Field theory as a string theory

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Work with E. Casali & D. Skinner [arXiv:1409.5656] and ongoing with others

Basic idea:

Over last 10+ years, many advances in study of *field theory* scattering amplitudes

Pervasive feature of these advances:

- Apparently unrelated to perturbation theory of space-time actions (e.g., Yang-Mills, Einstein-Hilbert)

Hints at new formulation(s) of certain QFTs

Today...

Discuss one particular example of these ideas:

New information about tree-level S-matrix of a field theory

$$\Rightarrow$$

New (non-linear) formulation of the classical field theory

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Our example: GR (type II SUGRA)

Nice analogy & contrast with similar story in *string theory*

Starting Point: Tree-level S-matrix

CHY formula: tree-level scattering amplitudes of gravitons, B-fields, and dilatons in *any* number of space-time dimensions [Cachazo-He-Yuan] :

$$\mathcal{M}_{n,0} = \int \frac{1}{\mathrm{vol}\,\mathrm{SL}(2,\mathbb{C})} \frac{|z_1 z_2 z_3|}{\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}z_3} \prod_{i=4}^{n} \bar{\delta}\left(\mathrm{d}z_i \sum_{j\neq i} \frac{k_i \cdot k_j}{z_i - z_j}\right) \mathcal{I}_n$$

$\{z_i\} \subset \Sigma \cong \mathbb{CP}^1$, $\{k_i\}$ null momenta,

Integrand $\mathcal{I}_n \in \bigotimes_{i=1}^{n} K_i^2$ compactly encodes kinematic data.

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Integrals over positions $\{z_i\}$ fixed by delta functions, imposing the *scattering equations* [Fairlie-Roberts, Gross-Mende, Witten] :

$$i \in \{4, \ldots, n\}\,, \qquad \sum_{j\neq i} \frac{k_i \cdot k_j}{z_i - z_j} = 0$$

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Indeed! Akin to (holomorphic) complexification of spinning worldline action [Mason-Skinner] :

$$S = \frac{1}{2\pi} \int_{\Sigma} P_\mu \, \bar{\partial} X^\mu + \bar{\psi}_\mu \bar{\partial} \psi^\mu - \bar{\chi} P_\mu \psi^\mu - \chi \, \eta^{\mu\nu} P_\mu \bar{\psi}_\nu - \frac{e}{2} \eta^{\mu\nu} P_\mu P_\nu$$

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Gauge fields $e, \chi, \bar{\chi}$ (non-trivial conformal weights) enforce constraints

$$P^2 = 0 \,, \qquad \psi \cdot P = 0 \,, \qquad \bar{\psi} \cdot P = 0 \,.$$

$$(P^2 = 0 \leftrightarrow \text{scattering eqns})$$

Gauging these constraints + worldsheet gravity gives action and BRST charge:

$$S = \frac{1}{2\pi}\int_\Sigma P_\mu\,\bar\partial X^\mu + \bar\psi_\mu\bar\partial\psi^\mu + b\,\bar\partial c + \tilde b\,\bar\partial\tilde c + \beta\,\bar\partial\gamma + \bar\beta\,\bar\partial\bar\gamma$$

$$Q = \oint c\,T + \frac{\tilde c}{2}\eta^{\mu\nu}P_\mu P_\nu + \bar\gamma P\cdot\psi + \gamma\eta^{\mu\nu}P_\mu\bar\psi_\nu\,.$$

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Vertex operators are in the cohomology of Q.

Fact: vertex operators in 1:1-correspondence with massless spectrum of type II SUGRA in $d = 10$ [TA-Casali-Skinner] .

(No XX OPE $\leftrightarrow$ no massive states)

Example: (fixed) graviton vertex operator

$$V = c\,\tilde{c}\,\delta(\gamma)\,\delta(\bar{\gamma})\,\epsilon_{\mu\nu}\psi^{\mu}\bar{\psi}^{\nu}\,\mathrm{e}^{ik\cdot X}\,,$$

Q-closure $\Leftrightarrow k^2 = 0 = \epsilon \cdot k$ (double contractions w/ P^2, $\psi \cdot P$, $\bar{\psi} \cdot P$)

Note: these are the Einstein field equations, linearized around Minkowski space

So far:

We have a worldsheet model for a flat target space

- Spectrum = type II SUGRA (massless)
- For $\Sigma \cong \mathbb{CP}^1$ correlators = CHY formulae [Mason-Skinner, Ohmori]
- Scattering equations $\Leftrightarrow$ $P^2 = 0$ constraint [TA-Casali-Skinner]
- Naturally extends to higher genus Σ $\Leftrightarrow$ higher-loop amplitudes (integrands) of type II SUGRA (?) [TA-Casali-Skinner, Casali-Tourkine, TA-Casali]

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How do we get to fully non-linear GR/SUGRA on a curved background?

Analogy with string theory

Story so far actually mirrors string theory:

- CHY formula $\sim$ Virasoro-Shapiro amplitude
- Complexified worldline action $\sim$ Polyakov action

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String theory

- Sphere amps $\xrightarrow{\alpha' \to 0}$ SUGRA tree-level S-matrix
- linearized EFEs $\leftrightarrow$ anomalous conformal weights

Worldsheet theory

- Sphere amps $=$ SUGRA tree-level S-matrix
- linearized EFEs $\leftrightarrow$ anomalies w/ currents P^2, $\psi \cdot P$, $\bar{\psi} \cdot P$

How to get non-linear statement in string theory?

- Formulate non-linear sigma model on curved target space
- Demand worldsheet conformal invariance $\rightarrow$ compute β-functions
- Conformal anomaly vanishes as $\alpha' \rightarrow 0 \Leftrightarrow$ non-linear field eqns.

[Callan-Martinec-Perry-Friedan, Banks-Nemeschansky-Sen]

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- Must work perturbatively in α' (background field expansion)
- Higher powers of $\alpha' \leftrightarrow$ higher-curvature corrections to field equations [Gross-Witten, Grisaru-van de Ven-Zanon]

Wish List

Based on contrast with string theory, we want to

- Formulate the worldsheet theory on a curved target space
- Do it so theory is solveable (no background field/perturbative expansion required)
- See non-linear field equations as some anomaly cancellation condition

Put CFT on a curved background $g_{\mu\nu}$:

$$S = \frac{1}{2\pi} \int_{\Sigma} P_{\mu}\, \bar{\partial} X^{\mu} + \bar{\psi}_{\mu}\, \bar{\partial} \psi^{\mu} + \bar{\psi}_{\mu}\, \Gamma^{\mu}_{\nu\rho} \bar{\partial} X^{\rho}\, \psi^{\nu} + \frac{1}{4} R_{\Sigma} \log\left(\sqrt{g}\right)$$

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$$\xrightarrow{\text{field redef'n}} \frac{1}{2\pi}\int_{\Sigma}\Pi_{\mu}\,\bar{\partial}X^{\mu} + \bar{\psi}_{\mu}\,\bar{\partial}\psi^{\mu} + \frac{1}{4}R_{\Sigma}\,\log\left(\sqrt{g}\right)$$

So free worldsheet OPEs! (crucial difference with NLSM)

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Curved background currents: $\psi \cdot P \rightarrow \mathcal{G}$, $\bar{\psi} \cdot P \rightarrow \bar{\mathcal{G}}$, $P^2 \rightarrow \mathcal{H}$

BRST charge

$$Q = \oint c\, T + \frac{\tilde{c}}{2} \mathcal{H} + \bar{\gamma}\, \mathcal{G} + \gamma\, \bar{\mathcal{G}}$$

Curved space currents take form

$$\begin{aligned}
\mathcal{G} &= \psi^{\mu}\Pi_{\mu} + \partial\left(\psi^{\mu}\Gamma^{\nu}_{\nu\mu}\right) \\
\bar{\mathcal{G}} &= g^{\mu\nu}\bar{\psi}_{\nu}\left(\Pi_{\mu} - \Gamma^{\kappa}_{\mu\lambda}\bar{\psi}_{\kappa}\psi^{\lambda}\right) + g^{\mu\nu}\partial\left(\bar{\psi}_{\kappa}\Gamma^{\kappa}_{\mu\nu}\right)
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'Quantum' terms here and in the action ensure space-time and worldsheet diffeo invariance *at quantum level*.

(*Not obvious* – related to properties of curved $\beta\gamma$-systems [Nekrasov, Witten, Frekel-Losev-Nekrasov,...])

Anomaly calculation

$Q^2 = 0$ iff

- $d = 10$ (conformal anomaly)
- Other symmetry currents obey

$$\mathcal{G}(z)\,\mathcal{G}(w) \sim 0 \sim \bar{\mathcal{G}}(z)\,\bar{\mathcal{G}}(w)\,, \qquad \mathcal{G}(z)\,\bar{\mathcal{G}}(w) \sim \frac{\mathcal{H}}{z-w}$$

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Remarkably, the only obstructions are [TA-Casali-Skinner] :

$$R_{\mu\nu} = 0 = R\,,$$

the vacuum Einstein equations!

$B_{\mu\nu}$ and Φ easily incorporated, leading to FEs for the NS-NS sector of type II SUGRA

Summary

So, we've accomplished our initial aim:

CHY formulae

$$\Leftrightarrow$$

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Where's this story going?

Future directions

Short-term:

- Full *super*gravity – not just NS-NS sector
- Other field theories

Medium-term:

- 'Scattering' on non-flat backgrounds (e.g., 'integrable' RR-backgrounds)
- Amplitudes at higher genus (role of modular invariance, no. of solutions to scattering eqns.)

Long-term/Pipe-dream:

- Connection with *string theory*