# Constraining Tree Level Gravitational Scattering

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# Based on

- ArXiv:1910.14392 S. Duttachowdhury, A. Gadde, I. Halder, T. Gopalka, L. Janagal and S. M.
- ArXiv 2001.07117, S. Chakraborty , S. Duttachowdhury, T. Gopalka S. Kundu, A. Mishra and S.M.
- Plus work in progress with D. Chandorkar, S. Duttachowdhury, S. Kundu and S. Chakraborty

# (Provocative) Introduction

- Consider type II string compactifications of the form $R^p \times M$ where $M$ is an internal manifold. E.g.
  $p = 4$ and $M =$ a $CY$ manifold
  or $p = 6$ and $M =$ a $K_3$ manifold.
- Such a compactification defines a quantum theory of gravity on $R^p$. At the quantum (loop) level the graviton scattering amplitudes depend on details of the compactification manifold $M$ (e.g. the value of the moduli of the CY manifold).
- It is an obvious - but nonetheless remarkable - fact that this dependence drops out at classical (tree) level. This follows immediately from the fact that graviton vertex operators lie entirely in the $R^p$ part of the worldsheet CFT. Tree level scattering amplitudes are also the same for IIA, IIB and Type I theory.

# Introduction

- Restated, classical type II theory on $R^p \times M$ admits a consistent truncation to a universal (i.e. $M$ independent) theory which describes the interaction of gravitons and an infinite number of additional fields. This is true at all energies in string units.
- Heterotic (and Bosonic) compactifications also admit consistent truncations to their own universal sectors.
- Finally, there is another, more elementary example of a 'classical' S matrix (defined as having only poles and no cuts). This is the classical Einstein S matrix.
- As far as I am aware, these examples exhaust the set of classical S matrices that emerge in any parameteric limit of string theory. The parametric limits relevant to the enumerated examples is $g_s \to 0$ (with no restriction on energy) for the string S matrices and $E/m_p \to 0$ with no restrictions on $g_s$ for the Einstein S matrix. The last limit is classical theory provided all low energy degrees of freedom is vanishingly small at low energies.

# Provocative Conjecture

- The observations of the previous transparency motivate the following bold conjecture.
- Conjecture I: The classical Einstein S matrix, the tree level type II S matrix, and the tree level Heterotic S matrix constitute the exhaustive list of 'consistent' tree level S matrices of gravity. Restated, every 'consistent' classical gravitational theory admits a consistent truncation to one of these three universal sectors.
- By 'tree level S matrix' in the conjecture above we mean an S matrix whose only singularities are poles corresponding to the exchange of a massive or massless particle transforming in some representation of the Little group.
- The word 'consistent' in the conjecture above is yet to be completely defined, but in particular it means that the theory in question obeys all the good properties we usually demand of classical theories, in particular causality and boundedness of energy.

# Attribution

- That something like Conjecture I should hold has been suggested on several occasions by Nima Arkani Hamed (though perhaps not in print).
- The validity - or otherwise - of conjecture I was also one of my "Two Questions About Gravity" in the talk by that title that I gave in the 50th Anniversary of String Theory session at Strings 2018.

# Implied Conjectures

- The (perhaps crazy sounding) Conjecture I implies the following two successively weaker results.
- Conjecture II: The only consistent tree level gravitational S matrix with poles of bounded spin is the Einstein S matrix
- Conjecture III: The only consistent tree level gravitational S matrix with only gravitational poles is the Einstein S matrix.
- This is a heirarchy of Russian dolls of conjectures. $I \implies II \implies III$ but the reverse implications do not hold.
- In this talk we will study aspects of the weaker conjectures III and II. We will have nothing further to say about the fascinating conjecture *I*, which, however, forms part of the motivational framework for this talk.

# Review: Three graviton scattering

- The conjectures of the previous subsection apply to the scattering of $n$ gravitons, for all $n = 3, 4, 5 \ldots$. The case $n = 3$ is especially simple.
- This simplicity has its root in the fact that 3 graviton S matrices are highly kinematically constrained. The most general 3 graviton S matrix - classical or quanutm - is necessrily a linear combination of three structures.

$$T_1 = (\epsilon_1.\epsilon_2\epsilon_3.p_1 + \text{perm})^2 \qquad \text{2 der : Einstein}$$

$$T_2 = (\epsilon_1 \wedge \epsilon_2 \wedge \epsilon_3 \wedge p_1 \wedge p_2)^2 \qquad \text{4 der : GaussBonnet}$$

$$T_3 = \text{tr}(f_1 f_2)\text{tr}(f_2 f_3)\text{tr}(f_1 f_3) \qquad \text{6 der : Reimann}^3$$

$$f_i^{\mu\nu} = p_i^\mu \epsilon_i^\nu - p_i^\nu \epsilon_i^\mu .$$

The formulae above are actually valid only for $D \geq 5$. In $D = 4$, $T_2$ vanishes but a new parity odd structure appears.)

- Note in particular that all 3 graviton scattering amplitudes - classical or quantum - are always analytic in momenta.

# Review: CEMZ result

- The most general 3 graviton S matrix takes the form

$$aT_1 + bT_2 + cT_3$$

  where $a$, $b$ and $c$ are numbers (they have mass dimensions but are independent of momenta).
- Camanho, Edelstein, Maldacena and Zhiboedov (CEMZ) demonstrated that any theory in which either $b$ or $c$ is nonzero is necessarily acausal unless it couples to higher spin particles of arbitrarily high spin.
- In other words, CEMZ have already established Conjecture II at the level of 3 graviton scattering.
- This is very encouraging. However note that 3 graviton scattering is special as it is parameterized by finite data. Scattering with 4 or more gravitons has qualitatively greater complexity. I turn, in the rest of the talk, to the study of 4 graviton S matrices.

# Analytic Structure

- In this talk we study tree level 4 graviton S matrices generated by local Lagrangians with a finite number of fields. All such S matrices are given by the sum of a finite number of exchange pole plus polynomials in momenta.
- The most general pole contribution is obtained by using the propagator of an intermediate particle $R$ to sew together two $ggR$ onshell three particle S matrices together.
- It follows that a complete classification of $ggR$ three point couplings, for all little group reprentations $R$, effectively gives us a complete classification of pole contributions to the S matrix.
- While $ggR$ couplings had previously been listed in special cases (e.g. $D = 4$ using the spinor helicity formalism) we were not able to locate a systematic ennumeration in general dimension, so we undertook this exercise ourselves.

# 3 point couplings: Ennumeration

In more detail, for every massive particle $S$ we enumerated

- The full list of $SO(D-1)$ little group representations $R$ that can couple onshell to two gravitons.
- The independent number of $ggS$ couplings (parity even and parity odd) for every representation $S$. (This number turns out to be bounded from above by 8).
- A completely explicit basis for each of these couplings.
- A completely explicit basis for the local lagrangians that generate these couplings.

Though we will not need them in this talk, we also have similar results for arbitrary $ppS$ couplings, where $p$ is a photon.

# 3 point couplings: Method

- I briefly outline the method we used to carry out this (simple) ennumeration exercise.
- The momenta $k_1$ and $k_2$ of the two gravitons span an $R^{1,1}$ subspace of $R^{D-1,1}$. We call this the scattering two plane.
- The massive particle transforms in its little group representation of the $SO(D-1)$ that stabilizes $k_1 + k_2$. The graviton polarizations transform in the traceless symmetric represention of the $SO(D-2)$ that stabilizes the scattering two plane.
- The Bose symmetric $SO(D-2)$ invariants formed from the product of these three representations are easily enumerated and constructed using $SO(D-2)$ group theory. Note the same group theory counting also enumerates distinct $T_{\mu\nu} T_{\mu\nu} S$ correlators in a $D-1$ dimensional CFT.
- Finally, each of these invariants can be systematically lifted to a Lorentz and gauge invariant onshell S matrix.

# Three Point Couplings: Important take away

- Observation: Every coupling on our list is of fourth or higher order in derivatives.
- This fact has a simple physical interpretation. Every *genuine ggS* interaction term is constructed out of a product of (derivatives of) two Reimann tensors with the field $S$.
- Note that two derivative interaction terms like $\int \sqrt{-g} S R$ or $\int \sqrt{-g} R_{\mu\nu\alpha\beta} S^{\mu\nu\alpha\beta}$ also generate $hhS$ couplings. However these couplings are fake. The same Lagrangians also generate quadratic $hS$ couplings. The field redefinitions that we need to make to diagonalize the propagator also kill these fake $hhS$ couplings. In the special case of $\int \sqrt{-g} S R$ this field redefinition is simply 'moving to the Einstein Frame'.

# Consistent Truncation

- It follows that (atleast as at the cubic level) that every 2 derivative classical theory of gravity interacting with other fields necessarily admits a consistent truncation to Einstein gravity. This is the first hint that Conjecture II is at all plausible.
- Note that electromagnetism behave very different from gravity in this regard. Photon three point couplings are easily constructed out of two field strength operators and so exist at two derivative order.
- For this reason an electromagnetic analogue of Conjecture II cannot hold: exchange interactions generated by $\int \phi F_{\mu\nu} F^{\mu\nu}$ are a simple counter example.
- With the poles under control we turn to the polynomial part of the amplitude. This involves genuine 4 particle interactions and so is more complicated.

# Polynomial 4 particle S matrices

The most general polynomial S matrix is simply the most general polynomial built out of polarizations and momenta that is

- 1: Lorentz Invariant
- 2: Separately quadratic in all polarzations
- 3: Gauge Invariant
- 4: Bose symmetric, i.e. invariant under $S_4$ permutations.

We have arrived at a completely explicit listing of all such S matrices. In the next few slides I explain some of the ingredients that went into our ennumeration.

# Module Structure

- To begin with let us put aside the requirement of $S_4$ invariance. Let us call the most general polynomial of momenta and polarizations that obeys conditions 1-3 (but not necessarily condition 4) of the previous page as the space of ‘unsymmetrized polynomial S matrices’.
- Let $M$ be any unsymmetrized polynomial $S$ matrix. It is then obvious that $P(s,t)M$ is also such an $S$ matrix (here $P$ is any polynomial of the Mandlestam variables).
- In mathematical language, the space of unsymmetrized polynomial S matrices is a ‘module over the the ring of polynomials of Mandlestam variables’.

# The Module of Quasi Invariant S matrices

- Now it is not difficult to verify that the $Z_2 \times Z_2$ subgroup of $S_4$, consisting of $I$, $P_{12}P_{34}$, $P_{13}P_{24}$ and $P_{14}P_{23}$ leaves the Mandlestam variables $s$, $t$ and $u$ invariant.
- Let us call the collections of polynomials of polarizations and momenta that obey conditions 1-3 above - but are also $Z_2 \times Z_2$ invariant - the space of 'Qasi Invariant' polynomial S matrices.
- The $Z_2 \times Z_2$ invariance of Mandlestam variables immeditely tells us that the space of Quasi Invariant Polynomial S matrices is also a module over the ring of polynomials of $s$, $t$ and $u$.
- The space of Quasi Invariant S matrices can be decomposed into irreps of $S_4/(Z_2 \times Z_2) = S_3$.

Recall that $S_3$ has 3 irreps; the one dimensional completely symmetric irrep **S**, the one dimensional completely antisymmetric irrep **A**. And the two dimensional mixed representation **M**.

# Characterizing these modules

- A module can be completely characterized by its generators (e.g. ‘Virasoro Primaries’) and the generators of its relations (e.g. ‘Null States’). In our paper we have explicitly listed both the generators and the null states (when they exist) for all the quasiprimary S matrix module in every dimension $D$
- With the module of quasi invariant S matrices under complete control, it is now a simple matter to enumerate all polynomial S matrices. They are simply the projection of the quasi invairant module onto the space of $S_3$ singlets.
- In our paper we have provided a completely explicit listing of all such S matrices - both parity even and parity odd - in every dimension. We have also explicitly listed the Lagrangians that generate these S matrices.

# Results: Counting

| dimension | Even partition function | Odd partition function |
|---|---|---|
| $D \geq 10$ | $x^8(x^{-2} + 6 + 9x^2 + 10x^4 + 3x^6)\mathrm{D}$ | 0 |
| $D = 9$ | $x^8(x^{-2} + 6 + 9x^2 + 10x^4 + 3x^6)\mathrm{D}$ | 0 |
| $D = 8$ | $x^8(x^{-2} + 6 + 9x^2 + 10x^4 + 3x^6)\mathrm{D}$ | 0 |
| $D = 7$ | $x^8(x^{-2} + 6 + 9x^2 + 10x^4 + 3x^6)\mathrm{D}$ | $x^8(2x^{-1} + 3x + 2x^3)\mathrm{D}$ |
| $D = 6$ | $x^8(6 + 9x^2 + 10x^4 + 3x^6)\mathrm{D}$ | $3x^{10}(x^2 + x^4 + x^6)\mathrm{D}$ |
| $D = 5$ | $x^8(4 + 7x^2 + 8x^4 + 3x^6)\mathrm{D}$ | $x^{11}(x^2 + x^4 + x^6)\mathrm{D}$ |
| $D = 4$ | $x^8(2 + 2x^2 + 3x^4 - x^6 - x^8)\mathrm{D}$ | $x^8(1 + x^2 + 2x^4 - x^6 - x^8)\mathrm{D}$ |

Table: Partition function over 4 graviton S-matrices. $\mathrm{D} = \frac{1}{(1-x^4)(1-x^6)}$. The coefficient of $x^m$ in these expressions gives the number of independent polynomial S matrices at $m$ derivative order.

Though we do not need them in this talk, we have also explicitly constructed the quasi invariant modules for 4 photon scattering and have evaluated the analogues of the partition functions evaluated above for that case also.

# Tests

- We are very confident that our results are correct because we have two different ways of counting the number of polynomial S matrices graded by dimension.
- The first way is to construct the generators (and relations) and then count parameters using the module structure, as described above.
- The second independent method proceeds by explicitly ennumerating local Lagrangians upto field refinitions and total derivatives rather than S matrices.
- We perform this ennumeration by evaluating an $SO(D)$ matrix integral that projects the 'four graviton letter' partition function onto the space of $SO(D)$ singlets after removing total derivatives. The computation is not completely trivial, but we managed to carry it through. Both methods give exactly the same final results, giving a highly nontrivial test of our module constructions.

# Constraining Polynomial S matrices

- Recall that the CEMZ programme for constraining 3 graviton scattering had 2 steps. The first step was to use symmetry considerations to minimally parameterize the S matrix. We are now done with the analogous step for the 4 graviton S matrix.
- As you can see the result here is much more complicated; as opposed to 3 numbers it is given in high enough dimensions in terms of terms of 10 unknown functions of $s$ and $t$.
- We now turn to the second step of the programme, namely to use a physical principle to constrain the parameters that appear in the S matrix.

# Chaos Bound

- Recall the following result obtained by Maldacena, Shenker and Stanford. Consider a large $N$ CFT. Consider the (ordinary time ordered) four point function of four idnentical operators inserted on the $x, t$ plane as follows.
- The first two operators are inserted at the point $t = 0$, $x = 1$ but then boosted respectively with boost parameters $e^{\frac{\tau}{2}}$ and $e^{-\frac{\tau}{2}}$. The next two operators are first placed at $t = 0$ and $x = -1$ and then boosted with the same two boosts.
- MSS considered the limit $N \to \infty$ first and then $\tau \to \infty$. They demonstrated that the four point function described above is allowed to grow as $\tau \to \infty$, but no faster than $e^{\tau}$.
- We will now examine the consequences of this result for the holographic dual of such a CFT.

# The Chaos Bound and S matrices

- Let us now consider a situation in which the holographic dual to our CFT has a scalar field with a local Lagrangian - and more generally couplings that would generate a local S matrix.
- Such lagrangians are paremeterized in precisely the manner described earlier in this talk. Heemskerk, Penedones, Polchinski and Sully used the usual AdS/CFT dictionary to explicitly construct the boudary 4 point function that arises out of any given bulk Lagrangian.
- Using their results one can verify the following result. Consider a term in the bulk Lagrangian that would lead to a flat space S matrix that scales in the Regge limit (large $s$, fixed $t$) like $s^{m+1}$. The four point function that follows from the same bulk term scales like $e^{m\tau}$.
- It follows immediately that any local bulk term that leads to an S matrix that grows faster than $s^2$ in the Regge limit violates the chaos bound and so must be unphysical.

# Conjecture for bounds on Regge Scattering

- The observations above lead us to make the following Classical Regge Growth (CRG) conjecture.
- Classical theories whose S matrices grow faster than $s^2$ in the Regge limit are unphysical.
- That something like the above should be true has been suggested - perhaps a bit implicitly - by many people including CEMZ, MSS, Caron-Huot, Zhibeodov, Arkani-Hamed ...
- The sharp link between the CRG conjecture and the Chaos bound has so far been most clearly established for scalars. In ongoing work we have made a fair amount of progress in verifying that this connection continues to hold in a sharp manner also for spinning particles like gravitons.
- Note that the CRG bound is saturated by scattering in Einstein gravity, and that $\alpha'$ effects in string theory change the power 2 to a power less than 2.

# Implications of CRG: contact graviton interactions

- We can now use our painstakingly constructed explicit parameterization of polynomial graviton S matrices to list the most general S matrid of this form that obeys CRG scaling. We find that there is only one such S matrix namely

$$a\left(\epsilon_1 \wedge \epsilon_2 \wedge \epsilon_3 \wedge \epsilon_4 \wedge p_1 \wedge p_2 \wedge p_3\right)^2$$

- This 6 derivative S matrix - which, (roughly speaking) scales like *stu* and so is CRG allowed is generate by the Lagrangian

$$\chi_6 = \int \sqrt{-g}\left(\frac{1}{8}\ \delta^{g}_{[a}\delta^{h}_{b}\delta^{i}_{c}\delta^{j}_{d}\delta^{k}_{e}\delta^{l}_{f]}\ R_{ab}{}^{gh} R_{cd}{}^{ij} R_{ef}{}^{kl}\right)$$

- In summary, that the most general purely gravitational CRG action (upto terms that cannot affect 4 graviton scattering) is

$$a(Einstein) + b(GB) + c(Reimann^3) + d\chi_6$$

- Recall again

$$\chi_6 = \int \sqrt{-g} \left( \frac{1}{8} \, \delta^{g}_{[a} \delta^{h}_{b} \delta^{i}_{c} \delta^{j}_{d} \delta^{k}_{e} \delta^{l}_{f]} \; R_{ab}{}^{gh} R_{cd}{}^{ij} R_{ef}{}^{kl} \right)$$
$$= \int \sqrt{-g} \Big( 4 R_{ab}{}^{cd} R_{cd}{}^{ef} R_{ef}{}^{ab} - 8 R_{a}{}^{c}{}_{b}{}^{d} R_{c}{}^{e}{}_{d}{}^{f} R_{e}{}^{a}{}_{f}{}^{b} - 24 R_{abcd} R^{ab}$$
$$+ 24 R_{abcd} R^{ac} R^{bd} + 16 R_{a}{}^{b} R_{b}{}^{c} R_{c}{}^{a} - 12 R_{a}{}^{b} R_{b}{}^{a} R + R^3 \Big) \quad (1)$$

# $D \leq 6$

- $$\chi_6 = \int \sqrt{-g} \left( \frac{1}{8} \, \delta^{g}_{[a}\delta^{h}_{b}\delta^{i}_{c}\delta^{j}_{d}\delta^{k}_{e}\delta^{l}_{f]} \; R_{ab}{}^{gh} R_{cd}{}^{ij} R_{ef}{}^{kl} \right)$$
- It is obvious that $\chi_6$ vanishes identially for $D \leq 5$. In $D = 6$ this term is a total derivative. The term is classically nontrivial only for $D \geq 7$. This fact is already apparent from the form of its S matrix.
- In $D \leq 6$ it thus follows that the most general CRG allowed purely gravitational action (upto terms that cannot impact the four graviton scattering) in $D \leq 6$ is

$$a(Einstein) + b(GB) + c(Reimann^3).$$

# Exchange Contributions: Graviton Exchange

- So far we have considered the implications of the CRG conjecture on the polynomial contributions to 4 graviton scattering. As we have already discussed above the most general contribution also has exchange contributions.
- The exchange contributions relevant to Conjecture III are graviton poles, so lets study those first.
- These amplitudes can be thought of as a quadratic form in the coefficients $a$, $b$ and $c$ (of the allowed 3 point structures of 3 graviton scattering).
- We have explicitly constructed the most general exchange contribution of this nature (and also decomposed it in terms of our 'generator' index structures above). The final answer is a bit complicated. Main important result, however, is that this pole contribution to the amplitude grows faster than $s^2$ in the Regge limit unless $b = c = 0$. It follows that the most general CRG allowed purely gravitational action in $D \leq 6$ is Einstein. In particular we CRG implies CEMZ + more.

# $s^J$?

- In the last slide I claimed that exchange contributions from, e.g., two GB vertices grows faster than $s^2$. On the other hand we often hear the following claim: the contribution to the S matrix from the exchange of paricles of spin $J$ scales like $s^J$. Given that GB exchange contributions capture only gravity exchange (i.e. $J = 2$) don't we have a contradiction?
- The resolution is the following. The contribution of spin $J$ particles to the S matrix scales like $s^J$ only in the $t$ channel. The contribution from the $s$ and $u$ channels is not universal. They depend on the details of the three point couplings. These contributions grow faster than $s^2$ in for GB-GB exchange.
- Note that $t$ channel contributions are special. They are non polynomial in $t$ even in the Regge limit. These are thus the only contributions that contribute to scattering at nonzero impact parameter in the Regge limit. In other words GB 4 graviton scattering violates the CRG conjecture - but not in a way that can be seen at nonzero impact parameter.

- On the other hand $s$ and $u$ channel contributions are typically analytic in $t$ in the Regge limit, and (like contact terms) contribute to scattering only at zero impact parameter.
- In order to conclude that a GB coupling is unphysical, it is thus not sufficient to check that the exchange contribution from two GB vertices grows faster than $s^2$. We must also check that this growth is of the form that that cannot be cancelled by addition of a local counterterm. We have indeed checked this.

# Massive Exchanges: General Analysis

- Using the fact that *ggP* couplings always involve 4 derivatives, in our paper we have presented a general argument that all massive exchange contributions to 4 graviton scattering grow faster than $s^2$ atleast for $D \leq 6$, and this growth is of the form that cannot be cancelled by a local counterterm.
- Moreover exchange contributions - unilike contact term contributions - always come with a definite sign (this follows from the reality of three point couplings and the fact that propagators have to have the right sign). For this reason the contribution of various different exchange contributions cannot cancel each other.
- It follows that the CRG conjecture excludes all exchange contributions. Assuming the CRG conjecture we thus more or less have a proof of conjecture II for $D \leq 6$ for the special case of four graviton scattering.

# Discussions and Conclusions

- In this talk we first presented a complete classification of 4 graviton (and four photon) classical S matrices in the theory whose Lagrangian has a finite number of derivatives and has a finite number of fields.
- We then presented a conjecture about the allowed growth of S matrices in classical theories. We then used this conjecture to completely classify allowed classical theories of gravity, upto Lagrangian terms of order Riemann$^5$ or higher that do not impact 4 graviton scattering.
- It would be very nice to understand our $s^2$ conjecture better - and if possible to replace it with a clear physical argument directly in flat space. We have some ideas that we are working on.
- It would also be interesting to understand the status of the ambiguity of the action in $D \geq 7$. Is this a genuine ambiguity, or does another physical argument set $a$ to zero?

# Discussions and Conclusions

- Using AdS/CFT one can turn our results into a constraint on stress tensor four point functions in the large $N$ limit. Our results suggest that the only Chaos bound allowed large $N$ $TTTT$ four point function that receives contributions from a finite number of single trace exchange blocks (in addition to double stress tensor exchanges) is the result generated by the pure Einstein action in the bulk.
- It would be very interesting to generalize the results of this talk to the scattering of more than 4 gravitions, and complete the process of characterizing the most general classical local theory of gravity consistent with general principles.
- Finally, if all this works out we could get more ambitious and generalize the study of this talk beyond local S matrices, with the hope of establishing Conjecture I: i.e the uniqueness of string scattering.

# Rough Work

# Detail: Universality of String Scattering 1

- Consider type II string compactifications of the form $R^p \times M$ where $M$ is an internal manifold. E.g. $p = 4$ and $M =$ a $CY$ manifold.
- Such a compactification defines a quantum theory of gravity on $R^p$. The spectrum of the theory includes $p$ dimensional gravitons.
- The scattering amplitudes of these gravitons are given by the following schematic formula

$$\mathcal{A} = \sum_g \int d\tau_i dz_i < V_1(z_1) \ldots V_n(z_n) >$$

where $V_n$ are the graviton vertex operators, $z_i$ are their insertion locations, and $\tau_i$ are the moduli of the genus $g$ Reimann surface. Expectations values are taken in the sigma model on $R^p \times M$.

# Detail: Universality of String Scattering II

- As graviton vertex operators all lie in the $R^p$ part of the CFT, the formula above can be simplified to

$$\mathcal{A} = \sum_g \int d\tau_i \, Z_M(\tau_i) \, C_{R^p}(\tau_i)$$

$$C(\tau_i) = \int dz_i < V_1(z_1) \ldots V_n(z_n) > |_{R^p}$$

where $Z_M(\tau_i)$ is partition function of the sigma model on $M$ on the Reimann surface. The vertex operator expectation values are taken purely in the $R^p$ part of the CFT.

- Even though $C(\tau_i)$ are universal - independent of $M$ - $Z(\tau_i)_M$ - and hence the integral over $\tau_i$ above - clearly depends on $M$. It follows that graviton scattering amplitudes at generic values of $g$ depend on details of the compactification manifold $M$ (e.g. are intricate functions the CY moduli).

# Detail: Universality of String Scattering III

- The story above holds for generic $g$. Let us now, however, focus on the special case $g = 0$. As the Reimann sphere has no moduli, the integral over $\tau_i$ is absent at $g = 0$. It follows that
$$\mathcal{A}_{g=0} = Z_M^{S^2} \; C_{R^p}^{S^2}$$
Here $Z_M^{S^2}$ is the partition function of the $M$ CFT on the 2 sphere. $Z_M^{S^2}$ is a multiplicative factor for all scattering amplitudes. It is an overall number that sets the value of the effective $p$ dimensional Newton constant.
- $C_{R^p}^{S^2}$, the nontrivial part of the scattering amplitude is universal (i.e. independent of $M$). It is not hard to convince oneself that $C_{R^p}^{S^2}$ is the same for type IIA, IIB and Type I theory.

# Detail: Three particle scattering, Counting 1

- Consider the scattering of 2 gravitons and a massive particle in a representation $R$ of the little group $SO(D-1)$. Spacetime can be divided up into the 'scattering 2 plane' spanned by $k_1$, $k_2$ and its orthogonal compliment.
- The polarization $\epsilon_1$ of the graviton with momentum $k_1$ obeys $k_1.\epsilon_1 = 0$. Implies $\epsilon_1 = \epsilon_1^\perp + a_1 k_1$ where $\epsilon_1^\perp$ lies in the orthogonal compliment. As far as gauge invariant amplitudes go, $\epsilon_1 = \epsilon_1^\perp$. Sim for $\epsilon_2$. It follows that graviton polarization states are labelled by traceless symmetric tensors of $SO(D-2)$ that stabilizes the scattering two plane.
- On the other hand the representation $R$ of $SO(D-1)$ (which stabilizes $k_1 + k_2$) descends, via $SO(D-1)$ branching rules, to a finite set of representations of the $SO(D-2)$ that also stabilizes $k_1 - k_2$, and so the full scattering two plane

# Detail: Three particle scattering: Counting 2

In order to enumerate all possible *ggP* three point functions we

- Enumerate all $SO(D-2)$ singlets in the product of two $SO(D-2)$ tranceless symmetric tensors and one copy of any of the $SO(D-2)$ reps that descend from $R$.
- Retain only those singlets that respect the Bose symmetry of gravitons.

This exercise is not difficult to undertake. Turns out that the number of three point structures - as we vary over $R$ - ranges from zero to three. Once we have enumerated all structures it is also easy to explicitly construct them all, and also to list the Lagrangians from which they follow.

# Detail: 3 particle scattering, Listing 1

- It is also not difficult to explicitly construct all relevant 3 point functions. For instance for $D \geq 8$. Yellow boxes denote indices effectively contracted with $k_1 - k_2$ in order to facilitate comparison with counting outlined above.

| | |
|---|---|
| $\begin{array}{\|c\|c\|}\hline a & b \\ \hline\end{array}$ : $\nabla_a \nabla_b R_{cdef} R_{cdef} S_{ab}$ | $\begin{array}{\|c\|c\|}\hline a & b \\ \hline\end{array}$ : $R_{efga} R_{efgb} S_{ab}$ |
| $\begin{array}{\|c\|c\|c\|c\|}\hline a & b & \alpha & \beta \\ \hline\end{array}$ : $R_{cadb} R_{c\alpha d \beta} S_{ab\alpha\beta}$ | |

| | |
|---|---|
| $\begin{array}{\|c\|c\|}\hline a & b \\ \hline c \\ \cline{1-1}\end{array}$ : $\nabla_d R_{acef} R_{bdef} S_{[ac]b}$ | $\begin{array}{\|c\|c\|c\|c\|}\hline a & b & c & d \\ \hline e \\ \cline{1-1}\end{array}$ : $\nabla_h R_{aedi} R_{hbci} S_{[ae]bcd}$ |
| $\begin{array}{\|c\|c\|c\|}\hline a & b & c \\ \hline d \\ \cline{1-1}\end{array}$ : $R_{efch} \nabla_b \nabla_h R_{efad} S_{[ad]bc}$ | $\begin{array}{\|c\|c\|c\|c\|c\|}\hline a & b & c & d & e \\ \hline f \\ \cline{1-1}\end{array}$ : $\nabla_\beta R_{\alpha bch} \nabla_h \nabla_e \nabla_\alpha R_{\beta daf} S_{[af]bcde}$ |

| | |
|---|---|
| $\begin{array}{\|c\|}\hline a \\ \hline b \\ \hline c \\ \hline\end{array}$ : $\nabla_f R_{abde} R_{cfde} S_{[abc]}$ | $\begin{array}{\|c\|c\|c\|}\hline a & d & e \\ \hline b \\ \cline{1-1} c \\ \cline{1-1}\end{array}$ : $\nabla_h R_{abdf} R_{chef} S_{[abc]de}$ |

# Detail: 3 point structures, Listing 2

| | |
|---|---|
| r | t |
| s | u |

and

| | |
|---|---|
| r | t |
| s | u |

: $R_{pqrs}R_{pqtu}S_{[rs][tu]}$ **and** $R_{prqt}R_{psqu}S_{[rs][tu]}$

| | | | |
|---|---|---|---|
| a | c | e | f |
| b | d | | |

: $R_{abeh}R_{cdfh}S_{[ab][cd]ef}$

| | | |
|---|---|---|
| c | i | d |
| a | j | |

: $R_{abdk}\nabla_k R_{bcij}S_{[ca][ij]d}$

| | | | |
|---|---|---|---|
| a | d | f | i |
| b | e | | |
| c | | | |

: $\nabla_i R_{abfj}R_{cjde}S_{[abc][de]fi}$

| | | |
|---|---|---|
| a | d | f |
| b | e | |
| c | | |

: $R_{abfi}R_{cide}S_{[abc][de]f}$

| | |
|---|---|
| a | d |
| b | e |
| c | f |

: $R_{abdh}R_{chef}S_{[abc][def]}$

# Detail: 3 point structures, Listing 3

| | | | |
|---|---|---|---|
| c | i | k | e |
| a | j | d | |

: $R_{abkd} \nabla_e R_{bcij} S_{[ca][ij][kd]e}$

| | | | | |
|---|---|---|---|---|
| a | b | c | d | e |
| f | i | j | | |

: $\nabla_k R_{afcj} \nabla_e R_{bidk} S_{[af][bi][cj]de}$

| | | |
|---|---|---|
| a | d | f |
| b | e | i |
| c | | |

: $\nabla_j R_{abde} R_{cjfi} S_{[abc][de][fi]}$

| | | | |
|---|---|---|---|
| a | c | e | i |
| b | d | f | j |

: $R_{abcd} R_{efij} S_{[ab][cd][ef][ij]}$

# Detail: Decomposition of Polynomials into $S_3$ reps

- The space of polynomials of $s$, $t$, $u$ of any given degree can be decomposed into representations of $S_3$. Let the number of representations of type $\alpha$ at degree $n/2$ be denoted as $n_\alpha(n)$. ($n$ counts the number of derivatives). Then

$$\text{Let}\quad Z_\alpha(x) = \sum_n n_\alpha(n) x^n,$$
$$Z_{\mathbf{S}}(x) = \mathcal{D}, \qquad Z_{\mathbf{A}}(x) = x^6 \mathcal{D}, \qquad Z_{\mathbf{M}}(x) = (x^2 + x^4)\mathcal{D},$$
$$\mathcal{D} = \frac{1}{(1-x^4)(1-x^6)} \tag{2}$$

- Below we will also need a formula for the partition function, $Z_{Z_2}(x)$, over the space of polynomials that are symmetric under interchange of $s$ and $t$ only. Using the fact that every particle exchange is represented as unity in the **S** representation, as $-1$ in the **A** representation and as a $2 \times 2$ matrix with eigenvalues $\pm 1$ in the **M** representation it follows that

$$Z_{Z_2}(x) = Z_{\mathbf{S}}(x) + Z_{\mathrm{M}}(x).$$

# Detail: Fusion Rules of $S_3$

- The $S_3$ fusion rules are

$$\begin{aligned}&\mathbf{S}\times R = R, \quad \mathbf{A}\times\mathbf{A} = S, \quad \mathbf{A}\times\mathbf{S} = \mathbf{A}, \quad \mathbf{A}\times\mathbf{M} = \mathbf{M},\\ &\mathbf{M}\times\mathbf{M} = \mathbf{S}+\mathbf{A}+\mathbf{M}\end{aligned} \tag{3}$$

- Note that the $S$ representation only appears on the RHS of the product of two equal representations. It follows that the most general polynomial S matrix takes the form

$$\sum_I f_I^a(s,t,u)E_I^a$$

where $E_I^a$ run over the all 'primitive' or 'generator' index structures, each of which transform in some representation of $S_3$ ($a$ is an index label within representations), and $f_I^a$ are any polynomials that transform in the same representation of $S_3$ (the sum over $a$ projects onto singlets).

# Detail: Descendent Lagrangians

- As we have mentioned above Generators of the Local S matrix module are not always $S_3$ invariant. On the other hand Lagrangians always give rise to $S_3$ invariant S matrices. Nonetheless there an interesting way to associate Lagrangians with generators.
- We say a Lagrangian structure $A$ is a descendent of a structure $B$ if first $A$ has more derivatives than $B$, but all the extra derivatives that are in $A$ but not in $B$ have indices that contract with each other. Second, if we remove all these contracted derivatives $A$ reduces to $B$.
- We say a Lagrangian structure corresponds to a given generator if the set of all desendents from that Lagrangian yield a set of S matrices that coincides with the restriction to $S_3$ singlets of the set of S matrices generated by the generator in question.

# Detail: Parity even photon S matrices in $D \geq 5$.

- For $D \geq 5$ the most general local parity invariant $S$ matrix for 4 photons is freely generated.
- The most general S matrix parameterized by by three polynomials in the **S** representation and two in the **M** representations. Equivalently - and more conveniently for some purposes, these set of polynomials may be shown to be characterized by 2 $Z_2$ invariant functions (i.e. functions that are symmetric under $u$ goes to $t$ interchange) $A^{0,1}(t,u)$ and a single $S_3$ invariant function $A^{2,1}(s,t,u)$.
- $A^{0,1}$ and $A^{0,2}$ parameterize descendents of the four derivative structures $(TrF^2)^2$ and $Tr(F^4)$ respectively while $A^{1,2}$ parameterizes descendents of the six derivative term

$$F_{ab}\mathrm{Tr}(\partial_a F \partial_b F F)$$

# Detail: Explicit parameterization of 4 photon S matrices 1

- Explicitly the most general parity even 4 photon S matrix in $D \geq 5$ is given by the sum of
- $$\begin{aligned} & A^{0,1}(t,u)\left(p^1_\mu\epsilon^1_\nu - p^1_\nu\epsilon^1_\mu\right)\left(p^2_\mu\epsilon^2_\nu - p^2_\nu\epsilon^2_\mu\right)\left(p^3_\alpha\epsilon^3_\beta - p^3_\beta\epsilon^3_\alpha\right)\left(p^4_\alpha\epsilon^4_\beta - p^4_\beta\epsilon^4_\alpha\right) \\ + & A^{0,1}(s,u)\left(p^1_\mu\epsilon^1_\nu - p^1_\nu\epsilon^1_\mu\right)\left(p^3_\mu\epsilon^3_\nu - p^3_\nu\epsilon^3_\mu\right)\left(p^2_\alpha\epsilon^2_\beta - p^2_\beta\epsilon^2_\alpha\right)\left(p^4_\alpha\epsilon^4_\beta - p^4_\beta\epsilon^4_\alpha\right) \\ + & A^{0,1}(t,s)\left(p^1_\mu\epsilon^1_\nu - p^1_\nu\epsilon^1_\mu\right)\left(p^4_\mu\epsilon^4_\nu - p^4_\nu\epsilon^4_\mu\right)\left(p^3_\alpha\epsilon^3_\beta - p^3_\beta\epsilon^3_\alpha\right)\left(p^2_\alpha\epsilon^2_\beta - p^2_\beta\epsilon^2_\alpha\right) \end{aligned} \tag{4}$$
- and

$$\begin{aligned} & A^{0,2}(t,u)\left(p^1_\mu\epsilon^1_\nu - p^1_\nu\epsilon^1_\mu\right)\left(p^3_\nu\epsilon^3_\alpha - p^3_\alpha\epsilon^3_\nu\right)\left(p^2_\alpha\epsilon^2_\beta - p^2_\beta\epsilon^2_\alpha\right)\left(p^4_\beta\epsilon^4_\mu - p^4_\mu\epsilon^4_\beta\right) \\ + & A^{0,2}(s,u)\left(p^1_\mu\epsilon^1_\nu - p^1_\nu\epsilon^1_\mu\right)\left(p^2_\nu\epsilon^2_\alpha - p^2_\alpha\epsilon^2_\nu\right)\left(p^3_\alpha\epsilon^3_\beta - p^3_\beta\epsilon^3_\alpha\right)\left(p^4_\beta\epsilon^4_\mu - p^4_\mu\epsilon^4_\beta\right) \\ + & A^{0,2}(t,s)\left(p^1_\mu\epsilon^1_\nu - p^1_\nu\epsilon^1_\mu\right)\left(p^3_\nu\epsilon^3_\alpha - p^3_\alpha\epsilon^3_\nu\right)\left(p^4_\alpha\epsilon^4_\beta - p^4_\beta\epsilon^4_\alpha\right)\left(p^2_\beta\epsilon^2_\mu - p^2_\mu\epsilon^2_\beta\right) \end{aligned} \tag{5}$$

# Detail: Explicit parameterization of 4 photon S matrices 2

- $$\begin{aligned}
&\left(A^{2,1}(s,t) + A^{2,1}(t,u) + A^{2,1}(u,s)\right) \times \\
&\Big[\left(p^1_a\epsilon^1_b - p^1_b\epsilon^1_a\right) p^2_a \left(p^2_\mu\epsilon^2_\nu - p^2_\nu\epsilon^2_\mu\right) p^3_b \left(p^3_\nu\epsilon^3_\alpha - p^3_\alpha\epsilon^3_\nu\right) \left(p^4_\alpha\epsilon^4_\mu - p^4_\mu\epsilon^4_\alpha\right) \\
&+ \left(p^2_a\epsilon^2_b - p^2_b\epsilon^2_a\right) p^1_a \left(p^1_\mu\epsilon^1_\nu - p^1_\nu\epsilon^1_\mu\right) p^4_b \left(p^4_\nu\epsilon^4_\alpha - p^4_\alpha\epsilon^4_\nu\right) \left(p^3_\alpha\epsilon^3_\mu - p^3_\mu\epsilon^3_\alpha\right) \\
&+ \left(p^3_a\epsilon^3_b - p^3_b\epsilon^3_a\right) p^4_a \left(p^4_\mu\epsilon^4_\nu - p^4_\nu\epsilon^4_\mu\right) p^1_b \left(p^1_\nu\epsilon^1_\alpha - p^1_\alpha\epsilon^1_\nu\right) \left(p^2_\alpha\epsilon^2_\mu - p^2_\mu\epsilon^2_\alpha\right) \\
&+ \left(p^4_a\epsilon^4_b - p^4_b\epsilon^4_a\right) p^3_a \left(p^3_\mu\epsilon^3_\nu - p^3_\nu\epsilon^3_\mu\right) p^2_b \left(p^2_\nu\epsilon^2_\alpha - p^2_\alpha\epsilon^2_\nu\right) \left(p^1_\alpha\epsilon^1_\mu - p^1_\mu\epsilon^1_\alpha\right)\Big]
\end{aligned} \tag{6}$$

- The most general local S matrices are given by the form listed above with $A^{0,1}$, $A^{0,2}$ and $A^{1,2}$ polynomials of $s$, $t$ and $u$. We have counted the data in such S matrices above- our photon S matrix has 7 degrees of freedom. The most general S matrices - not necessarily local - are also given by the forms above allowing for more general (not necessarily polynomial) dependences of the unknown

# S matrices for 4 identical gravitons

- As another example we present the most general parity even gravity S matrix in $D \geq 7$.
- This S matrix turns out to be parameterized by 7 $Z_2$ invariant, one function that enjoys no permutation symmetry and two functions that are completely permutation symmetric. or a total of 29 degrees of freedom.
- In more detail we have one completely symmetric generator at 6 derivatives (Riemann$^3$) term

$$\chi_6 = \int \sqrt{-g} \left( \frac{1}{8} \, \delta^{g}_{[a}\delta^{h}_{b}\delta^{i}_{c}\delta^{j}_{d}\delta^{k}_{e}\delta^{l}_{f]} \; R_{ab}{}^{gh} R_{cd}{}^{ij} R_{ef}{}^{kl} \right) \tag{7}$$

Second Lovelock term. One d.o.f.

# S matrices for 4 identical Gravitions: parity even $D \geq 7$.

- At 8 derivative order we have 5 generators in the **3** and one generator in the **6** rep of $S_3$. Total 21 dofs.
- At 10 derivative order there are 2 generators in the **3** rep. 6 degrees of freedom.
- Finally at 12 derivative order there is a single generator in the **S** rep. One d.o.f.
- Note: If we set $g_{\mu\nu}(k) = \eta_{\mu\nu} + \epsilon_\mu(k)\epsilon_\nu(k)e^{ik.x}$ with $k^2 = 0$ then it turns out that $R_{abmn}$ evaluated to linearized order is proportional to $F_{ab}(k)F_{mn}(k)$ where $F_{mn} = k_m\epsilon_n - k_n\epsilon_n$. In our Lagrangian terms below we will sometimes replace $R_{abmn}$ with $F_{ab}F_{mn}$.

# Explicit parameterization of the general 4 graviton S matrix: 1

- Explicitly, the most general 4 gravition S matrix is given by the sum of

$$S_1 = 3B^{0,0}(s,t,u)\,(\epsilon_1 \wedge \epsilon_2 \wedge \epsilon_3 \wedge \epsilon_4 \wedge p_1 \wedge p_2 \wedge p_3)^2 \qquad (8)$$

  with $B^{0,0}(s,t,u)$ completely symmetric (this is from descendents of the Reimann$^3$ structure) and

- $$B^{0,1}(s,t)\left[(p^1_p\epsilon^1_q - p^1_q\epsilon^1_p)\,(p^2_p\epsilon^2_q - p^2_q\epsilon^2_p)\,(p^3_r\epsilon^3_s - p^3_s\epsilon^3_r)\,(p^4_r\epsilon^4_s - p^4_s\epsilon^4_r)\right.$$
$$\left.(p^1_a\epsilon^1_b - p^1_b\epsilon^1_a)\,(p^2_b\epsilon^2_c - p^2_c\epsilon^2_b)\,(p^3_c\epsilon^3_d - p^3_d\epsilon^3_c)\,(p^4_d\epsilon^4_a - p^4_a\epsilon^4_d)\right]$$
$$+ B^{0,1}(s,u)\,[3 \leftrightarrow 4] + B^{0,1}(t,s)\,[2 \leftrightarrow 3] + B^{0,1}(t,u)\,[2 \leftrightarrow 3 \text{ then } 2 \leftrightarrow 4]$$
$$+ B^{0,1}(u,t)\,[2 \leftrightarrow 4] + B^{0,1}(u,s)\,[2 \leftrightarrow 4 \text{ then } 2 \leftrightarrow 3] \qquad (9)$$

  where $B^{0,1}$ has no special symmetry property; this term is from descendents of $\mathrm{Tr}(F^1F^2)\mathrm{Tr}(F^3F^4)\mathrm{Tr}(F^1F^2F^3F^4)$

# Explicit parameterization of the gravity S matrix:2

- $$B^{0,2}(t,u)\left[\left(p^1_p\epsilon^1_q - p^1_q\epsilon^1_p\right)\left(p^2_p\epsilon^2_q - p^2_q\epsilon^2_p\right)\left(p^3_r\epsilon^3_s - p^3_s\epsilon^3_r\right)\left(p^4_r\epsilon^4_s - p^4_s\epsilon^4_r\right)\right.$$
  $$\left.\left(p^1_a\epsilon^1_b - p^1_b\epsilon^1_a\right)\left(p^3_b\epsilon^3_c - p^3_c\epsilon^3_b\right)\left(p^2_c\epsilon^2_d - p^2_d\epsilon^2_c\right)\left(p^4_d\epsilon^4_a - p^4_a\epsilon^4_d\right)\right]$$
  $$+ B^{0,2}(s,u)\,[3 \leftrightarrow 2] + B^{0,2}(s,t)\,[2 \leftrightarrow 4] \tag{10}$$

  where

  $$B^{0,2}(t,u) = B^{0,2}(u,t) \tag{11}$$

  From descendents of $\mathrm{Tr}(F^1F^2)\mathrm{Tr}(F^3F^4)\mathrm{Tr}(F^1F^3F^2F^4)$.

- $$B^{0,3}(s,u)\left[\left(p^1_a\epsilon^1_b - p^1_b\epsilon^1_a\right)\left(p^2_b\epsilon^2_c - p^2_c\epsilon^2_b\right)\left(p^3_c\epsilon^3_d - p^3_d\epsilon^3_c\right)\left(p^4_d\epsilon^4_a - p^4_a\epsilon^4_d\right)\right.$$
  $$\left.\left(p^1_p\epsilon^1_q - p^1_q\epsilon^1_p\right)\left(p^2_q\epsilon^2_r - p^2_r\epsilon^2_q\right)\left(p^3_r\epsilon^3_s - p^3_s\epsilon^3_r\right)\left(p^4_s\epsilon^4_p - p^4_p\epsilon^4_s\right)\right]$$
  $$+ B^{0,3}(t,u)\,[3 \leftrightarrow 2] + B^{0,3}(s,t)\,[3 \leftrightarrow 4] \tag{12}$$

  $$B^{0,3}(s,u) = B^{0,3}(u,s) \tag{13}$$

  (from descendents of $\mathrm{Tr}(F^1F^2F^3F^4)\mathrm{Tr}(F^1F^2F^3F^4)$)

# Explicit parameterization of the Gravity S matrix: 3

- $$B^{0,4}(s,t)\left[\left(p^1_a\epsilon^1_b - p^1_b\epsilon^1_a\right)\left(p^2_b\epsilon^2_c - p^2_c\epsilon^2_b\right)\left(p^3_c\epsilon^3_d - p^3_d\epsilon^3_c\right)\left(p^4_d\epsilon^4_a - p^4_a\epsilon^4_d\right)\right.$$
$$\left.\left(p^1_p\epsilon^1_q - p^1_q\epsilon^1_p\right)\left(p^3_q\epsilon^3_r - p^3_r\epsilon^3_q\right)\left(p^2_r\epsilon^2_s - p^2_s\epsilon^2_r\right)\left(p^4_s\epsilon^4_p - p^4_p\epsilon^4_s\right)\right]$$
$$+ B^{0,4}(s,u)\,[3 \leftrightarrow 4] + B^{0,4}(u,t)\,[2 \leftrightarrow 4] \tag{14}$$
$$B^{0,4}(s,t) = B^{0,4}(t,s) \tag{15}$$
from descendents of $\mathrm{Tr}(F^1F^2F^3F^4)\mathrm{Tr}(F^1F^3F^2F^4)$

- $$B^{0,5}(t,u)\left[\left(p^1_p\epsilon^1_q - p^1_q\epsilon^1_p\right)\left(p^2_p\epsilon^2_q - p^2_q\epsilon^2_p\right)\left(p^3_r\epsilon^3_s - p^3_s\epsilon^3_r\right)\left(p^4_r\epsilon^4_s - p^4_s\epsilon^4_r\right)\right.$$
$$\left.\left(p^1_a\epsilon^1_b - p^1_b\epsilon^1_a\right)\left(p^2_a\epsilon^2_b - p^2_b\epsilon^2_a\right)\left(p^3_c\epsilon^3_d - p^3_d\epsilon^3_c\right)\left(p^4_c\epsilon^4_d - p^4_d\epsilon^4_c\right)\right]$$
$$+ B^{0,5}(s,u)\,[3 \leftrightarrow 2] + B^{0,5}(s,t)\,[2 \leftrightarrow 4] \tag{16}$$
$$B^{0,5}(t,u) = B^{0,5}(u,t) \tag{17}$$
from descendents of $\mathrm{Tr}(F^1F^2)\mathrm{Tr}(F^3F^4)\mathrm{Tr}(F^1F^2)\mathrm{Tr}(F^3F^4)$

- $$B^{0,6}(s,u)\left[(p^1_p\epsilon^1_q - p^1_q\epsilon^1_p)(p^4_p\epsilon^4_q - p^4_q\epsilon^4_p)(p^2_r\epsilon^2_s - p^2_s\epsilon^2_r)(p^3_r\epsilon^3_s - p^3_s\epsilon^3_r)\right.$$
$$\left.(p^1_a\epsilon^1_b - p^1_b\epsilon^1_a)(p^2_a\epsilon^2_b - p^2_b\epsilon^2_a)(p^3_c\epsilon^3_d - p^3_d\epsilon^3_c)(p^4_c\epsilon^4_d - p^4_d\epsilon^4_c)\right]$$
$$+ B^{0,6}(t,u)[3 \leftrightarrow 2] + B^{0,6}(s,t)[3 \leftrightarrow 4] \tag{18}$$

$$B^{0,6}(s,u) = B^{0,6}(u,s) \tag{19}$$

from descendents of $\mathrm{Tr}(F^1F^2)\mathrm{Tr}(F^3F^4)\mathrm{Tr}(F^1F^4)\mathrm{Tr}(F^2F^3)$

- This completes the listing of the S matrices of denscendents of 6 and 8 derivative terms. We now turn to the listing of S matrices that follow from descendents of the two 10 derivative and one 12 derivative terms.

# Explicit prameterization of the general 4 graviton S matrix: 5

- $$+\left(B^{2,1}(s,u)\,(p^1_p\epsilon^1_q - p^1_q\epsilon^1_p)\,(p^2_q\epsilon^2_r - p^2_r\epsilon^2_q)\,(p^3_r\epsilon^3_s - p^3_s\epsilon^3_r)\,(p^4_s\epsilon^4_p - p^4_p\epsilon^4_s)\right.$$
$$B^{2,1}(t,u)\,(p^1_p\epsilon^1_q - p^1_q\epsilon^1_p)\,(p^3_q\epsilon^3_r - p^3_r\epsilon^3_q)\,(p^2_r\epsilon^2_s - p^2_s\epsilon^2_r)\,(p^4_s\epsilon^4_p - p^4_p\epsilon^4_s)$$
$$\left.+B^{2,1}(t,s)\,(p^1_p\epsilon^1_q - p^1_q\epsilon^1_p)\,(p^3_q\epsilon^3_r - p^3_r\epsilon^3_q)\,(p^4_r\epsilon^4_s - p^4_s\epsilon^4_r)\,(p^2_s\epsilon^2_p - p^2_p\epsilon^2_s)\right)$$
$$\left((p^1_a\epsilon^1_b - p^1_b\epsilon^1_a)\,p^2_a\,(p^2_\mu\epsilon^2_\nu - p^2_\nu\epsilon^2_\mu)\,p^3_b\,(p^3_\nu\epsilon^3_\alpha - p^3_\alpha\epsilon^3_\nu)\,(p^4_\alpha\epsilon^4_\mu - p^4_\mu\epsilon^4_\alpha)\right.$$
$$+(p^2_a\epsilon^2_b - p^2_b\epsilon^2_a)\,p^1_a\,(p^1_\mu\epsilon^1_\nu - p^1_\nu\epsilon^1_\mu)\,p^4_b\,(p^4_\nu\epsilon^4_\alpha - p^4_\alpha\epsilon^4_\nu)\,(p^3_\alpha\epsilon^3_\mu - p^3_\mu\epsilon^3_\alpha)$$
$$+(p^3_a\epsilon^3_b - p^3_b\epsilon^3_a)\,p^4_a\,(p^4_\mu\epsilon^4_\nu - p^4_\nu\epsilon^4_\mu)\,p^1_b\,(p^1_\nu\epsilon^1_\alpha - p^1_\alpha\epsilon^1_\nu)\,(p^2_\alpha\epsilon^2_\mu - p^2_\mu\epsilon^2_\alpha)$$
$$\left.+(p^4_a\epsilon^4_b - p^4_b\epsilon^4_a)\,p^3_a\,(p^3_\mu\epsilon^3_\nu - p^3_\nu\epsilon^3_\mu)\,p^2_b\,(p^2_\nu\epsilon^2_\alpha - p^2_\alpha\epsilon^2_\nu)\,(p^1_\alpha\epsilon^1_\mu - p^1_\mu\epsilon^1_\alpha)\right) \quad (20)$$

$$B^{2,1}(s,u) = B^{2,1}(u,s) \quad (21)$$

from descendents of $\mathrm{Tr}(F^1F^2F^3F^4)F^1_{ab}\mathrm{Tr}(p^2_aF^2p^3_bF^3F^4)$.

# Explicit parameterization of the general 4 graviton S matrix: 6

- $$\begin{aligned}
&\left(B^{2,2}(t,u)\,(p^1_p\epsilon^1_q - p^1_q\epsilon^1_p)\,(p^2_p\epsilon^2_q - p^2_q\epsilon^2_p)\,(p^3_r\epsilon^3_s - p^3_s\epsilon^3_r)\,(p^4_r\epsilon^4_s - p^4_s\epsilon^4_r)\right.\\
&+ B^{2,2}(s,u)\,(p^1_p\epsilon^1_q - p^1_q\epsilon^1_p)\,(p^3_p\epsilon^3_q - p^3_q\epsilon^3_p)\,(p^2_r\epsilon^2_s - p^2_s\epsilon^2_r)\,(p^4_r\epsilon^4_s - p^4_s\epsilon^4_r)\\
&\left.+ B^{2,2}(t,s)\,(p^1_p\epsilon^1_q - p^1_q\epsilon^1_p)\,(p^4_p\epsilon^4_q - p^4_q\epsilon^4_p)\,(p^3_r\epsilon^3_s - p^3_s\epsilon^3_r)\,(p^2_r\epsilon^2_s - p^2_s\epsilon^2_r)\right)\\
&\left((p^1_a\epsilon^1_b - p^1_b\epsilon^1_a)\,p^2_a\,(p^2_\mu\epsilon^2_\nu - p^2_\nu\epsilon^2_\mu)\,p^3_b\,(p^3_\nu\epsilon^3_\alpha - p^3_\alpha\epsilon^3_\nu)\,(p^4_\alpha\epsilon^4_\mu - p^4_\mu\epsilon^4_\alpha)\right.\\
&+ (p^2_a\epsilon^2_b - p^2_b\epsilon^2_a)\,p^1_a\,(p^1_\mu\epsilon^1_\nu - p^1_\nu\epsilon^1_\mu)\,p^4_b\,(p^4_\nu\epsilon^4_\alpha - p^4_\alpha\epsilon^4_\nu)\,(p^3_\alpha\epsilon^3_\mu - p^3_\mu\epsilon^3_\alpha)\\
&+ (p^3_a\epsilon^3_b - p^3_b\epsilon^3_a)\,p^4_a\,(p^4_\mu\epsilon^4_\nu - p^4_\nu\epsilon^4_\mu)\,p^1_b\,(p^1_\nu\epsilon^1_\alpha - p^1_\alpha\epsilon^1_\nu)\,(p^2_\alpha\epsilon^2_\mu - p^2_\mu\epsilon^2_\alpha)\\
&\left.+ (p^4_a\epsilon^4_b - p^4_b\epsilon^4_a)\,p^3_a\,(p^3_\mu\epsilon^3_\nu - p^3_\nu\epsilon^3_\mu)\,p^2_b\,(p^2_\nu\epsilon^2_\alpha - p^2_\alpha\epsilon^2_\nu)\,(p^1_\alpha\epsilon^1_\mu - p^1_\mu\epsilon^1_\alpha)\right)
\end{aligned} \tag{22}$$

$$B^{2,2}(t,u) = B^{2,2}(u,t) \tag{23}$$

from descendents of $\mathrm{Tr}(F^1F^2)\mathrm{Tr}(F^3F^4)F^1_{ab}\mathrm{Tr}(p^2_aF^2p^3_bF^3F^4)$

# Explicit prameterization of the general 4 graviton S matrix:7

- $$\begin{aligned}(B^{4,1}(s,t)+B^{4,1}(t,u)+B^{4,1}(u,s))\times\\ \left[\left(p^1_a\epsilon^1_b-p^1_b\epsilon^1_a\right)p^2_a\left(p^2_\mu\epsilon^2_\nu-p^2_\nu\epsilon^2_\mu\right)p^3_b\left(p^3_\nu\epsilon^3_\alpha-p^3_\alpha\epsilon^3_\nu\right)\left(p^4_\alpha\epsilon^4_\mu-p^4_\mu\epsilon^4_\alpha\right)\right.\\ \left(p^1_p\epsilon^1_q-p^1_q\epsilon^1_p\right)p^2_p\left(p^2_\beta\epsilon^2_\gamma-p^2_\gamma\epsilon^2_\beta\right)p^3_q\left(p^3_\gamma\epsilon^3_\delta-p^3_\delta\epsilon^3_\gamma\right)\left(p^4_\delta\epsilon^4_\beta-p^4_\beta\epsilon^4_\delta\right)\\ \left.+(1\leftrightarrow 2)+(1\leftrightarrow 3)+(1\leftrightarrow 4)\right]\end{aligned} \tag{24}$$

  $$B^{4,1}(s,t)=B^{4,1}(u,t)=B^{4,1}(t,s)=B^{4,1}(u,s)=B^{4,1}(s,u)=B^{4,1}(t,u) \tag{25}$$

  from descendents of
  $F^1_{pq}\mathrm{Tr}(p^2_pF^2p^3_qF^3F^4)F^1_{ab}\mathrm{Tr}(p^2_aF^2p^3_bF^3F^4)$

# Detail: Examples of use of 4 photon scattering parameterization

- The tree level scattering of 4 photons in type 1 theory (or in type II theory on D branes) has a single index structure - the structure that follows from the Lagrangian structure

$$L_{4V}^{ss} \propto \frac{1}{16}\left(\mathrm{Tr}(F^4) - \frac{1}{4}(\mathrm{Tr}(F^2))^2\right) \tag{26}$$

  which itself can be obtained by expanding the Born Infeld action to quartic order in $F_{\mu\nu}$. Consequently this scattering amplitude can be cast into our general form with $A^{2,1} = 0$ and $A^{0,2} = -\frac{1}{4}A^{0,1}$. The actual expression for $A^{0,1}$ is a well knonwn Veneziano type function.
- We have also recast the formula for tree level scattering in the open bosonic string into our general form. The final result is more complicated - and we do not write it here, but simply note that it involves all three of our structures.

# Detail: Examples of Use of the S matrix

- Gravitons. The 4 graviton S matrix from the Einstein Lagrangian, which is given by

$$A^{EG}_{4h} = \frac{-4\kappa^2}{stu}\Big(\frac{1}{2}\epsilon_2.\epsilon_3\left(s\epsilon_1.k_3\epsilon_4.k_2 + t\epsilon_1.k_2\epsilon_4.k_3\right) + \frac{1}{2}\epsilon_1.\epsilon_4\left(s\epsilon_2.k_4\epsilon_3.k_1 + t\epsilon_2.k_1\epsilon_3.k_4\right)$$
$$+\frac{1}{2}\epsilon_2.\epsilon_4\left(s\epsilon_1.k_4\epsilon_3.k_2 + u\epsilon_1.k_2\epsilon_3.k_4\right) + \frac{1}{2}\epsilon_1.\epsilon_3\left(s\epsilon_2.k_3\epsilon_4.k_1 + u\epsilon_2.k_1\epsilon_4.k_3\right)$$
$$+\frac{1}{2}\epsilon_3.\epsilon_4\left(t\epsilon_1.k_4\epsilon_2.k_3 + u\epsilon_1.k_3\epsilon_2.k_4\right) + \frac{1}{2}\epsilon_1.\epsilon_2\left(t\epsilon_3.k_2\epsilon_4.k_1 + u\epsilon_3.k_1\epsilon_4.k_2\right)$$
$$-\frac{1}{4}st\epsilon_1.\epsilon_4\epsilon_2.\epsilon_3 - \frac{1}{4}su\epsilon_1.\epsilon_3\epsilon_2.\epsilon_4 - \frac{1}{4}tu\epsilon_1.\epsilon_2\epsilon_3.\epsilon_4\Big)^2 \quad (27)$$

- This turns out to be proportional to $\frac{1}{stu}$ times the S matrix generated the Lagrangian

$$L^{EG}_{4h} \propto \frac{1}{32}(R_{pqrs}R^{pqrs})^2 - \frac{1}{2}R^{pqrs}R_{pqr}{}^{t}R^{uvw}{}_{s}R_{uvwt} + \frac{1}{16}R^{pqrs}R_{pq}{}^{tu}R_{tu}{}^{vw}R_{rsvw}$$
$$-\frac{1}{4}R^{pqrs}R_{pq}{}^{tu}R_{rt}{}^{vw}R_{suvw} - R^{pqrs}R_{p}{}^{t}{}_{r}{}^{u}R_{tvws}R_{q}{}^{v}{}_{u}{}^{w} + \frac{1}{2}R^{pqrs}R_{p}{}^{t}{}_{r}{}^{u}R_{t}{}^{v}{}_{u}{}^{w}R_{qvsw} \quad (28)$$

which, in turn, is easily written as a linear combination of the six 4 Reimann structures listed above.

# Detail: Examples of use of the 4 graviton S matrix

- Next, the 4-graviton amplitude in Type *II* superstring theory is proportional (in the sense of index structure) to the S matrix for Einstein gravity, and so can also be easily written in our basis.

$$A_{4h}^{ss} = h(s,t,u,\alpha')A_{4h}^{EG} \tag{29}$$

- The tree level S matrices for the heterotic string and the bosonic string are more complicated, but also can each be written as a linear combination of the last 9 structures we discussed above.
- The first structure - descendents of the 6 derivative term - never appears in tree level string amplitudes. It would be interesting to check whether this structure appears in string loop amplitudes. We have not yet tried this.

# Detail: Implications of CRG: contact scalar interactions

- It is easy to verify that there are exactly three local scalar S matrices that obey the conjectured bound on growth of Regge amplitudes
- These S matrices and their corresponding Lagrangian structures are

$$a_0 + a_2(st + tu + us) + a_3 stu \tag{30}$$

They come from the local Lagrangian

$$a_0\phi^4 + a_2\left(\partial_\mu\partial_\nu\phi\partial_\mu\phi\partial_\nu\phi\phi\right) + a_3\left(\partial_\mu\partial_\nu\partial_\alpha\phi\partial_\mu\phi\partial_\nu\phi\partial_\alpha\phi\right) \tag{31}$$

- (31) is also precisely the terms that characterize that part of the 4 point function that is undetermined by Caron Huot's formula (see e.g. a paper by Zhibedeov and Turaci from 2 years ago). This is not a concidence, as the chaos bound was a key phyiscal input into Caron Huot's formula.

# Detail: Implications of CRG: Contact photon interactions

- It is not difficult to use our explicit parameterization of photon S matrices to ennumerate all local photon 4 point S matrices that grow no faster than $s^2$ in the Regge limit.
- We find these are given in terms of four constants $a$, $b$, $c$ and $d$ by $A^{01}(t,u) = a$, $A^{0,2}(t,u) = b + cs$, $A^{1,0} = d$ corresponding to the four parameter set of Lagrangians

$$a(\mathrm{Tr}F^2)^2 + b\mathrm{Tr}F^4 + c\mathrm{Tr}\left(\partial_\mu FF\partial_\mu FF\right) + dF_{ab}\mathrm{Tr}\left(\partial_a F\partial_b FF\right) \tag{32}$$

- Note that this allowed set of Lagrangians includes the expansion of the Born Infeld action to quadratic order
- In analogy with the scalar case we expect (32) to parameterize the ambiguity in the large $N$ version of Caron-Huots formula for vectors.

# Detail: Non Polynomial S matrices

- As a brief aside we note that our final expression for the most general polynomial S matrix is parameterized by collection of polynomials of $s, t, u$. These polynomials are required to have certain specified symmetry properties under permutations of $s$, $t$ and $u$ but are otherwise arbitrary.
- Simply replacing the polynomials in the parameterizations above by the most general functions of $s$, $t$ and $u$ with the same symmetry properties yields a kinematical parameterization of the most general 4 graviton S matrix (classical or quantum).
- In particular all the pole exchange S matrices described earlier in this talk can also be recast in this form, with suitable choices for the arbitrary functions.

# Detail: Massive exchanges: Examples

- Recall that the Einstein graviton exchange contribution to graviton scattering grew like $s^2$. This fact might make us suspect that there exists a massive spin to contribution to this process that also grows in the same manner.
- Remarkably enough, however, it turns out that there are 2 (rather than 3) couplings of 2 gravitons to massive spin two particles. The couplings in question are the analogues of the 4 and 6 derivative couplings, and always lead to exchange contributions to graviton scattering that grows faster than $s^2$
- It is also easy to verify that other simple examples of exchange scattering contributions - like contributions due to the exchage of scalar particles - also violate the CRG conjecture in a way that cannot be compensated for by local counterterms.