HIGHER SPIN CORRECTIONS TO ENTANGLEMENT ENTROPY

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INTRODUCTION AND MOTIVATION

- Consider a field theory which admits higher spin symmetry.

Introduce chemical potentials conjugate to the conserved higher spin currents.

Can one systematically obtain corrections to the partition function, entanglement entropy as an expansion in the chemical potentials?

• Why is this question interesting ?

∗ Higher spin theories are toy examples of the large gauge degree of freedom present in string theory.

∗Holographic duals to higher spin theories in $d = 2, 3$ have been constructed.
The answer to the question provides more insight to the thermodynamics/entanglement entropy of the dual theories.

∗ Thermodynamics of field theories deformed by chemical potentials corresponding to $U(1)$ currents are well studied.

Can we generalize this study for deformations of chemical potentials corresponding to higher spin currents?

• Consider conformal field theories in $d = 2$ with which admits a conserved spin 3 current j_{μ_1,μ_2,μ_3} .
It is traceless and symmetric.
There are only 2 independent components.

$$j_{000}, \qquad j_{111}$$

This can be organized as

$$\begin{aligned} W(z) &= W_{zzz} = -\frac{1}{4}(j_{000} + i\, j_{111}), \\ \bar{W}(\bar{z}) &= W_{\bar{z}\bar{z}\bar{z}} = -\frac{1}{4}(j_{000} - i\, j_{111}) \end{aligned}$$

• Deforming the theory by a chemical potential corresponding to the charge

$$Q = \int dt \int dx \, j_{000}$$

is equivalent to adding the term

$$\delta S = \mu \left(\int d^2 z W(z) + \int d^2 z \bar{W}(\bar{z}) \right) = -\frac{1}{2}\mu \int d^2 z \, j_{000}$$

in the Lagrangian with the understanding we perform the spatial integration in $d^2 z$ first.

$$\int d^2 z = \int dt \int dx$$

• Therefore we need to study conformal field theories perturbed by

$$\delta S = \mu\left(\int d^2 z W(z) + \int d^2 z \bar{W}(\bar{z})\right)$$

• The perturbation theory is more special than conventional conformal perturbation theory where one considers operators of dimensions $(h, \bar{h})$.

Here the operator is holomorphic.

• The operator W has dimension 3, therefore the perturbation is irrelevant.

We need to carefully choose a prescription to ensures the answers are physical.

• The resulting answers must coincide with deforming the Hamiltonian of the theory

$$H \to H + \mu \int dx\, j_{000} = H + \mu Q$$

We will see how to achieve this as we proceed.

• Our main focus will be to use the perturbation theory developed

to evaluate corrections to the single interval entanglement entropy when the theory is deformed by a spin-3 current.

• To put this in context let us recall what is known about single interval entanglement entropy.

SINGLE INTERVAL

ENTANGLEMENT ENTROPY

• Consider a $1+1$ dimensional conformal field theory with central charge c on a real line.

and at finite temperature . β

Let the sub space A be an interval of length Δ.

Then EE is given by

$$S_A = \frac{c}{3} \ln \left(\frac{\beta}{\pi\epsilon} \sinh(\frac{\pi\Delta}{\beta}) \right)$$

• There is a similar expression for the Rényi entropy eg.

$$S_A = \frac{c}{6}(n + \frac{1}{n}) \left(\frac{\beta}{\pi\epsilon} \sinh(\frac{\pi\Delta}{\beta}) \right)$$

• The replica trick relates the reduced density matrix involved in the the Rényi entropy to the partition function of the CFT on n copies of the plane branched along the interval A.

$$\mathrm{Tr}(\rho_A^n) = \frac{Z_n[A]}{Z^n}$$

• This partition function can be evaluated using the uniformization map from the n sheeted branched surface to the plane.

$$w = \left(\frac{z - y_1}{z - y_2}\right)^{\frac{1}{n}}$$

z is the co-ordinate on the multi-sheeted Riemann surface R^n,

w is the co-ordinate on the complex plane.

- It can be shown that

$$\mathrm{Tr}(\rho_A^n) = \frac{Z_n[A]}{Z^n}$$

is equal to the 2 point function of branch point twist operator τ_n of conformal dimensions

$$d_n = \bar{d}_n = \frac{c}{24}\left(n - \frac{1}{n}\right)$$

inserted at the end points of the interval.

Result:

• For any CFT which admits a spin 3 current we show that the entanglement entropy for a single interval is corrected

$$S_A(\beta,\mu,\Delta) = \frac{c}{3}\ln\left(\frac{\beta}{\epsilon}\sinh\frac{\pi\Delta}{\beta}\right) + \frac{\mu^2}{\beta^2} S_A^{(2)}(\beta,\Delta) + O(\mu^4)$$

We evaluate $S_A^{(2)}(\beta,\Delta)$ and prove it is universal.

• Contrast this result with situation when the system is on a ring of radius R and at finite temperature β.

The CFT is on a torus. The EE of an interval of size Δ

$$S_A = \frac{c}{3} \ln \left(\frac{\beta}{\epsilon} \sinh \frac{\pi \Delta}{\beta} \right) + \exp \left(-\frac{R}{\beta} \right) f(\Delta/\beta)$$

These are finite size corrections.

Finite size corrections capture information of the spectrum of the theory

and are not universal.

What can we use this result for ?

• There is a proposal by deBoer, Jottar and Ammon, Castro, Iqbal 2013 to evaluate entanglement entropy in holographic higher spin theories generalizing the proposal of Ryu, Takayanagi 2006.

•The proposal uses the Chern-Simons formulation of the higher spin theory.

The EE is written in terms of a Wilson line in the bulk joining the end points of the entangling interval.

A formula for the EE $S_A(\mu, \beta, \lambda = 3)$,
($\lambda = 3$ theory contains only spins 3 and spin 2),
has been written down resulting from this proposal.

• The CFT calculation and the fact the $O(\mu^2)$ term is universal will enable a precise check on the holographic proposal for entanglement entropy in these theories.

We have two tasks:

• Develop conformal perturbation theory for a theory deformed by the holomorphic higher spin current.

• Use it to evaluate corrections to entanglement entropy.

Simple theories we have to explicitly preform/check our tasks:

• The theory of N complex free fermions realizes a $\mathcal{W}_{1+\infty}$ algebra.

After removing the over all $U(1)$ it realizes the $\mathcal{W}_{\infty}[\lambda = 0]$ algebra.

It admits a spin-3 current.

$$J = \psi^*_a \psi^a \,, \qquad\qquad T = \frac{1}{2}\left(\partial\bar{\psi}^*_a \psi^a - \psi^*_a \partial\psi^a\right) \,,$$

$$W = i\frac{\sqrt{5}}{12\pi}\left(\partial^2\psi^*_a\psi^a - 4\,\partial\psi^*_a\partial\psi^a + \psi^*_a\partial^2\psi^a\right) .$$

- For a theory of complex free fermions,

there is an explicit construction of the twist operator in terms of the bosonized fields

$$\psi = \exp(i\varphi)$$

Therefore we can use correlations function of twist operators to evaluate entanglement entropy and its corrections.

- The theory of N free bosons realizes the $\mathcal{W}_\infty[\lambda = 1]$ algebra.

$$
\begin{aligned}
T(z) &= -\partial X_a \partial \bar{X}^a, \\
W(z) &= \sqrt{\frac{5}{12\pi^2}}(\partial^2 \bar{X}_i \partial X_i - \partial \bar{X}_i \partial^2 X_i).
\end{aligned}
$$

- For a theory of complex free bosons,

though there is no explicit realization of the twist fields, all correlators involving it can be evaluated using OPEs and knowledge of the singularities.

Dixon, Friedan, Martinec, Shenker 1987

HIGHER SPIN CORRECTIONS:

PARTITION FUNCTION

• To order μ^2, the deformed CFT partition function is given in conformal perturbation theory as

$$
\begin{aligned}
Z = \quad & Z^{(0)}_{\mathrm{CFT}} \times \\
& \Big(1 - \mu \int d^2 z \, \langle W \rangle_{\mathrm{CFT}^{(0)}} \\
& + \tfrac{1}{2}\mu^2 \int d^2 z_1 \int d^2 z_2 \, \langle W(z_1)\, W(z_2) \rangle_{\mathrm{CFT}^{(0)}} \\
& + \ldots \mathrm{h.c.}\Big)
\end{aligned}
$$

The CFT is on the infinite spatial line, and at temperature β.

- The one point function of W on the cylinder vanishes:

It is a conformal primary.

The one point function vanishes on the plane: translational invariance,

by conformal transformation, it vanishes on the cylinder.

- So we examine

$$\langle W(z_1)\, W(z_2)\rangle_{\mathbb{R}\times S^1_\beta} \;=\; \mathcal{N}\, \frac{\pi^6}{\beta^6 \sinh^6\left(\frac{\pi}{\beta}(z_1 - z_2)\right)}\,.$$

with

$$\mathcal{N} \;=\; (2\pi i)^{-2}\, \frac{10\, c}{3}\,.$$

• The first correction to the partition function appears at order μ^2:

$$\frac{\ln Z}{L} = \frac{1}{L}\Big[\ln Z^{(0)}_{\rm CFT} +$$
$$+\frac{1}{2}\mu^2\int_0^\beta d\tau_2\int_{-\infty}^{\infty} d\sigma_2\int_0^\beta d\tau_1\int_{-\infty}^{\infty} d\sigma_1\,\frac{\mathcal{N}\pi^6}{\beta^6\sinh^6\left(\frac{\pi}{\beta}(z_1-z_2)\right)}$$
$$+\,\mathrm{h.c.}\ldots\Big]\,,$$
$$z_{1,2}\equiv\sigma_{1,2}+i\tau_{1,2}\,.$$

L, the size of the system which is being taken to infinity.

The integral can be analytically obtained.

• We perform the integral first on the spatial direction and then along the compact direction.

Integration over (σ_1, τ_1) yields a constant independent of (σ_2, τ_2).
Hence the second integration gives rise to an extensive scaling with the volume of the cylinder:

Lets illustrate this with a simple integral of the same type.

$$\int_0^\beta d\tau \int_{-\infty}^{\infty} d\sigma \frac{1}{\sinh^2(\sigma + i\tau - a)} = -\int_0^\beta d\tau \coth(\sigma + i\tau - a)|_{-\infty}^{\infty} = -2\beta$$

The result for the partition function is

$$\frac{1}{L}\ln Z = \frac{\pi\, c}{6}\,\beta^{-1} + \frac{8\pi^3\, c}{9}\,\mu^2\,\beta^{-3} + \ldots.$$

There is a correction to Cardy's universal result for the high temperature partition function.

Note that this correction is also universal, independent of λ.

• The $O(\mu^2)$ correction agrees with the leading the first correction to the high temperature partition function of the spin-three black hole.
Gutperle, Kraus (2011)

• It also agrees with the CFT computations of Gaberdiel, Hartman, Jin (2012) and Kraus, Perlmutter (2011).

• The work of Kraus, Perlmutter took advantage of the free field realization of $\mathcal{W}_\infty[\lambda]$ symmetry for $\lambda = 0, 1$ and computed the exact thermal partition function.

• Gaberdiel, Hartman Jin utilized the modular transformation properties of torus amplitudes when the Hamiltonian is deformed by the higher spin charge, taking their high temperature limits and evaluating the resulting integrals.

• We have employed conformal perturbation theory, directly used the two-point correlator of W-currents, and calculated its integral on the cylinder $\mathbb{R} \times S^1_\beta$.

• This strategy has been pushed further to obtain the $\mathcal{O}(\mu^4)$ correction using the 4-point function of W-currents on the cylinder.
Jiang Long (2014).

CORRECTIONS TO ENTANGLEMENT ENTROPY

• We are now ready to set up the the perturbation expansion of the Rényi entropy, EE in terms of the chemical potential μ.

• We need to evaluate

$$\mathrm{Tr}(\rho_A^n) = \frac{1}{Z^n(\mu)} \int_{R^n} [d\varphi_i] \exp\left(- \sum_{i=1}^{n} S[\varphi_i, \mu] \right)$$

where

$$S[\varphi, \mu] = S^{(0)}[\varphi] + \mu \int d^2 z (W(z) + \bar{W}(\bar{z}))$$

Note that there is a spin 3 current perturbation for each copy of the sheet.

• Expanding in μ we need to evaluate

$$\frac{1}{Z^n(\mu)}\left(\langle\tau_n(y_1)\bar{\tau}_n(y_2)\rangle_{(0)}+\mu\langle\tau_n(y_1)\bar{\tau}_n(y_2)\int d^2(W(z)+\bar{W}(\bar{z}))\rangle_{(0)}\right.$$
$$\left.+\frac{1}{2}\mu^2\langle\tau_n(y_1)\bar{\tau}_n(y_2)\left[\int d^2z(W(z)+\bar{W}(\bar{z}))\right]^2\rangle_{(0)}\right)$$

These correlators need to be evaluated on the cylinder and the integrals need to be performed.

At $O(\mu^2)$ there is a double integral to be performed.

• The linear term in μ vanishes. This can be seen by using the uniformization map

$$
\begin{aligned}
\langle \tau_n \bar{\tau}_n W \rangle &= \langle W \rangle_{R^n}, \\
&= \left(\frac{\partial w}{\partial z} \right)^3 \langle W \rangle_{w-\mathrm{plane}} \\
&= 0
\end{aligned}
$$

The last step uses translational invariance in the w-plane.

This result can be explicitly checked either using the free fermion or the free boson realization.

• The holomorphic and the anti-holomorphic currents are decoupled,
the same argument can be used to show the correlators involving cross terms between holomorphic and anti-holomorphic spin-3 currents at $O(\mu^2)$ vanish.

• Therefore the only correlator left to evaluate is

$$\langle \tau_n(y_1)\bar{\tau}_n(y_2)W(z_1)W(z_2)\rangle$$

• We will now present the outline of the calculation of this correlator for any CFT which admits a W current and show it is universal.

- By conformal invariance

$$\langle \tau_n(y_1)\bar{\tau}_n(y_2)W(z_1)W(z_2)\rangle = -\frac{5c}{6\pi^2}\frac{F(x)}{z_{12}^6|y_{12}|^{4d_n}}$$

where

$$x = \frac{(z_1-y_2)(z_2-y_1)}{(z_1-y_1)(z_2-y_2)},$$

$$z_{12} = z_1 - z_2, \qquad y_{12} = y_1 - y_2 = \Delta$$

Determining $F(x)$

- From the fact the currents are holomorphic we have $F(x)$ is a holomorphic function in x.

- $z_1 \to z_2$ is a symmetry of the correlator, therefore

$$F(x) = F\left(\frac{1}{x}\right)$$

- We fix the normalization so that $z_1 \to z_2$ limit

$$F(x = 1) = 1$$

- Look at limits

$$z_1 \to y_1 \, , z_2 \to y_2 \, , \qquad x \to \infty$$

or the limits

$$z_2 \to y_2 \, , z_2 \to y_1 \, , \qquad x \to 0$$

In these limits the W current come close to the twist operator.

- We need the information of the following OPE

$$W(z)\tau_n(y) = \frac{\mathcal{O}}{(z-y)^M}$$

Note that $\mathcal{O}$ belongs to the twisted sector.

By definition τ_n creates the ground state of the twisted sector.

But $\mathcal{O}$ cannot be τ_n since $\langle W(z)\tau_n\bar{\tau}_n\rangle = 0$.

Therefore

$$\dim \mathrm{O} > \dim \tau_n = d_n$$

By dimensional analysis

$$M = 3 + d_n - \dim \mathrm{O} < 3$$

• Note that the W current involves the sum over the W currents over all the copies.

Therefore

$$W\tau_n = \sum_i W_i \tau_n$$

should not have branch cuts.

The insertion of the twist operator τ_n just cyclically permutes the copy label i and W is invariant under this.

Therefore M is an integer.

- This together with the fact $F(x) = F(1/x)$ results in

$$F(x) = a_0 + a_1(x + \frac{1}{x}) + a_2(x^2 + \frac{1}{x^2})$$

such that $a_0 + 2a_1 + 2a_2 = 1$.

We can re-write this by introducing the modified cross ratio

$$\begin{aligned} \eta &= x + \frac{1}{x} - 2, \\ &= \frac{(z_1 - z_2)^2 (y_1 - y_2)^2}{(z_1 - y_1)(z_1 - y_2)(z_2 - y_1)(z_2 - y_2)} \end{aligned}$$

In terms of η we have

$$F(\eta) = 1 + f_1 \eta + f_2 \eta^2$$

- To fix f_1, f_2 use the $W(z_1)W(z_2)$ OPE.

$$
\begin{aligned}
-\frac{1}{\pi^2} W(z_1)W(z_2) \quad = \quad & \frac{5c}{6z_{12}^6} + \frac{5T(z_2)}{z_{12}^4} + \frac{5T'(z_2)}{z_{12}^3} \\
& + \frac{1}{z_{12}^2}\left(4U^{(4)}(z_2) + \frac{16\Lambda^{(4)}(z_2)}{c+\frac{22}{5}} + \frac{3}{4}T''(z_2)\right) \\
& \frac{1}{z_{12}}\left(2\partial U(z_2) + \frac{8}{c+\frac{22}{5}}\partial\Lambda^{(4)} + \frac{1}{6}T'''(z_2)\right)
\end{aligned}
$$

Where

$$
\Lambda^{(4)} =: TT : -\frac{3}{10}\partial^2 T
$$

and $U^{(4)}$ is the spin 4 current, it is a conformal primary.

The coefficient of this term contains λ dependence.

• To fix f_1, f_2 substitute the OPE into the correlator.

then one obtains an expansion in $(z_1 - z_2)$ with coefficients involving the three point functions

$$\langle U^{(4)} \tau_n \bar{\tau}_n \rangle = \langle \partial U^{(4)} \tau_n \bar{\tau}_n \rangle = 0,$$

$$\langle T \tau_n \bar{\tau}_n \rangle, \quad \langle \Lambda^{(4)} \tau_n \bar{\tau}_n \rangle, \quad \ldots$$

All of which can be evaluated by using the uniformization map.

• We can then perform the expansion in $(z_1 - z_2)$ in RHS of the expression

$$\begin{aligned}\langle \tau_n(y_1)\bar{\tau}_n(y_2)W(z_1)W(z_2)\rangle &= -\frac{5c}{6\pi^2}\frac{F(x)}{z_{12}^6|y_{12}|^{4d_n}}, \\ &= -\frac{5c}{6\pi^2}\frac{1}{z_{12}^6|y_{12}|^{4d_n}}\left(1 + f_1\eta + f_2\eta^2\right)\end{aligned}$$

Matching the coefficients fixes f_1, f_2.

In fact there are 4 equations but only 2 parameters.

But there exists a unique solution

$$f_1 = \frac{n^2-1}{4n^2}, \qquad f_2 = \frac{(n^2-1)^2}{120n^4} - \frac{n^2-1}{40n^4}$$

• This determines the required correlation function of the spin 3 currents in presence of the twists.

• We have performed the following cross-checks on the result for the 4-point function.

$*$ The 4-point function agrees with that obtained for the free field theories.

$*$ Using the same procedure one can obtain the correlator of the stress tensor in presence of the twists $\langle TT\tau_n\bar{\tau}_n\rangle$.

We can also obtain this correlator using the conformal Ward identity on the 3-point function $\langle T\tau_n\bar{\tau}_n\rangle$.

Both results agree.

* Finally one can also evaluate the 4 point function $\langle W W \tau_n \bar{\tau}_n \rangle$ using the uniformization map.

$$\begin{aligned}\langle W W \tau_n \bar{\tau}_n \rangle &= \sum_{i,j} \langle W_i W_j \tau_n \bar{\tau}_n \rangle, \\ &= \sum_{i,j} \langle W_i W_j \rangle_{R^n}\end{aligned}$$

Using the uniformization map, this reduces to the two point function of the spin 3 currents inserted at the images of the respective Riemann sheet in the w-plane.

Then one needs to sum over the images.

The resultant correlator agrees with the OPE method.

• Now that one has the expression for the correlator, we can go over to the cylinder using the conformal map

$$u = \frac{\beta}{2\pi} \ln z$$

The we obtain

$$\frac{\mathrm{Tr}\rho_{\mathrm{A}}^{\mathrm{n}}}{1-n} = \frac{c(1+n)}{6n} \ln\left(\sinh\frac{\pi\Delta}{\beta}\right) + S_n^{(2)}$$

where

$$S_n^{(2)} = \frac{5\pi^4 c\mu^2}{6\beta^6(n-1)} \int d^2u_1 d^2u_2 \frac{f_1\eta_\beta + f_2\eta_\beta^2}{\sinh^6(\frac{\pi u_{12}}{\beta})}$$

$$\eta_\beta = \frac{\sinh^2[\frac{\pi}{\beta}(u_1-u_2)]\sinh^2\frac{\pi\Delta}{\beta}}{\sinh\frac{\pi}{\beta}(u_1-y_1)\sinh\frac{\pi}{\beta}(u_1-y_2)\sinh\frac{\pi}{\beta}(u_2-y_1)\sinh\frac{\pi}{\beta}(u_2-y}$$

• The integrals are done on the cylinder using the following prescription.

Integrate along the spatial direction infinite direction first.

Separate possible coincident point by insertion of an $i\epsilon$.

• One can show, the result depends only on the

Residue of the double poles of the integrand

Residue of the simple poles and its location of the integrand.

• The result for the correction to the Rényi entropy to $O(\mu^2)$ is given by

$$S_n^{(2)} = \frac{5c\mu^2 n}{6\pi^2(n-1)}(f_1\mathcal{I}_1 + f_2\mathcal{I}_2)$$

$$\mathcal{I}_1(\Delta) = \frac{4\pi^4}{3\beta^2}\left(\frac{4\pi\Delta}{\beta}\coth\left(\frac{\pi\Delta}{\beta}\right) - 1\right) +$$
$$+\frac{4\pi^4}{\beta^2}\sinh^{-2}\left(\frac{\pi\Delta}{\beta}\right)\left\{\left(1-\frac{\pi\Delta}{\beta}\coth\left(\frac{\pi\Delta}{\beta}\right)\right)^2 - \left(\frac{\pi\Delta}{\beta}\right)^2\right\}$$

and

$$\mathcal{I}_2(\Delta) = \frac{8\pi^4}{\beta^2}\left(5 - \frac{4\pi\Delta}{\beta}\coth\left(\frac{\pi\Delta}{\beta}\right)\right) +$$
$$+\frac{72\pi^4}{\beta^2}\sinh^{-2}\left(\frac{\pi\Delta}{\beta}\right)\left\{\left(1-\frac{\pi\Delta}{\beta}\coth\left(\frac{\pi\Delta}{\beta}\right)\right)^2 - \frac{1}{9}\left(\frac{\pi\Delta}{\beta}\right)^2\right\}.$$

• Taking the limit $n \to 1$ results in the entanglement entropy.

If one takes further the limit $\frac{\Delta}{\beta} >> 1$, then one obtains the correction to the expected correction to the thermal entropy as evaluated by

Gaberdiel, Hartman, Jin 2012.

• The result agrees with the $O(\mu^2)$ correction evaluated using the Wilson line proposal of deBoer, Jottar.

• The $O(\mu^4)$ correction has been evaluated by these methods. Jiang Long (2014)

MORE CONFORMAL

PERTURBATION THEORY

• Consider the theory of M free fermions on the torus.
Introduce the following chemical potentials.

∗ $U(1)$ chemical potential

$$S \to S + \mu \int d^2z(J(z) + \bar{J}(z))$$

∗ The stress tensor

$$S \to S + \mu \int d^2z(T(z) + \bar{T}(z))$$

∗ The spin-3 chemical potential

$$S \to S + \mu \int d^2z(W(z) + \bar{W}(z))$$

• The spin 3-perturbation for the free fermion theory on the torus has been considered earlier in the context of large N 2-dimensional Yang-Mills seen as a string theory.
Douglas (1993), Dijkgraaf (1997) .

• Evaluate the corrections to the

partition function, single interval Rényi entropy and entanglement entropy

in each of the cases using conformal perturbation theory.

• The results for the partition function in each case will enable us to test the prescription with the direct calculation in the Hamiltonian picture.

$$\ln \mathcal{Z} \sim 2M \sum_{m=0}^{\infty} \left[\ln(1 + e^{2\pi i \tau m + f(m)}) + \ln((1 + e^{2\pi i \tau m - f(m)}) \right]$$

$*$ For the $U(1)$ chemical potential

$$f(m) \sim \mu$$

The sum can can be done exactly.

$$\mathcal{Z} = \left| \frac{\vartheta_2(\pi \mu \beta | \tau)}{\eta(\tau)} \right|^{2M}$$

$\ast$ For the deformation by stress tensor

$$f(m) \sim \mu m$$

It is just a shift of τ.

For the deformation by higher spin chemical potential

$$f(m) \sim \mu m^2$$

The sum is not exactly doable.

There exists a formula for each term in the perturbative expansion in μ.

The term of order μ^n is a quasi-modular form of weight $3n$.
M. Kaneko and D. Zagier (1995)

• We can compare the results from conformal perturbation theory developed to these exact results.

• In each of the case we obtain agreement.

• We will detail the results for the deformation with higher spin chemical potential.

- The correction to the partition function:

$$\ln \mathcal{Z} = \ln \mathcal{Z}_{\mathrm{CFT}} - \beta F^{(2)} + \ldots$$
$$-\beta F^{(2)} = \tfrac{1}{2}\mu^2 \int_{\mathbb{T}^2} d^2 z_1 \int_{\mathbb{T}^2} d^2 z_2 \, \langle W(z_1) W(z_2) \rangle + \mathrm{h.c.}$$

- Evaluating the two point function on the torus and performing the integrals on the torus using the prescription results in:

$$
\begin{aligned}
-\beta F^{(2)} \;=&\\
&80 M\mu^2\beta^2 \frac{\pi^4}{L^4}\left[\frac{1}{2^5\cdot 3^4\cdot 5}\left(10E_2^3 - 6E_4E_2 - 4E_6\right)\right.\\
&\qquad - \frac{1}{2^5\cdot 3^2}(E_4 - E_2^2)\frac{\vartheta_\nu''}{\vartheta_\nu}\\
&\qquad \left. - \frac{1}{2^6\cdot 3^2}\left(\frac{\vartheta_\nu^{(6)}}{\vartheta_\nu} + 2E_2\frac{\vartheta_\nu^{(4)}}{\vartheta_\nu} + E_2^2\frac{\vartheta_\nu''}{\vartheta_\nu}\right)\right].\\
=&\; 160 M\mu^2\beta^2\frac{\pi^4}{L^4}\sum_{m=1}^{\infty}\frac{m^4\, q^m}{(1+q^m)^2}, \qquad (\nu = 2)
\end{aligned}
$$

$\nu = 2, 3$ corresponds to periodic, anti-periodic fermions on the spatial circle.

- The results for the corrections to the Rényi entropy can be written in terms of

quasi-elliptic functions of a weight 6

- All these corrections were shown to satisfy several non-trivial consistency checks. eg. cylinder limit, thermal entropy limit.

Lets illustrate one, which indicates the non-trivial nature of how checks are satisfied.

The Rényi entropy when the size of the interval equals the system size $\Delta = L$, must agree in the $n = 1$ limit to the thermal entropy.

The Rényi entropy correction at order μ^2, with $\Delta = L$ is given by

$$
\begin{aligned}
&S^{(2)}_{\mathrm{RE}}(\Delta, n)\Big|_{\Delta=L} \\
&= \frac{\mu^2 M}{(1-n)} \frac{80\pi^4\beta^2}{L^4} \sum_k \left[\frac{1}{2^5\cdot 3^2}\left(E_2^2 - E_4\right)\left(\frac{\vartheta''_\nu(x)}{\vartheta_\nu(x)} - \frac{\vartheta''_\nu}{\vartheta_\nu}\right) \right. \\
&- \frac{1}{2^6\cdot 3^2}\left\{ E_2^2 \left([\ln \vartheta_\nu(x)]'' - \frac{\vartheta''_\nu}{\vartheta_\nu} \right) \right. \\
&+ 2E_2 \left(\frac{\vartheta^{(4)}_\nu(x)}{\vartheta_\nu(x)} - \frac{\vartheta^{(3)}_\nu(x)}{\vartheta_\nu(x)} \frac{\vartheta'_\nu(x)}{\vartheta_\nu(x)} - \frac{\vartheta^{(4)}_\nu}{\vartheta_\nu} \right) \\
&\left.\left. + \frac{\vartheta^{(6)}_\nu(x)}{\vartheta_\nu(x)} - \left(\frac{\vartheta^{(3)}_\nu(x)}{\vartheta_\nu(x)} \right)^2 - \frac{\vartheta^{(6)}_\nu}{\vartheta_\nu} \right\} \right]_{x=\frac{k\pi}{n}} .
\end{aligned}
$$

The sum over k runs from $-(n-1)/2$ to $(n-1)/2$ in steps of unity.

We have

$$S_{\mathrm{RE}}^{(2)}(\Delta, n=1)\Big|_{\Delta=L} = \beta^2 \frac{\partial F^{(2)}}{\partial \beta}$$

(Non-trivial mathematical identity)

OTHER DIRECTIONS

- All these methods can be extended for evaluating Rényi entropy, entanglement entropy of excited states in CFT's deformed by holomorphic currents.

On going work with Surbhi Khetrapal and Prem Kumar

• Theories which admit higher spin currents in $d = 3$ are known.

Consider the theory at finite temperature, deform the Hamiltonian of the theory by the charge

$$\delta H = \mu \int d^2x \, j_{000}$$

This deformation preserves all the spatial symmetries.

For the $O(N)$ vector model this charge is just a bilinear in the fields. The path integral seems doable.

• It will be interesting to study this deformation and understand the implications for the dual Vasiliev theories.