A bound on chaos

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IAS

June 26, 2015

1. 1306.0622 w/ Shenker
2. 1312.3296 w/ Shenker
3. 1409.8180 w/ D. Roberts, Susskind
4. talks late 2014 Kitaev
5. 1412.6087 w/ Shenker
6. 1503.01409 w/ Maldacena, Shenker

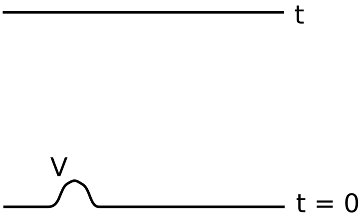
See also work by Leichenauer ('14) and by Polchinski ('15).

state: $|\beta\rangle$

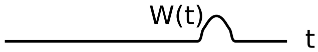
t

$t = 0$

state: $V|\beta\rangle$



state: $W(t)V|\beta\rangle$



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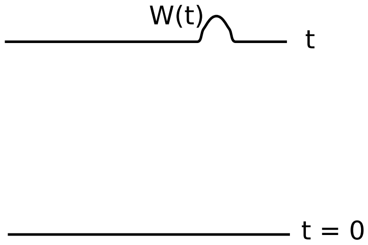
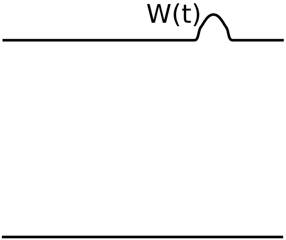
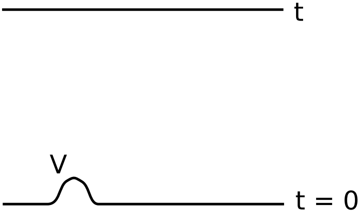
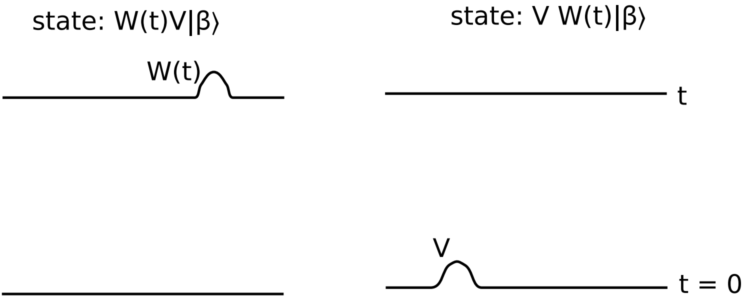


state: W(t)V|β⟩



state: V W(t)|β⟩

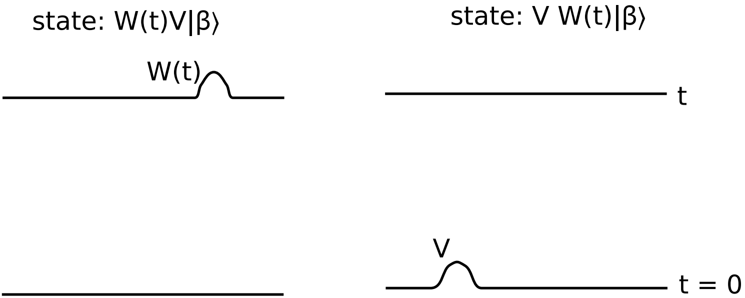




These states are distinguishable, so their overlap

$$\langle \beta | V\, W(t) V\, W(t) | \beta \rangle$$

should be small. The butterfly effect is the statement that the overlap will be small, at sufficiently large t, for any* choice of V, W.



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$$\langle \beta | V(-t/2) W(t/2) V(-t/2) W(t/2) | \beta \rangle$$

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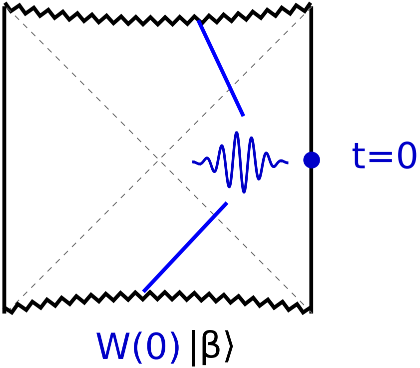
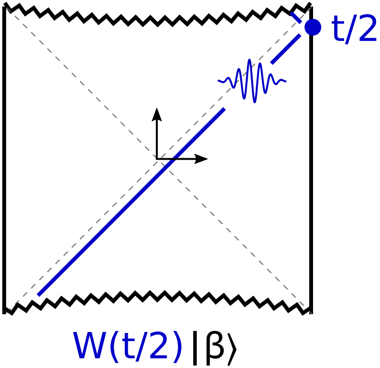
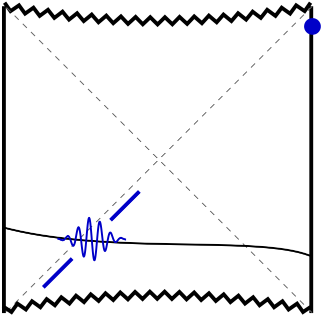


Energy of quantum is order thermal scale β^{-1}.

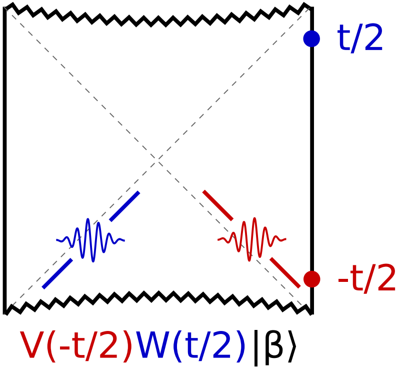


Rapidity of boost $= \frac{2\pi}{\beta}\frac{t}{2}$, so

$$p^+ \sim \beta^{-1} \exp \frac{\pi t}{\beta}$$

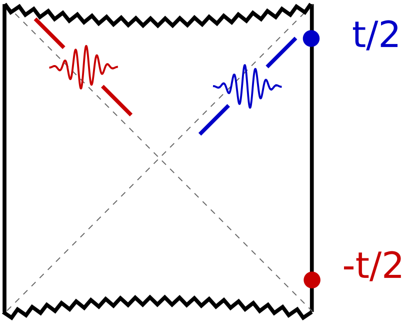


$W(t/2)|\beta\rangle$



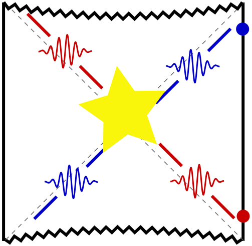
W quantum has $p^+ \sim \beta^{-1} \exp \frac{\pi}{\beta} t$

V quantum has $q^- \sim \beta^{-1} \exp \frac{\pi}{\beta} t$



$$W(t/2)\,V(-t/2)\,|\beta\rangle$$

The inner product is a scattering amplitude, weighted by wave functions, for W, V quanta colliding near the horizon.



Both p^+ and q^- are $\sim \beta^{-1} e^{\pi t/\beta}$, so the characteristic value of the Mandelstam s parameter is

$$s = p^+ q^- \sim \beta^{-2} e^{\frac{2\pi}{\beta} t}.$$

Representing the scattering amplitude as $e^{i\delta(s)}$, we have

$$\langle V\,W(t)V\,W(t)\rangle = \int dp^+ dq^- e^{i\delta(p^+q^-)}|\psi_W(p^+)|^2|\psi_V(q^-)|^2$$

It turns out that the relevant regime is the elastic eikonal regime, where using shock waves, one can calculate ['t Hooft]

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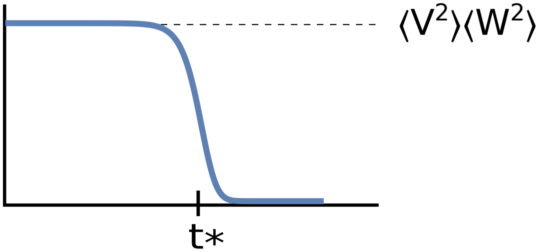
The growth with s is characteristic of the growth with energy of gravitational interactions. Expanding in G_N:

$$\langle ...\rangle = \langle V^2\rangle\langle W^2\rangle \left(1 + \# i G_N e^{\frac{2\pi}{\beta} t} + ...\right)$$

The correction is growing exponentially. We can use this to define a Lyapunov exponent [Kitaev] so that the growth is $e^{\lambda_L t}$. For gravity,

$$\lambda_L = \frac{2\pi}{\beta}.$$

At larger times, the integrand becomes highly oscillatory, and the overlap decays exponentially:



Crossover at

$$t_* = \frac{\beta}{2\pi} \log \frac{1}{G_N} = \frac{\beta}{2\pi} \log N^2$$

[Hayden/Preskill, Sekino/Susskind]

Regge behavior of string theory corrects this analysis:

$$\delta_{string}(s, b) \propto s^{1-\#\alpha'/\ell_{AdS}^2+\dots}$$

which leads to a slower Lyapunov exponent

$$\lambda_L = \frac{2\pi}{\beta}\left(1 - \#\frac{\alpha'}{\ell_{AdS}^2} + \dots\right).$$

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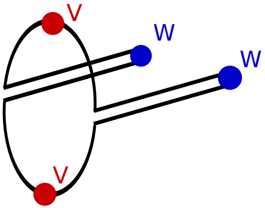
Evidence from [Camanho/Edelstein/Maldacena/Zhiboedov] suggests that $\delta(s, b) \propto s$ is the fastest growth allowed. Could there be a universal bound on λ_L, in the spirit of [Kovtun/Son/Starinets]?

A bound on chaos

$$F(t) \equiv tr[y\, V\, y\, W(t)\, y\, V\, y\, W(t)] \qquad y = \rho(\beta)^{1/4}$$



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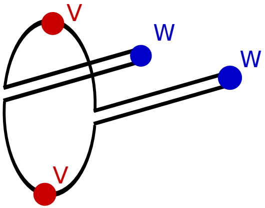


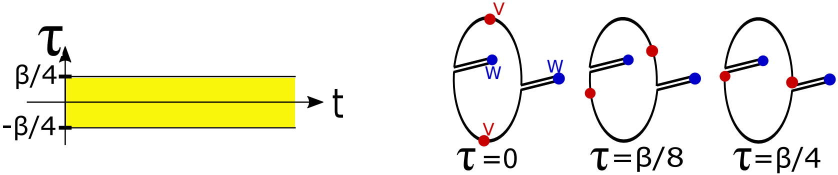
Special case: in a large N theory with $\langle \beta | V\, W | \beta \rangle = 0$,

$$F(t) = F_d + O(N^{-2}) \qquad F_d \equiv tr[y^2 V y^2 V] tr[y^2 W y^2 W].$$

A holographic calculation similar to above gives

$$F(t) = F_d - \frac{1}{N^2} e^{\frac{2\pi}{\beta} t} + O(N^{-4})$$

$F(t + i\tau)$ is analytic in a half-strip:



If we can show $|F| \leq F_d + \epsilon$ everywhere on boundary of strip, then complex analysis (Schwarz-Pick + Phragmen-Lindelof) $\implies$

$$\frac{d}{dt}(F_d - F(t)) \leq \frac{2\pi}{\beta}(F_d - F(t)) + \epsilon.$$

Left boundary: large N factorization implies

$$F(t) = F_d + O(N^{-2}).$$

At the left boundary, the time parameter is not large, so there is no competing parameter and the second term can be bounded by $\epsilon = const/N^2$.

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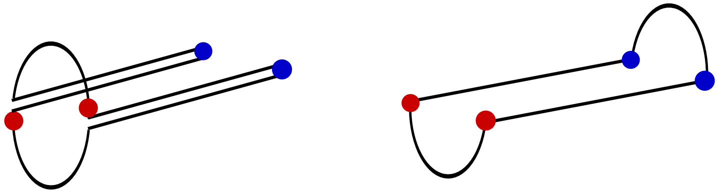
Top/bottom boundaries: here t can be large, and factorization fails. However, Cauchy-Schwarz implies

$$|F(t \pm i\beta/4)| \leq tr[V\, y^2 V\, W(t) y^2 W(t)]$$

We assume RHS approximately factorizes, $RHS \approx F_d \pm const/N^2$, uniform in time. (Reminder: $F_d \equiv tr[y^2 V y^2 V] tr[y^2 W y^2 W]$.)



The result of all of this is that

$$\frac{d}{dt}(F_d - F(t)) \leq \frac{2\pi}{\beta}(F_d - F(t)) + \frac{const}{N^2}.$$

Just to make the point, plug in $F(t) = F_d - N^{-2}e^{\lambda_L t}$. Then

$$\frac{d}{dt}e^{\lambda_L t} \leq \frac{2\pi}{\beta}e^{\lambda_L t} + const.$$

As soon as t is more than a few times β, the constant is unimportant and we get

$$\lambda_L \leq \frac{2\pi}{\beta}.$$

Summary

- Chaos can be diagnosed using correlators like $\langle V\, W(t) V\, W(t)\rangle$.
- In AdS/CFT, these are computed by high energy scattering near a black hole horizon.
- The initial exponential growth of the influence of chaos on this correlator can be used to define a Lyapunov exponent λ_L. There is a universal bound on the Lyapunov exponent

$$\lambda_L \leq \frac{2\pi k_B T}{\hbar},$$

saturated by systems dual to gravity.

Thank you!