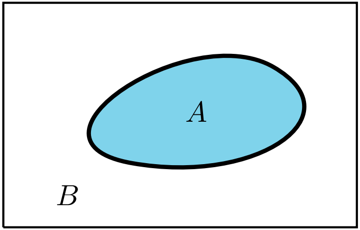

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Replica trick

Entanglement entropy

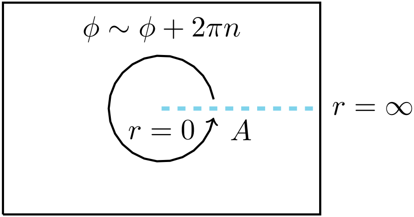
$$S_A = \lim_{n\to 1} \left(\partial_n - 1\right) W_n$$

- $W_n = -\log Z[\mathcal{M}_n]$: the Euclidean partition function
- $\mathcal{M}_n$: the n-fold cover with a deficit angle $2\pi(1-n)$ around the entangling surface $\Sigma \equiv \partial A$

The n-fold cover $\mathcal{M}_n$

- Suppose A is a semi-infinite line in two dimensions

$$A = \{0 \le r < \infty,\ \phi = 0\}$$

$\mathcal{M}_n \quad =$ 

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- CFT_2 with central charges $c_L \neq c_R$ has a gravitational anomaly

$$\nabla_\mu T^{\mu\nu} \sim (c_L - c_R) X^\nu \neq 0$$

- The boost (rotation) of the line A is not a symmetry

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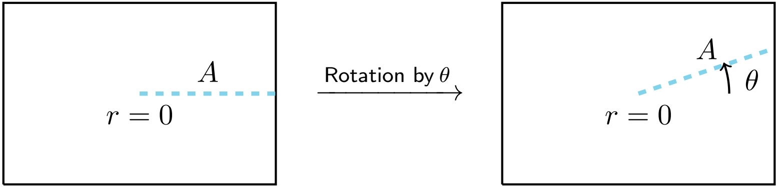
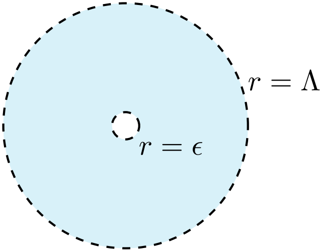


Conformal transformation



- $\mathcal{M}_n$ conformally mapped to a cylinder of circumference $2\pi n$

Cylinder partition function

$$Z[\mathcal{M}_n] = \langle 0| \exp\left[-\ell\, H + i\tilde{\theta}\, P\right] |0\rangle$$

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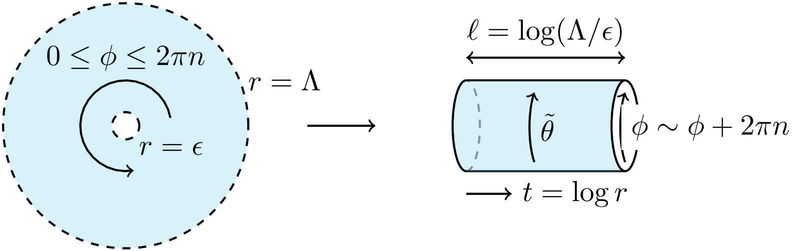

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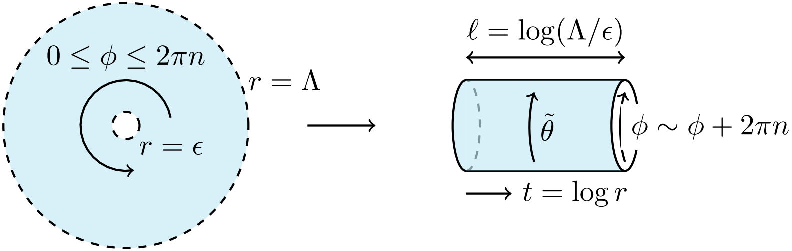

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$$S_A = \frac{c_L + c_R}{12} \log(\Lambda/\epsilon) - i\frac{c_L - c_R}{24}\theta$$

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The imaginary part depends on the rotation angle!

(Similar calculation for an interval in CFT$_2$ [Wall 11, Castro-Detournay-Iqbal-Perlmutter 14])

Chiral anomaly?

- The chiral anomaly in 2d takes the form

$$\nabla_\mu J^\mu \sim c_s\, \epsilon_{\mu\nu} F^{\mu\nu} \ , \qquad c_s \propto k_L - k_R$$

- Can it be seen through EE in not only CFT but also QFT?

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Thermodynamic partition function

- Consider a manifold $\mathcal{M}$ with $U(1)$ isometry

$$ds^2_{\mathcal{M}} = e^{2\sigma(x)}(d\tau + a_i(x)dx^i)^2 + g_{ij}(x)dx^i dx^j$$

- When τ has a period β, the local temperature is

$$T = e^{-\sigma(x)}/\beta \ ,$$

- The thermodynamic partition function [Banerjee-Bhattacharya-Bhattacharyya-Jain-Minwalla 12, Jensen-Kaminski-Kovtun-Meyer-Ritz-Yarom 12, Jain-Sharma 12]

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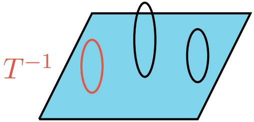

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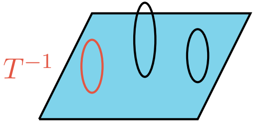

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Anomalous partition function

- Decompose W into gauge-invariant and anomalous parts

$$W = W_{\text{gauge-inv}} + W_{\text{anom}}$$

- W_{anom} can be fixed by the anomaly inflow mechanism [Jensen-Loganayagam-Yarom 12]

$$\delta W_{\text{anom}} = \delta \int_{\hat{\mathcal{M}}} i\, I_{CS}\ , \qquad \partial \hat{\mathcal{M}} = \mathcal{M}$$

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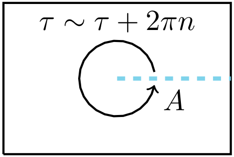

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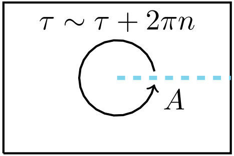

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Anomalous partition function in 2d

- Anomaly polynomial in 2d

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Anomalies in EE

- Rotation $\tau \to \tau + \theta$, $\xi^\tau = \theta$

Gravitational anomaly for a rotation by angle θ

$$\delta_\theta S_A \sim i\, c_g\, \theta$$

- $U(1)$ gauge variation $A_\mu \to A_\mu + \partial_\mu \chi$

Chiral anomaly for a gauge variation

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- Anomaly polynomial:

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Mixed anomaly

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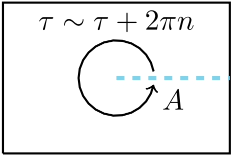
$\mathcal{M}_n =$



$\times \; \mathbb{R}^2_{x,y}$

- Magnetic field $B \equiv F_{xy}$ through the entangling surface

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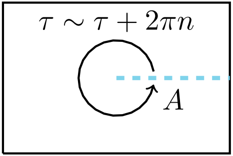

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- Similarly calculation to the mixed anomaly yields

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Summary

- Systematic treatment of anomalies in entanglement entropy using the thermodynamic partition function
- The partition function method does not need conformal symmetry
- Can be applied as well in higher even dimensions (including mixed anomalies), but how about other anomalies such as parity anomaly in odd dimensions?