The $\mathrm{AdS}_5 \times \mathrm{S}^5$ Superstring Action

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Introduction

Previous studies of the type IIB superstring in an $AdS_5 \times S^5$ background (Metsaev and Tseytlin, 1998) are based on the quotient space

$$PSU(2,2|4)/SO(4,1) \times SO(5).$$

I will present an alternative approach in which the Grassmann coordinates provide a nonlinear realization of $PSU(2,2|4)$ based on the quotient space

$$PSU(2,2|4)/SU(2,2) \times SU(4)$$

and the bosonic coordinates are described as a submanifold of $SU(2,2) \times SU(4)$.

The bosonic truncation

The unit-radius sphere:

$$\hat{z} \cdot \hat{z} = (z^1)^2 + (z^2)^2 + \ldots + (z^6)^2 = 1$$

The unit-radius anti de Sitter space:

$$\hat{y} \cdot \hat{y} = -(y^0)^2 + (y^1)^2 + \ldots + (y^4)^2 - (y^5)^2 = -1$$

The unit-radius metric:

$$ds^2 = d\hat{z} \cdot d\hat{z} + d\hat{y} \cdot d\hat{y}$$

The induced world-sheet metric:

$$G_{\alpha\beta} = \partial_\alpha \hat{z} \cdot \partial_\beta \hat{z} + \partial_\alpha \hat{y} \cdot \partial_\beta \hat{y}$$

The bosonic part of the radius R superstring action:

$$S = -\frac{R^2}{4\pi\alpha'} \int d^2\sigma \sqrt{-h} h^{\alpha\beta} G_{\alpha\beta}$$

AdS/CFT implies that

$$R^2 = \alpha' \sqrt{\lambda},$$

where $\lambda = g_{YM}^2 N$ is the 't Hooft parameter of the dual CFT, which is $\mathcal{N} = 4$ SYM with gauge group $U(N)$.

Supermatrices

$$M = \begin{pmatrix} a & \tau b \\ \tau c & d \end{pmatrix}, \quad \tau = e^{-i\pi/4}$$

a and d are even blocks referring to $SU(4)$ and $SU(2,2)$.
b and c are odd blocks that transform as bifundamentals.

The "superadjoint" is defined by

$$M^\dagger = \begin{pmatrix} a^\dagger & -\tau c^\dagger \\ -\tau b^\dagger & d^\dagger \end{pmatrix}.$$

This satisfies $(M_1 M_2)^\dagger = M_2^\dagger M_1^\dagger$.

A unitary supermatrix satisfies $MM^\dagger = I$ and an antihermitian supermatrix satisfies $M + M^\dagger = 0$.

The "supertrace" is defined (as usual) by

$$\mathrm{str} M = \mathrm{tr}\, a - \mathrm{tr}\, d.$$

The main virtue of this definition is that

$$\mathrm{str}(M_1 M_2) = \mathrm{str}(M_2 M_1).$$

The $\mathfrak{psu}(2,2|4)$ algebra is described by $\mathfrak{su}(2,2|4)$ matrices modded out by the equivalence relation

$$M \sim M + \lambda I.$$

Nonlinear realization of the superalgebra

The θ coordinates are 16 complex Grassmann numbers that transform under $SU(4) \times SU(2,2)$ as $(\mathbf{4}, \bar{\mathbf{4}})$.

It is natural to describe them by 4×4 matrices, rather than by 32-component spinors as we did for the flat-space limit. No Fierz transformations will be required!

The rule

$$\delta\theta = \omega\theta - \theta\tilde{\omega} + \varepsilon + \theta\varepsilon^{\dagger}\theta$$

closes precisely on the $\mathfrak{psu}(2,2|4)$ algebra. It is reminiscent of Volkov-Akulov Goldstino transformations.

We construct supermatrices $\Gamma(\theta) \in PSU(2,2|4)$ of the form

$$\Gamma = \begin{pmatrix} I & \tau\theta \\ \tau\theta^\dagger & I \end{pmatrix} \begin{pmatrix} f^{-1} & 0 \\ 0 & \tilde{f}^{-1} \end{pmatrix}$$

by choosing f and $\tilde{f}$ such that $\Gamma\Gamma^\dagger = I$. This is achieved for

$$f = \sqrt{I+u} = I + \frac{1}{2}u + \dots$$
$$\tilde{f} = \sqrt{I+\tilde{u}} = I + \frac{1}{2}\tilde{u} + \dots,$$

where

$$u = i\theta\theta^\dagger \quad \text{and} \quad \tilde{u} = i\theta^\dagger\theta$$

are hermitian matrices.

It then follows that

$$\delta_\varepsilon \Gamma = \begin{pmatrix} M(\varepsilon) & 0 \\ 0 & \tilde{M}(\varepsilon) \end{pmatrix} \Gamma + \Gamma \begin{pmatrix} 0 & \tau\varepsilon \\ \tau\varepsilon^\dagger & 0 \end{pmatrix},$$

where

$$M(\varepsilon) = (\delta_\varepsilon f - i f \varepsilon \theta^\dagger) f^{-1},$$
$$\tilde{M}(\varepsilon) = (\delta_\varepsilon \tilde{f} - i \tilde{f} \varepsilon^\dagger \theta) \tilde{f}^{-1}.$$

The natural interpretation is that θ and Γ describe the coset space

$$PSU(2,2|4)/SU(4) \times SU(2,2).$$

A flat connection

Now consider

$$A = \Gamma^{-1} d\Gamma = \begin{pmatrix} K & \tau\Psi \\ \tau\Psi^{\dagger} & \tilde{K} \end{pmatrix}.$$

This one-form supermatrix is constructed entirely out of θ. It is super-antihermitian and flat $(dA + A \wedge A = 0)$.

Under a supersymmetry transformation

$$\delta_{\varepsilon} A = -d \begin{pmatrix} M & 0 \\ 0 & \tilde{M} \end{pmatrix} - \left[A, \begin{pmatrix} M & 0 \\ 0 & \tilde{M} \end{pmatrix} \right].$$

Inclusion of bosonic coordinates

$$Z = \begin{pmatrix} 0 & u & v & w \\ -u & 0 & -\bar{w} & \bar{v} \\ -v & \bar{w} & 0 & -\bar{u} \\ -w & -\bar{v} & \bar{u} & 0 \end{pmatrix} = \Sigma_a z^a,$$

where $u = z^1 + iz^2$, $v = z^3 + iz^4$, and $w = z^5 + iz^6$.

Using $|u|^2 + |v|^2 + |w|^2 = 1$,

$$Z = -Z^T, \quad ZZ^\dagger = I, \quad \det Z = 1.$$

The only purpose in displaying all the elements of the matrix Z is to establish beyond any doubt the existence of a matrix with all of these properties. Otherwise, explicit representations are never used in this work. There is a very similar construction for Y.

The matrix Z defines a codimension 10 map of S^5 into $SU(4)$. Similarly, $Y : AdS_5 \to SU(2,2)$.

The supersymmetry transformations of the bosonic coordinates are

$$\delta_\varepsilon Z = MZ + ZM^T \quad \text{and} \quad \delta_\varepsilon Y = \tilde{M}Y + Y\tilde{M}^T.$$

The antihermitian connections

$$\Omega = ZdZ^{-1} - K - ZK^T Z^{-1},$$
$$\tilde{\Omega} = YdY^{-1} - \tilde{K} - Y\tilde{K}^T Y^{-1}$$

transform nicely under supersymmetry transformations

$$\delta_\varepsilon \Omega = [M, \Omega] \quad \text{and} \quad \delta_\varepsilon \tilde{\Omega} = [\tilde{M}, \tilde{\Omega}].$$

Therefore, the $PSU(2,2|4)$ invariant metric with the correct bosonic truncation is

$$ds^2 = -\frac{1}{4}\left(\mathrm{tr}(\Omega^2) - \mathrm{tr}(\tilde{\Omega}^2)\right).$$

Majorana–Weyl matrices

We wish to split objects transforming as $(\mathbf{4}, \bar{\mathbf{4}})$ into two pieces that correspond to MW spinors in the flat-space limit while respecting the group theory.

To do this, we define an involution

$$\Psi \to \Psi' = Z\Psi^{\star}Y^{-1}.$$

Then

$$\Psi = \Psi_1 + i\Psi_2 \quad \text{and} \quad \Psi' = \Psi_1 - i\Psi_2,$$

where Ψ_1 and Ψ_2 are MW matrices for which

$$\Psi_I' = \Psi_I \quad I = 1, 2.$$

Let us now define three supermatrix one-forms

$$A_1 = \begin{pmatrix} \Omega & 0 \\ 0 & \tilde{\Omega} \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & \tau\Psi \\ \tau\Psi^\dagger & 0 \end{pmatrix},$$
$$A_3 = \begin{pmatrix} 0 & \tau\Psi' \\ \tau\Psi'^\dagger & 0 \end{pmatrix}.$$

In all three cases

$$\delta_\varepsilon A_i = \left[\begin{pmatrix} M & 0 \\ 0 & \tilde{M} \end{pmatrix}, A_i \right].$$

By making the unitary transformation

$$J_i = \Gamma^{-1} A_i \Gamma \quad i = 1, 2, 3,$$

we obtain supermatrices that transform under $\mathfrak{psu}(2,2|4)$ in the natural way:

$$\delta_\Lambda J_i = [\Lambda, J_i].$$

The infinitesimal parameters are given by

$$\Lambda = \begin{pmatrix} \omega & -\tau\varepsilon \\ -\tau\varepsilon^\dagger & \tilde{\omega} \end{pmatrix}.$$

We will formulate the superstring action and its equations of motion entirely in terms of these three one-forms.

Maurer–Cartan equations

Using the explicit formulas that have been given, one obtains the Maurer–Cartan equations

$$dJ_1 = -J_1 \wedge J_1 + J_2 \wedge J_2 + J_3 \wedge J_3 - J_1 \wedge J_2 - J_2 \wedge J_1,$$

$$dJ_2 = -2J_2 \wedge J_2,$$

$$dJ_3 = -(J_1 + J_2) \wedge J_3 - J_3 \wedge (J_1 + J_2).$$

These imply that

$$J_\pm = J_1 + J_2 \pm iJ_3$$

and $2J_2$ are flat.

The Wess–Zumino term

The closed three-form that determines the WZ term for the fundamental string is also exact ($H_3 = dB_2$). Therefore, we can look for a suitable two-form B_2.

Consider the invariant two-forms

$$\mathrm{str}(J_i \wedge J_j) = -\mathrm{str}(J_j \wedge J_i).$$

The only one of these that is nonzero is

$$\mathrm{str}(J_2 \wedge J_3).$$

The correct coefficient will be determined (up to a sign) by requiring local kappa symmetry.

The superstring world-sheet action

The induced invariant metric is

$$G_{\alpha\beta} = -\frac{1}{4}\left(\mathrm{tr}(\Omega_\alpha \Omega_\beta) - \mathrm{tr}(\tilde{\Omega}_\alpha \tilde{\Omega}_\beta)\right) = -\frac{1}{4}\mathrm{str}(J_{1\alpha} J_{1\beta})$$

and the superstring world-sheet action is

$$S = -\frac{\sqrt{\lambda}}{4\pi}\int d^2\sigma \sqrt{-h}\, h^{\alpha\beta} G_{\alpha\beta} - \frac{\sqrt{\lambda}}{8\pi}\int \mathrm{str}(J_2 \wedge J_3).$$

This gives the $PSU(2,2|4)$ Noether current

$$J = J_1 + \star J_3.$$

Its conservation encodes equations of motion:

$$d \star J = d \star J_1 + dJ_3 = 0.$$

Theta variations

For an arbitrary variation $\delta\theta$

$$\delta A = -d \begin{pmatrix} \mathcal{M} & \tau\rho \\ \tau\rho^\dagger & \tilde{\mathcal{M}} \end{pmatrix} - \left[A, \begin{pmatrix} \mathcal{M} & \tau\rho \\ \tau\rho^\dagger & \tilde{\mathcal{M}} \end{pmatrix} \right],$$

where

$$\rho = f^{-1}\delta\theta\tilde{f}^{-1}$$

and

$$\mathcal{M} = -f^{-1}\delta f + i\rho\theta^\dagger, \quad \tilde{\mathcal{M}} = -\tilde{f}^{-1}\delta\tilde{f} + i\rho^\dagger\theta.$$

To get simple formulas, we simultaneously vary the bosonic coordinates

$$\delta Z = \mathcal{M} Z + Z \mathcal{M}^T \quad \text{and} \quad \delta Y = \tilde{\mathcal{M}} Y + Y \tilde{\mathcal{M}}^T.$$

In terms of supermatrices and differential forms, these variations give

$$\delta S_1 = -\frac{\sqrt{\lambda}}{4\pi} \int \mathrm{str}(R[A_1 \wedge \star A_2 + \star A_2 \wedge A_1]),$$

$$\delta S_2 = -\frac{\sqrt{\lambda}}{4\pi} \int \mathrm{str}(R[A_3 \wedge A_1 + A_1 \wedge A_3]),$$

where

$$R = \begin{pmatrix} 0 & \tau\rho \\ \tau\rho^\dagger & 0 \end{pmatrix}.$$

This provides additional equations of motion, which can be brought to the form

$$\star J_1 \wedge J_2 + J_2 \wedge \star J_1 = J_1 \wedge J_3 + J_3 \wedge J_1$$

or, equivalently,

$$\star J_1 \wedge J_3 + J_3 \wedge \star J_1 = J_1 \wedge J_2 + J_2 \wedge J_1.$$

Integrability

These equations, together with $d \star J_1 + dJ_3 = 0$ and the MC equations, imply that

$$J = c_1 J_1 + c_1' \star J_1 + c_2 J_2 + c_3 J_3$$

is flat ($i.e.$, $dJ + J \wedge J = 0$) for

$$c_1 = -\sinh^2 \lambda, \quad c_1' = \pm \sinh \lambda \cosh \lambda,$$

$$c_2 = 1 \mp \cosh \lambda, \quad c_3 = \sinh \lambda.$$

This is how Bena, Polchinski, and Roiban proved integrability in 2003.

Kappa Symmetry

The previous variations leave the action invariant provided that $\delta\theta$ is suitably restricted. The restriction is parametrized by a MW matrix κ.

Reexpressed in terms of MW matrices,

$$\delta(ds^2) = -2i \sum_{I=1}^{2} \mathrm{tr}(\Psi_I^\dagger[\Omega\rho_I - \rho_I\tilde{\Omega}])$$

and

$$\delta\mathrm{str}(J_2 \wedge J_3) = 4i\,\mathrm{tr}\left(\Psi_1^\dagger \wedge [\Omega\rho_1 - \rho_1\tilde{\Omega}]\right)$$
$$-4i\,\mathrm{tr}\left(\Psi_2^\dagger \wedge [\Omega\rho_2 - \rho_2\tilde{\Omega}]\right).$$

There exists another involution, $\rho \to \gamma(\rho)$ given by

$$\gamma(\rho) = -\frac{1}{2}\frac{\varepsilon^{\alpha\beta}}{\sqrt{-G}}\left(\Omega_\alpha\Omega_\beta\rho - 2\Omega_\alpha\rho'\tilde{\Omega}_\beta + \rho\tilde{\Omega}_\alpha\tilde{\Omega}_\beta\right).$$

The proof that $\gamma \circ \gamma = I$ involves forming a determinant in the numerator to cancel the one in the denominator.

This involution combines with the two other involutions $\rho \to \rho'$ and $\Omega \to \star\Omega$ to give the identity

$$\Omega\gamma(\rho') - \gamma(\rho)\tilde{\Omega} = \star(\Omega\rho' - \rho\tilde{\Omega}).$$

Putting these facts together,

$$\delta S = i\frac{\sqrt{\lambda}}{\pi}\int \mathrm{tr}\left(\Psi_1^\dagger \wedge [\Omega\gamma_+(\rho_1) - \gamma_+(\rho_1)\tilde{\Omega}]\right)$$
$$-\mathrm{tr}\left(\Psi_2^\dagger \wedge [\Omega\gamma_-(\rho_2) - \gamma_-(\rho_2)\tilde{\Omega}]\right),$$

where we have introduced projections

$$\gamma_\pm(\rho_I) = \frac{1}{2}\left(\rho_I \pm \gamma(\rho_I)\right).$$

Thus, recalling $\rho = f^{-1}\delta\theta\tilde{f}^{-1}$, S is invariant for

$$\delta\theta = f\big(\gamma_-(\kappa) + i\gamma_+(\kappa)\big)\tilde{f},$$

where $\kappa(\sigma)$ is an arbitrary MW matrix.

Conclusion

So far, the main achievement of this work is to reproduce well-known results. However, the formulation described here has some attractive features that are not shared by previous ones:

- The complete θ dependence of all quantities is described by simple analytic expressions.
- All formulas have manifest $SU(4) \times SU(2,2)$ symmetry, and many have manifest $PSU(2,2|4)$ symmetry.

The utility of this formalism for obtaining new results remains to be demonstrated. There are two main directions to explore.

- deriving new facts about this theory
- formulating (or reformulating) other theories in a similar way.

The End

Thank you for your attention.