Nonperturbative Mellin Amplitudes

EPFL

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arXiv:1912.11100 with **Joao Silva** and **Alexander Zhiboedov**
Work in progress with **Dean Carmi**, **Joao Silva** and **Alexander Zhiboedov**

Strings 2020
29 June - 03 July | Cape Town | South Africa

Motivation

QFT or Quantum Gravity

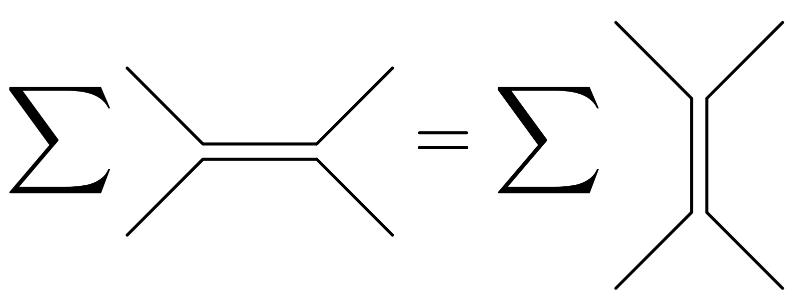
Anti-de Sitter | Minkowski

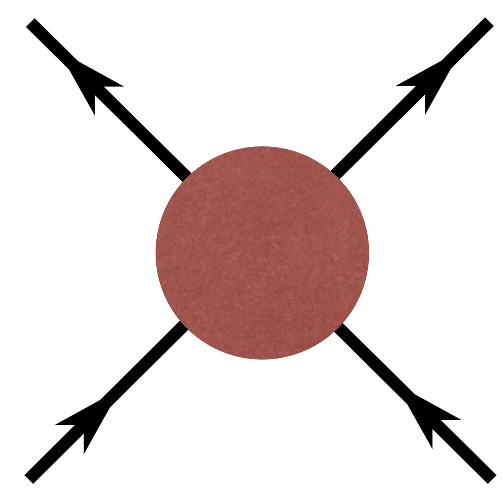
Conformal Bootstrap | **S-matrix Bootstrap**

Mellin Amplitudes $\xrightarrow[\text{Limit}]{\text{Flat Space}}$ Scattering Amplitudes

$\Updownarrow$

Boundary Correlators





Motivation

Constraining EFTs in AdS

Does any EFT in AdS_{d+1} can be UV completed into a consistent CFT_d?

YES | **NO**

Garden conjecture

Swampland conjecture

CFT sum rules give new handle into this question.

Motivation

Conformal Bootstrap in Mellin Space

Can we setup a more efficient conformal bootstrap using Mellin amplitudes?

Mellin Amplitudes are very useful for perturbative computations (holographic correlators, perturbative CFTs, O(N)-model, event shapes,…).

Are Mellin amplitudes useful **nonperturbatively**?

Outline

1. Existence and Analyticity
2. Non-perturbative CFT sum rules
    - Regge boundedness
    - Polyakov conditions
    - Dispersion relations
3. Applications
    - EFTs in AdS
    - ϵ-expansion

Mellin Amplitude Basics

Mellin Amplitude Basics

Correlation function of scalar primary operators

$$A(x_i) = \langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle$$

['09 Mack]

$$A(x_i) = \int [d\gamma] \boxed{M(\gamma_{ij})} \prod_{i<j}^{n} \Gamma(\gamma_{ij}) \left(x_{ij}^2\right)^{-\gamma_{ij}}$$

Constraints:

$$\sum_{j=1}^{n} \gamma_{ij} = 0 \ , \qquad \gamma_{ii} = -\Delta_i = -\mathrm{dim}[\mathcal{O}_i] \ , \qquad \gamma_{ij} = \gamma_{ji}$$

integration variables = # independent cross-ratios

Mellin Amplitude Basics

Four-point function of identical operators

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle = \frac{F(u,v)}{(x_{13}^2 x_{24}^2)^{\Delta}}$$

cross ratios

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$$

$$v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$$

$$F(u,v) = \int_{\mathcal{C}} \frac{d\gamma_{12} d\gamma_{14}}{(2\pi i)^2}\, u^{-\gamma_{12}} v^{-\gamma_{14}} \underbrace{\Gamma(\gamma_{12})^2 \Gamma(\gamma_{14})^2 \Gamma(\Delta - \gamma_{12} - \gamma_{14})^2 M(\gamma_{12},\gamma_{14})}_{\hat{M}(\gamma_{12},\gamma_{14})}$$

Crossing symmetry

$$F(u,v) = F(v,u) = v^{-\Delta} F\left(\frac{u}{v}, \frac{1}{v}\right) \iff M(\gamma_{12},\gamma_{14}) = M(\gamma_{14},\gamma_{12}) = M(\gamma_{12},\gamma_{13})$$

$$\gamma_{12} + \gamma_{13} + \gamma_{14} = \Delta$$

1.
Existence and Analyticity

Existence and Analyticity

$$F(u,v) = \int_{c-i\infty}^{c+i\infty} \frac{d\gamma_{12} d\gamma_{14}}{(2\pi i)^2} u^{-\gamma_{12}} v^{-\gamma_{14}} \hat{M}(\gamma_{12},\gamma_{14})$$

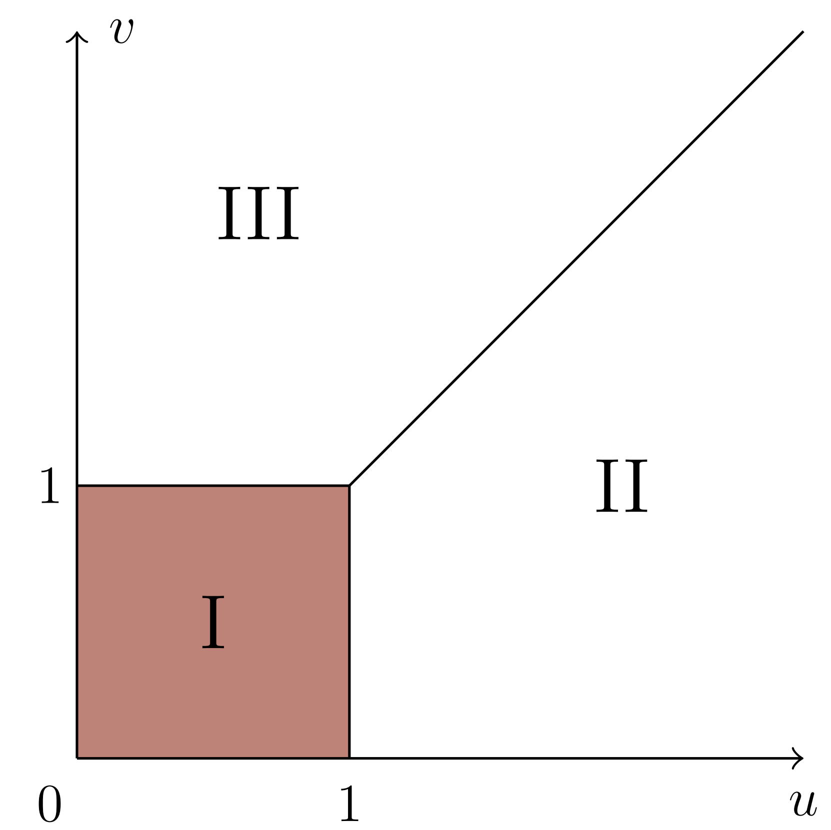
$$\Rightarrow \qquad \hat{M}(\gamma_{12},\gamma_{14}) = \int_0^\infty \frac{du dv}{uv} u^{\gamma_{12}} v^{\gamma_{14}} F(u,v)$$

Divergent integral

Idea: split the integral in 3 regions

$$K(\gamma_{12},\gamma_{14}) = \int_0^1 \frac{du dv}{uv} u^{\gamma_{12}} v^{\gamma_{14}} F(u,v)$$

Analytic for $\begin{array}{l} \mathrm{Re}\,\gamma_{12} > \Delta \\ \mathrm{Re}\,\gamma_{14} > \Delta \end{array}$

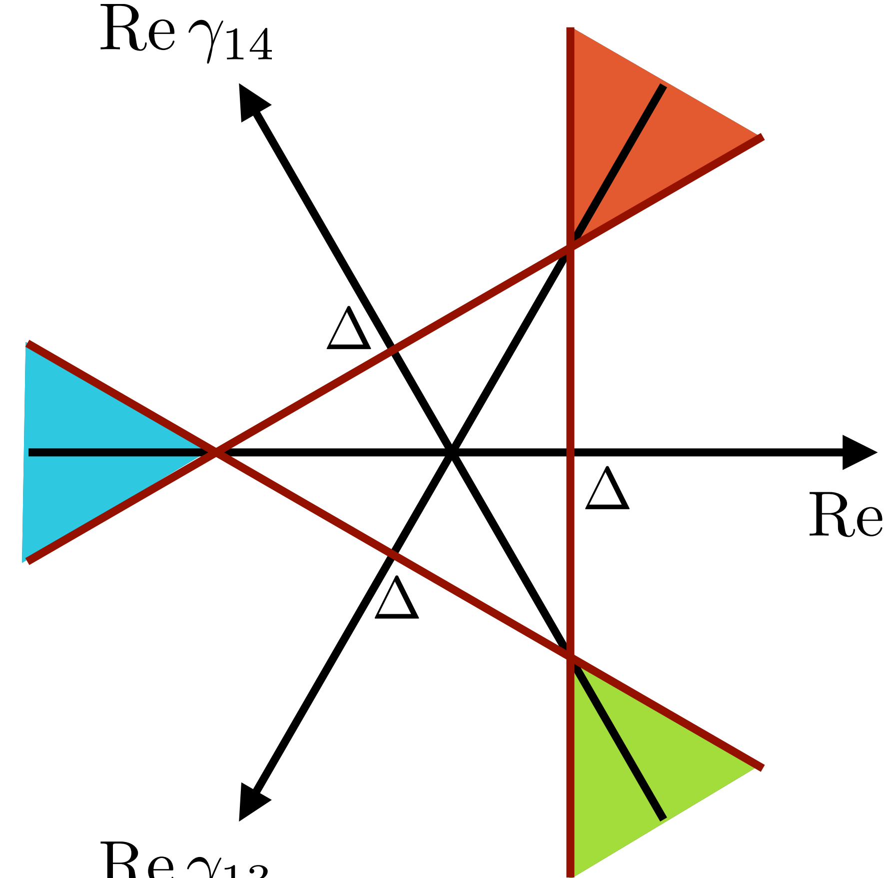


Existence and Analyticity

$$
\begin{aligned}
F(u,v) &= \int_{\mathrm{Re}(\gamma_{12},\gamma_{14})>\Delta} \frac{d\gamma_{12}d\gamma_{14}}{(2\pi i)^2} u^{-\gamma_{12}} v^{-\gamma_{14}} K(\gamma_{12},\gamma_{14}) && \text{I} \\
&+ \int_{\mathrm{Re}(\gamma_{13},\gamma_{14})>\Delta} \frac{d\gamma_{12}d\gamma_{14}}{(2\pi i)^2} u^{-\gamma_{12}} v^{-\gamma_{14}} K(\gamma_{13},\gamma_{14}) && \text{II} \\
&+ \int_{\mathrm{Re}(\gamma_{12},\gamma_{13})>\Delta} \frac{d\gamma_{12}d\gamma_{14}}{(2\pi i)^2} u^{-\gamma_{12}} v^{-\gamma_{14}} K(\gamma_{12},\gamma_{13}) && \text{III}
\end{aligned}
$$

region

with $\gamma_{12} + \gamma_{13} + \gamma_{14} = \Delta$



If not for **different** contours

$$\hat{M}(\gamma_{12},\gamma_{14}) = K(\gamma_{12},\gamma_{14}) + K(\gamma_{13},\gamma_{14}) + K(\gamma_{12},\gamma_{13})$$

Existence and Analyticity

Strategy: analytically continue K-function to bring the 3 integrals to the same contour.

$$\Rightarrow \quad \boxed{\hat{M}(\gamma_{12},\gamma_{14}) = K(\gamma_{12},\gamma_{14}) + K(\gamma_{13},\gamma_{14}) + K(\gamma_{12},\gamma_{13})}$$

$$K(\gamma_{12},\gamma_{14}) = \int_0^1 \frac{du dv}{uv} u^{\gamma_{12}} v^{\gamma_{14}} F(u,v)$$

Example: Mean Field Theory

$$F(u,v) = 1 + u^{-\Delta} + v^{-\Delta}$$

$$\Rightarrow K(\gamma_{12},\gamma_{14}) = \frac{1}{\gamma_{12}\gamma_{14}} + \frac{1}{\gamma_{12}(\gamma_{14}-\Delta)} + \frac{1}{(\gamma_{12}-\Delta)\gamma_{14}} \quad \Rightarrow \quad \hat{M}(\gamma_{12},\gamma_{14}) = 0$$

Disconnected correlator does not contribute to Mellin amplitude.

Analyticity of K-function

Strategy: analytically continue K-function to bring the 3 integrals to the same contour.

$$\Rightarrow \boxed{\hat{M}(\gamma_{12},\gamma_{14}) = K(\gamma_{12},\gamma_{14}) + K(\gamma_{13},\gamma_{14}) + K(\gamma_{12},\gamma_{13})}$$

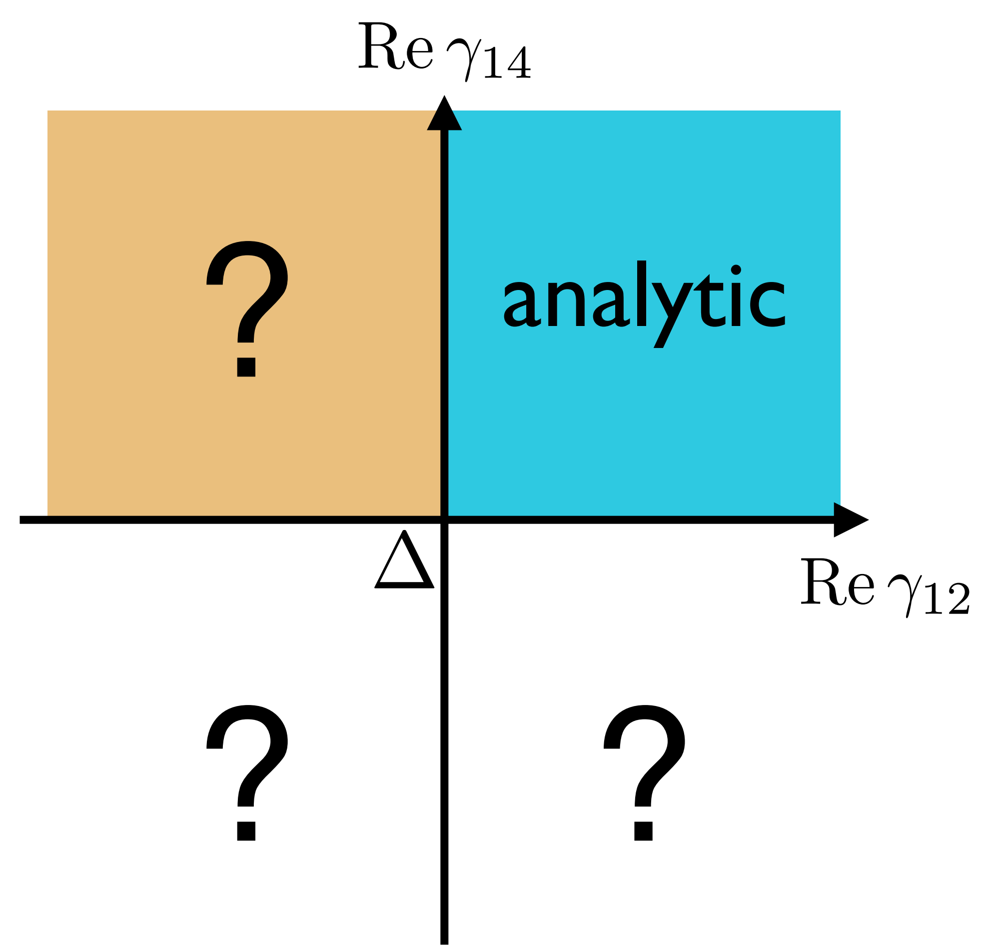
$$K(\gamma_{12},\gamma_{14}) = \int_0^1 \frac{du dv}{uv} u^{\gamma_{12}} v^{\gamma_{14}} F(u,v)$$

Operator Product Expansion (lightcone)

$$F(u,v) = \sum_{J,\tau\geq 0} C^2_{\tau,J} \underbrace{g_{\tau,J}(v) u^{\frac{\tau}{2}-\Delta}}_{\text{collinear block}}$$

$$\mathcal{O}\times\mathcal{O} = \sum_{J,\tau\geq 0} C_{\tau,J}\mathcal{O}_{\tau,J}$$

spin twist

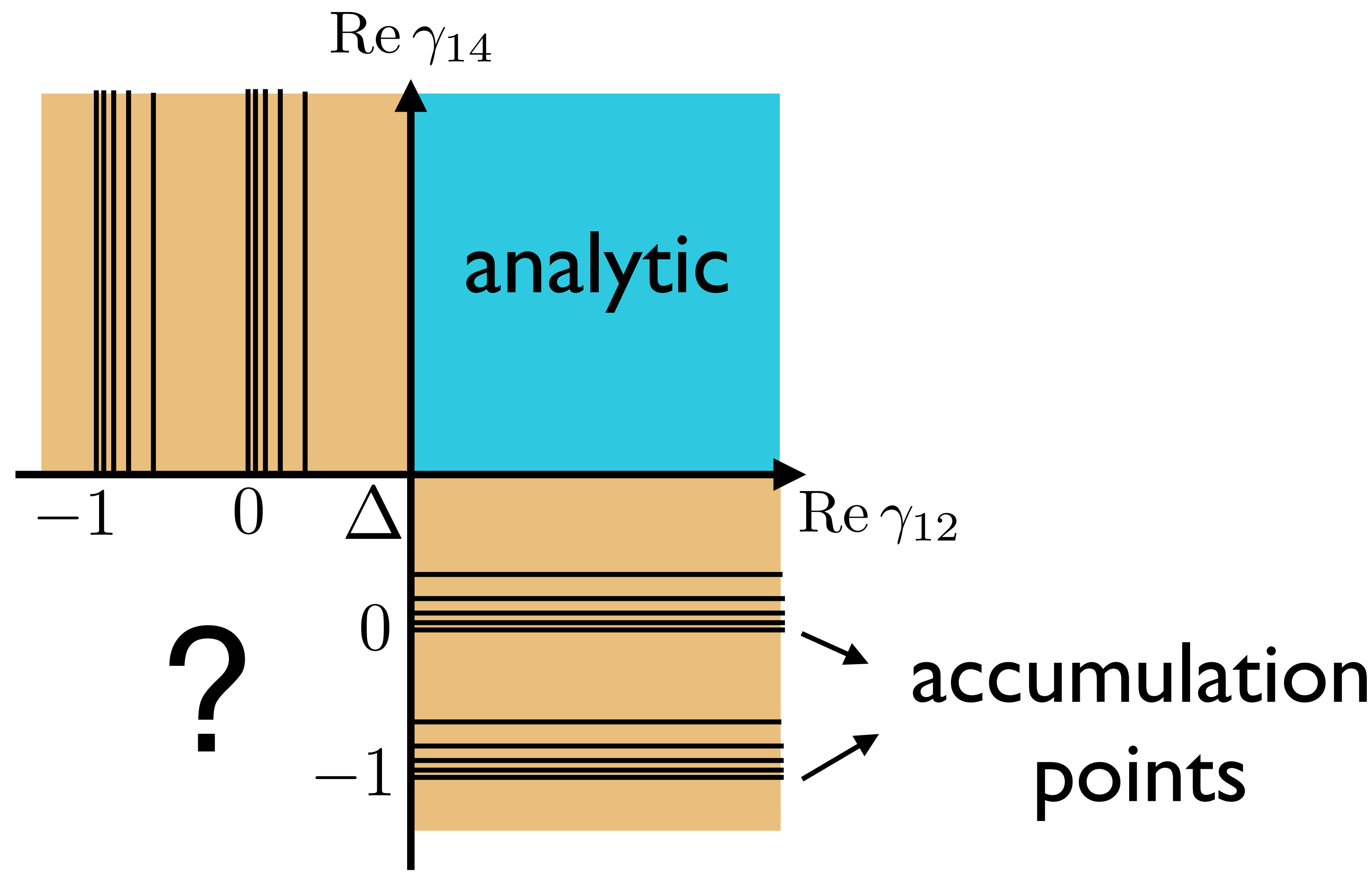


Analyticity of K-function

Poles at

$$\gamma_{12} = \Delta - \frac{\tau}{2}$$

$$\gamma_{14} = \Delta - \frac{\tau}{2}$$



Double-twist operators

$$\tau = 2\Delta + 2n + \gamma(n, J) \qquad n = 0, 1, 2, \ldots$$

$$\gamma(n,J) \xrightarrow[J \to \infty]{} 0$$

['12 Komargodski, Zhiboedov]
['12 Fitzpatrick, Kaplan, Poland, Simmons-Duffin]

Twist spectrum **discrete** or **continuous**?

It is continuous in irrational CFT_2

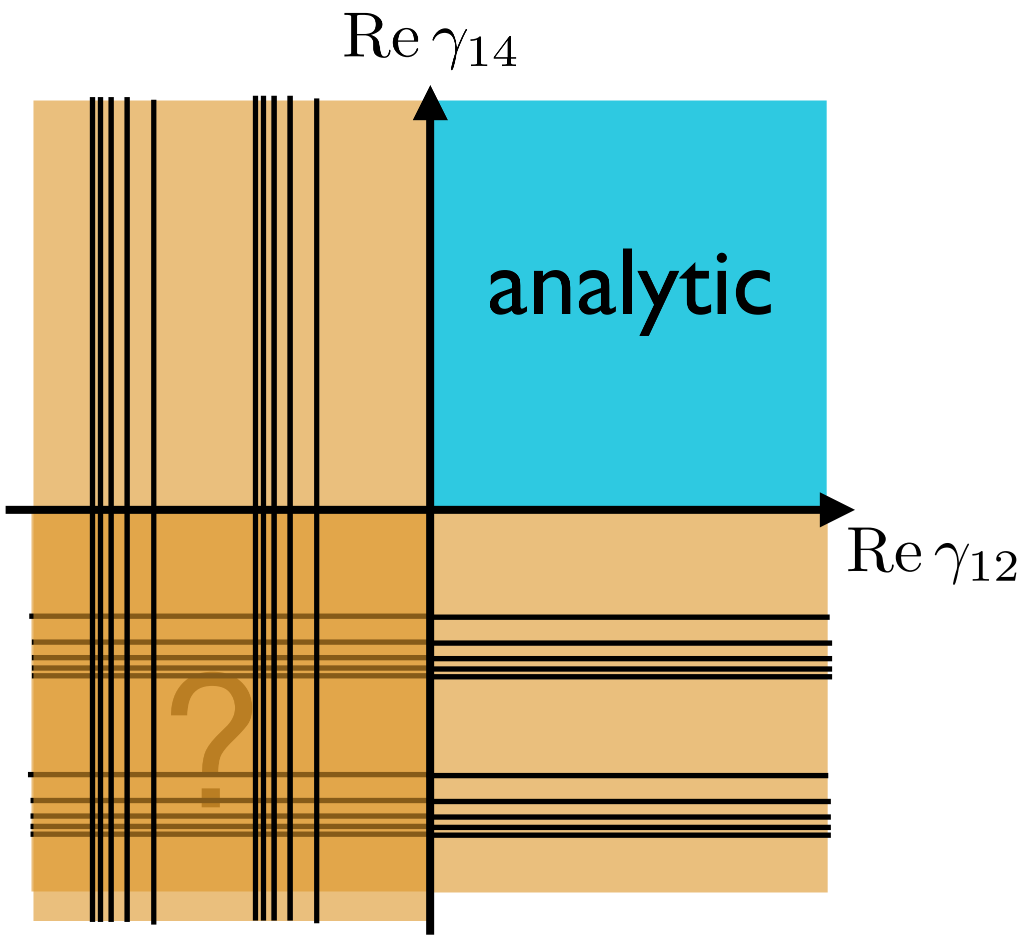
['18 Kusuki]
['18 Collier, Gobeil, Maxfield, Perlmutter]

Analyticity of K-function

Poles at

$$\gamma_{12} = \Delta - \frac{\tau}{2}$$

$$\gamma_{14} = \Delta - \frac{\tau}{2}$$



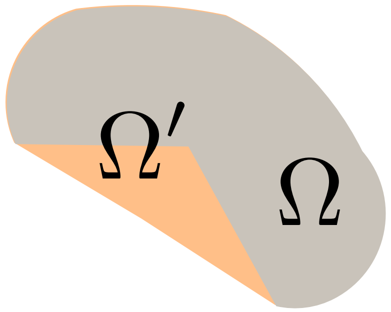
Maximal Mellin Analyticity conjecture: the OPE poles are the only singularities.

Tube theorem:

$F(x,y)$ analytic for $\mathrm{Re}(x,y) \in \Omega$

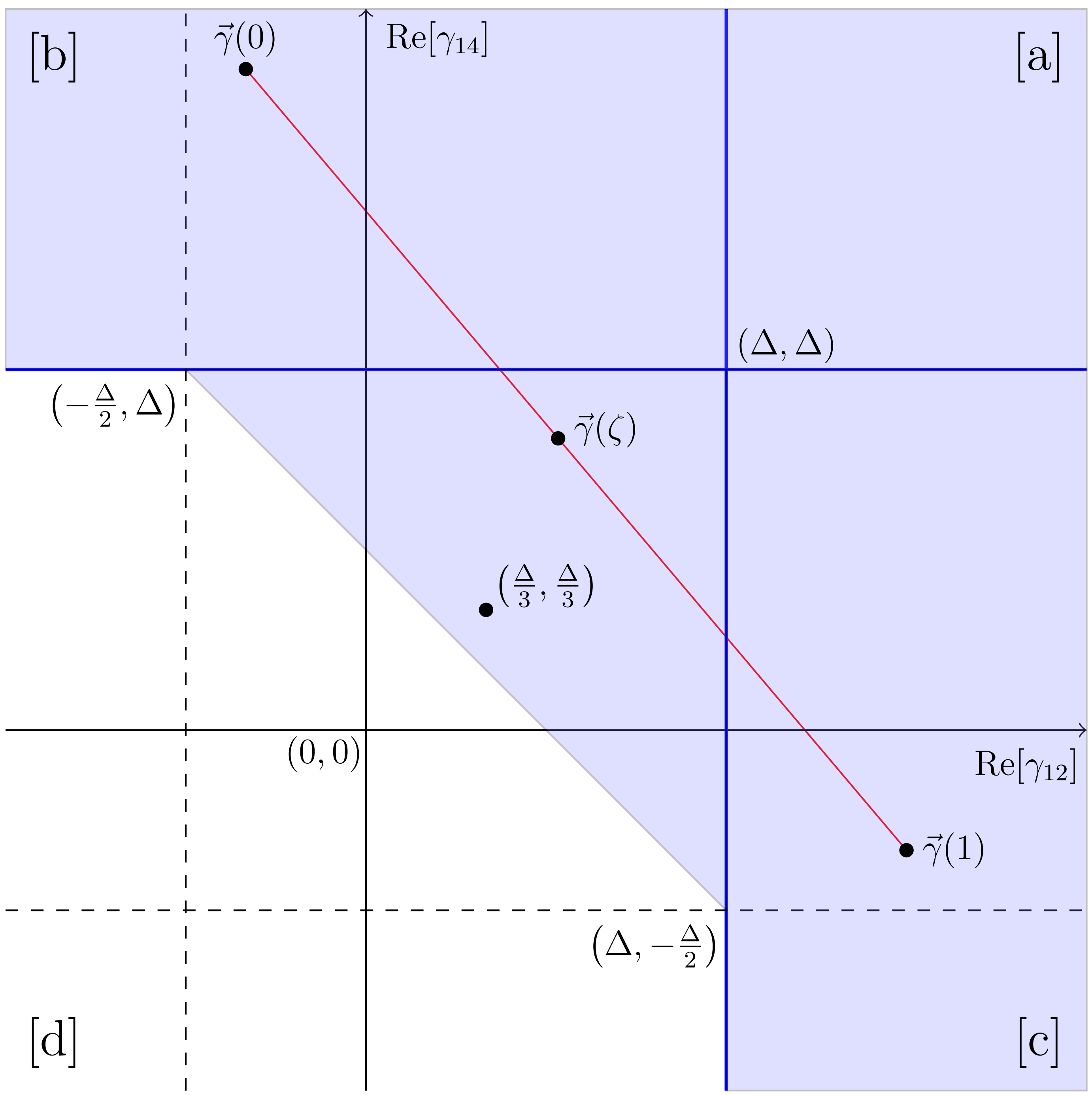
$$\Downarrow$$

$F(x,y)$ analytic for $\mathrm{Re}(x,y) \in \Omega' =$ convex hull of Ω



Analyticity of K-function

For Δ = minimal twist, we proved MMA in the following region:

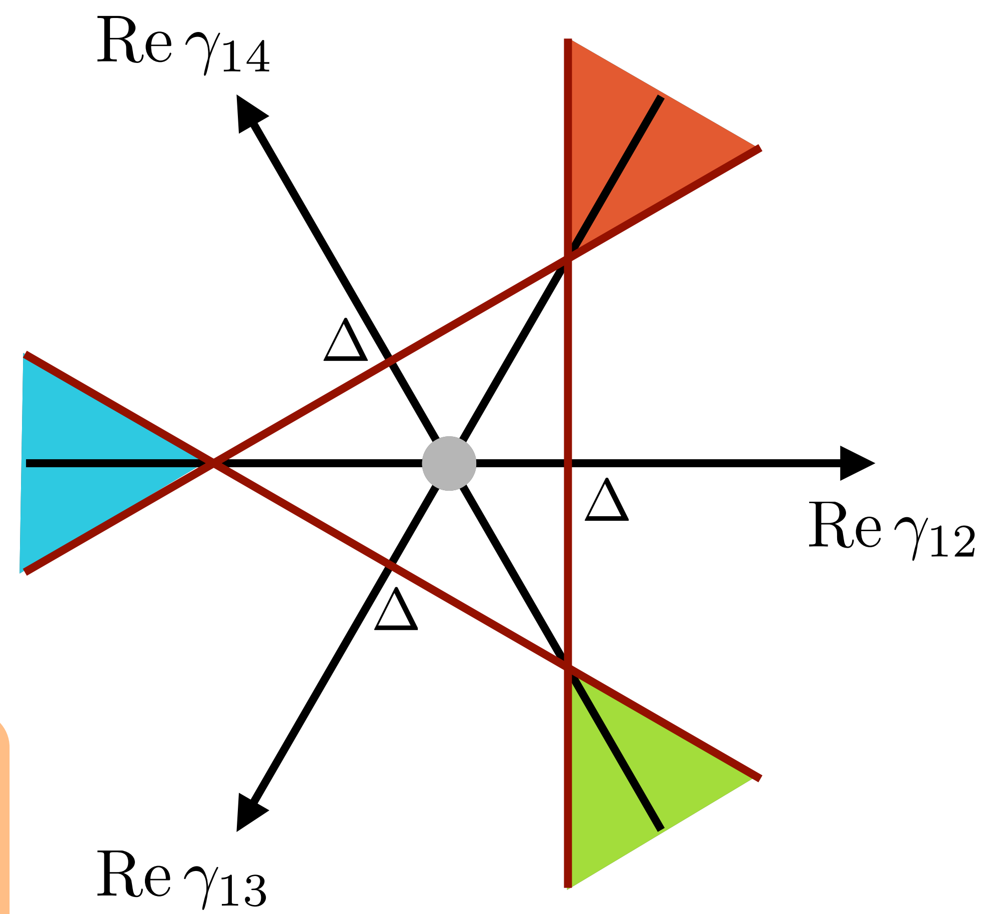


Crossing symmetric point: $\gamma_{12} = \gamma_{13} = \gamma_{14} = \dfrac{\Delta}{3}$

Straight contour formula

$$F(u,v) = \int_{\mathrm{Re}\gamma_{12}=\mathrm{Re}\gamma_{14}=\frac{\Delta}{3}} \frac{d\gamma_{12}d\gamma_{14}}{(2\pi i)^2} u^{-\gamma_{12}} v^{-\gamma_{14}} \hat{M}(\gamma_{12},\gamma_{14})$$

$$+ \sum_{J,\tau<\frac{4\Delta}{3}} C^2_{\tau,J} \left[u^{\frac{\tau}{2}-\Delta} g_{\tau,J}(v) + v^{\frac{\tau}{2}-\Delta} g_{\tau,J}(u) + v^{-\frac{\tau}{2}} g_{\tau,J}\left(\frac{u}{v}\right) \right]$$



Example: 3D Ising CFT

$$(x_{13}^2 x_{24}^2)^{\Delta} \langle \sigma(x_1)\sigma(x_2)\sigma(x_3)\sigma(x_4)\rangle_{\mathrm{connected}} =$$

$$= \int_{\mathrm{Re}\gamma_{12}=\mathrm{Re}\gamma_{14}=\frac{\Delta}{3}} \frac{d\gamma_{12}d\gamma_{14}}{(2\pi i)^2} u^{-\gamma_{12}} v^{-\gamma_{14}} \hat{M}(\gamma_{12},\gamma_{14})$$

2.
Nonperturbative CFT sum rules

Regge boundedness

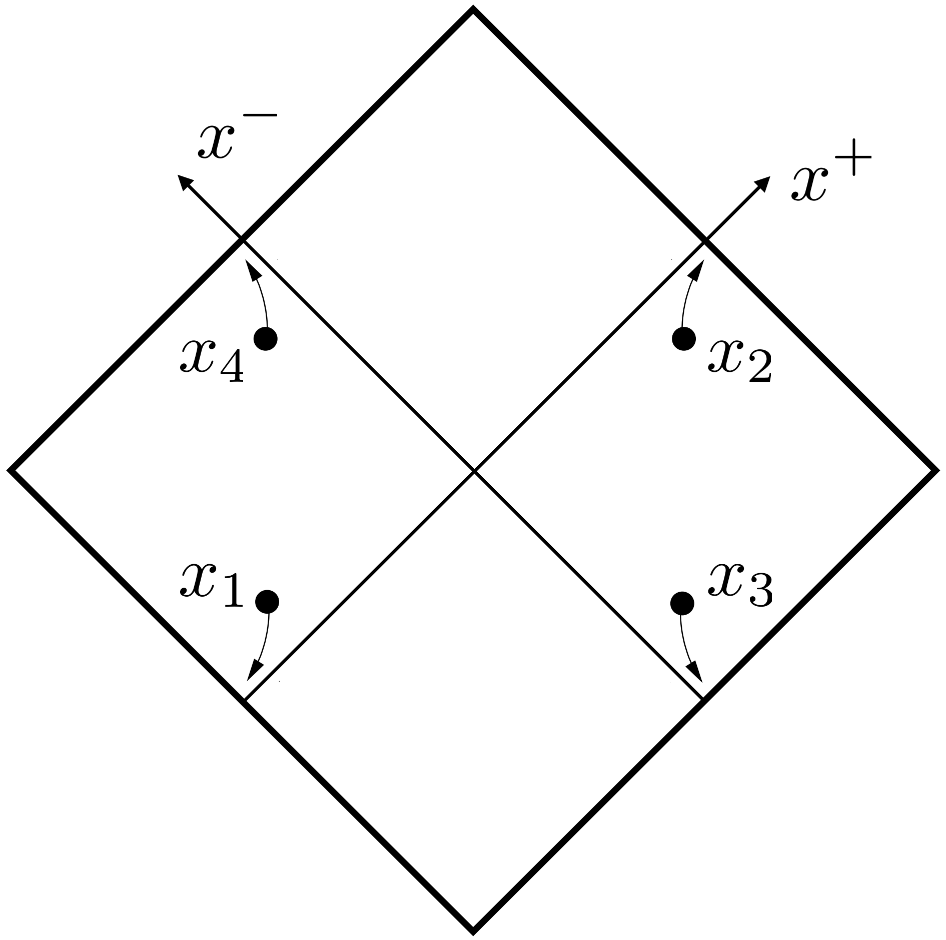
Regge boundedness

OPE + Unitarity:

$$\lim_{\substack{u\to 0\\ v\to 1}} \left| \frac{F(u, e^{2\pi i} v)}{F(u,v)} \right| \le 1$$

$$\Downarrow$$

$$\lim_{\operatorname{Im}\gamma_{12}\to\infty} M(\gamma_{12},\gamma_{14}) \le c|\gamma_{12}|$$



Conjecture:

$$\lim_{|\gamma_{12}|\to\infty} M(\gamma_{12},\gamma_{14}) \le c|\gamma_{12}| \qquad \operatorname{Re}\gamma_{14} > \Delta - \frac{\tau_{gap}}{2}$$

Polyakov conditions

Polyakov conditions

['74 Polyakov]

Absence of exact integer powers in the OPE: $(x_{12}^2)^n \qquad n = 0, 1, 2, \ldots$

$$M(\gamma_{12}, \gamma_{14}) = \frac{\hat{M}(\gamma_{12}, \gamma_{14})}{\Gamma^2(\gamma_{12})\Gamma^2(\gamma_{13})\Gamma^2(\gamma_{14})} \underset{\text{naive}}{\Longrightarrow} \quad \begin{array}{l} M(\gamma_{12}, \gamma_{14}) \xrightarrow[\gamma_{12} \to -n]{} 0 \\ \dfrac{M(\gamma_{12}, \gamma_{14})}{\gamma_{12} + n} \xrightarrow[\gamma_{12} \to -n]{} 0 \end{array}$$

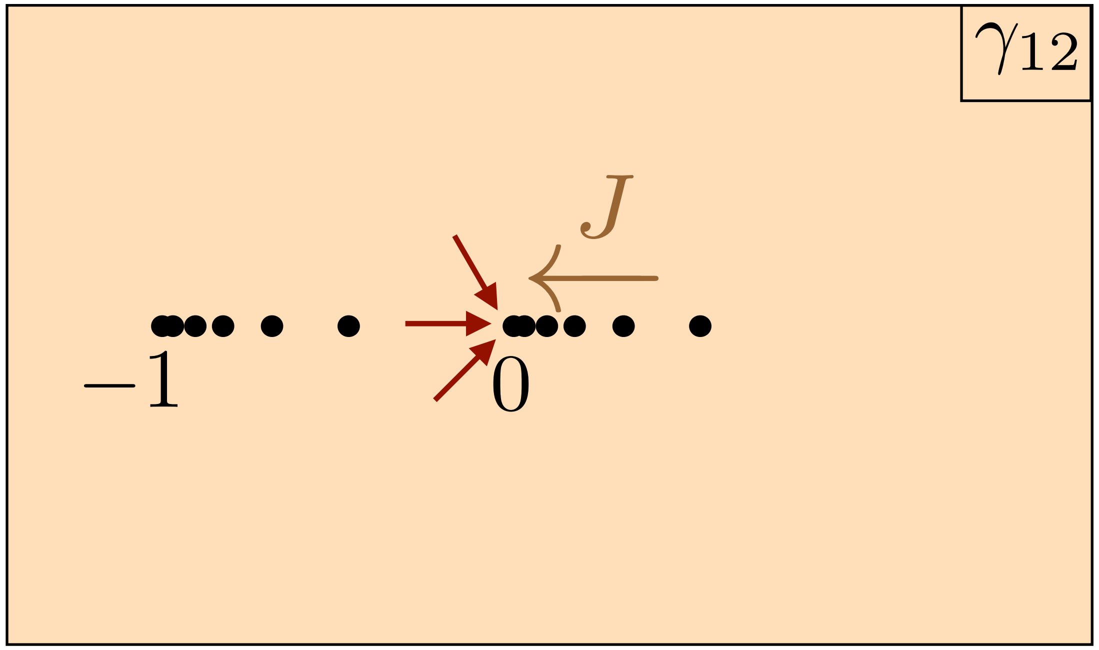
$\gamma_{12} = -n$ is an **accumulation point** of poles!

Poles at $\gamma_{12} = \Delta - \dfrac{\tau}{2}$

Double-twist operators

$$\tau = 2\Delta + 2n + \gamma(n, J) \qquad n = 0, 1, 2, \ldots$$

$$\gamma(n, J) \approx \frac{f(n)}{J^{\tau_{gap}}} \qquad J \to \infty$$



Analogous to Regge limit of **Veneziano amplitude**

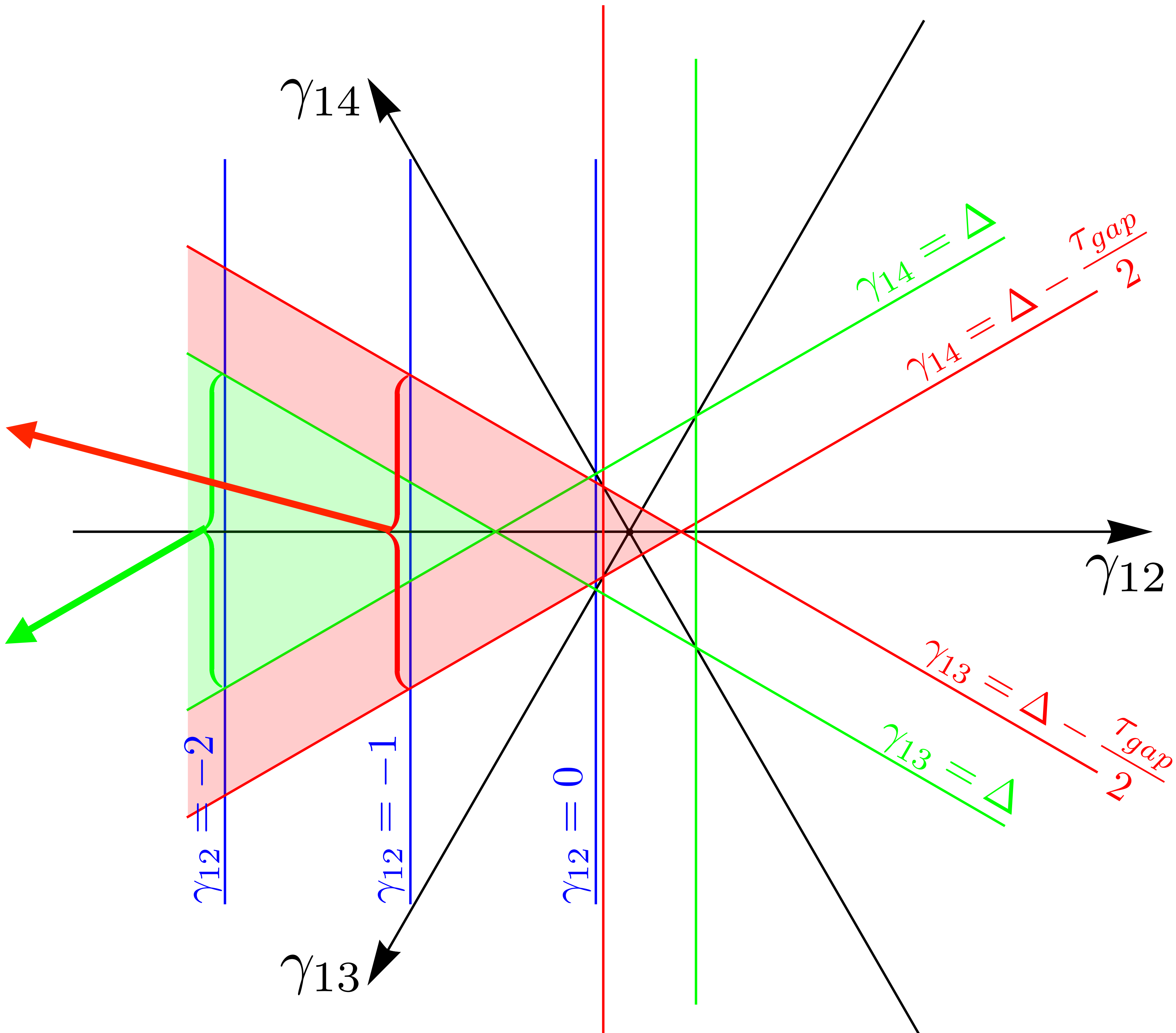
$$A(s, t) \sim s^{\alpha(t)} \qquad |s| \to \infty \qquad \arg s \neq 0, \pi$$

Polyakov conditions

$$M(\gamma_{12},\gamma_{14}) \xrightarrow[\gamma_{12}\to -n]{} (\gamma_{12}+n)^{1+2\frac{\gamma_{14}-\Delta}{\tau_{gap}}} + (\gamma_{12}+n)^{1+2\frac{\gamma_{13}-\Delta}{\tau_{gap}}} + O(\gamma_{12}+n)^2$$

$$M(\gamma_{12},\gamma_{14}) \xrightarrow[\gamma_{12}\to -n]{} 0$$

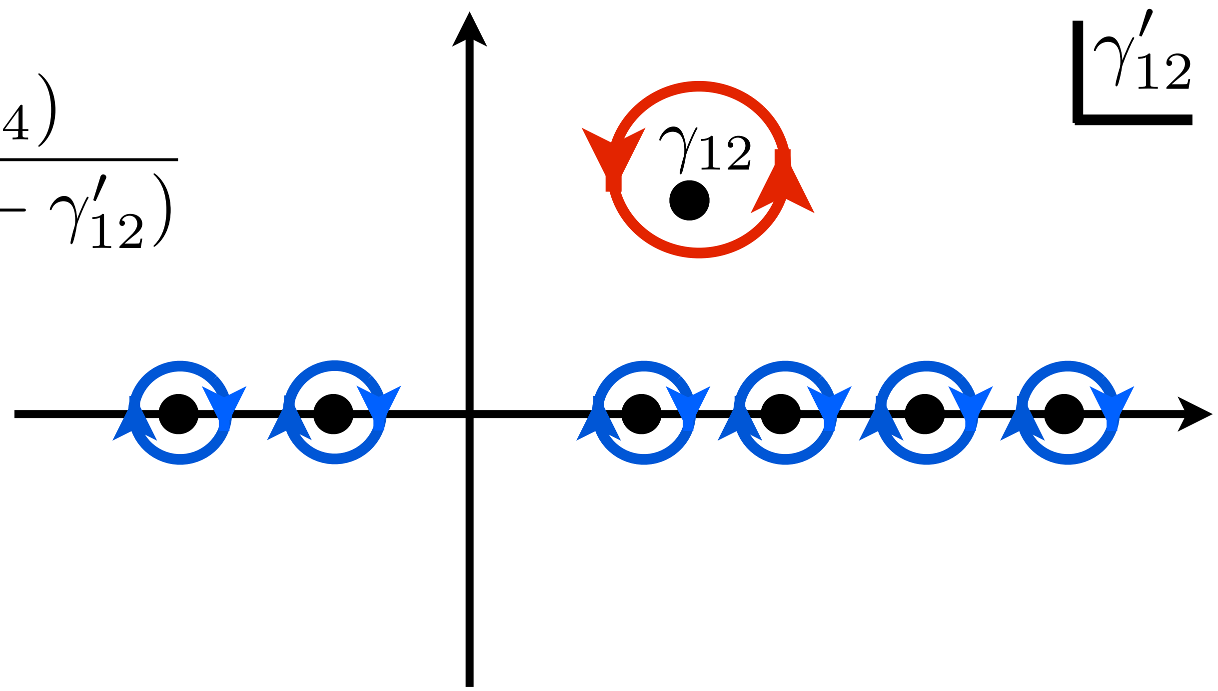
$$\frac{M(\gamma_{12},\gamma_{14})}{\gamma_{12}+n} \xrightarrow[\gamma_{12}\to -n]{} 0$$



Dispersion relations and sum rules

Dispersion relation

$$\frac{M(\gamma_{12},\gamma_{14})}{\gamma_{12}\gamma_{13}} = \oint \frac{d\gamma'_{12}}{\gamma'_{12}-\gamma_{12}} \frac{M(\gamma'_{12},\gamma_{14})}{\gamma'_{12}(\Delta-\gamma_{14}-\gamma'_{12})}$$



$$\Delta - \frac{\tau_{gap}}{2} < \gamma_{14} < \frac{\tau_{gap}}{2}$$

$\Longrightarrow$ Can use **Polyakov condition** at $\gamma_{12}=0$ and $\gamma_{13}=0$

$$\frac{M(\gamma_{12},\gamma_{14})}{\gamma_{12}\gamma_{13}} = \sum_{\tau} \frac{\mathrm{Res}_{\gamma_{12}=\Delta-\frac{\tau}{2}} M(\gamma_{12},\gamma_{14})}{(\Delta-\frac{\tau}{2})(\frac{\tau}{2}-\gamma_{14})}\left[\frac{1}{\gamma_{12}-\Delta+\frac{\tau}{2}}+\frac{1}{\gamma_{13}-\Delta+\frac{\tau}{2}}\right]$$

$\propto$ **Mack polynomial** $\qquad \gamma_{13}=\Delta-\gamma_{12}-\gamma_{14}$

$$M(\gamma_{12},\gamma_{14}) = \sum_{\tau,J} C^2_{\tau,J} Q_{\tau,J}(\gamma_{14}) \frac{\gamma_{12}\gamma_{13}}{(\Delta-\frac{\tau}{2})(\frac{\tau}{2}-\gamma_{14})}\left[\frac{1}{\gamma_{12}-\Delta+\frac{\tau}{2}}+\frac{1}{\gamma_{13}-\Delta+\frac{\tau}{2}}\right]$$

['19 Mazac, Rastelli, Zhou] ['19 Carmi, Caron-Huot] ['19 Sleight, Taronna]

Sum rules

$$M(\gamma_{12},\gamma_{14}) = \sum_{\tau,J} C^2_{\tau,J} Q_{\tau,J}(\gamma_{14}) \frac{\gamma_{12}\gamma_{13}}{(\Delta-\frac{\tau}{2})(\frac{\tau}{2}-\gamma_{14})}\left[\frac{1}{\gamma_{12}-\Delta+\frac{\tau}{2}}+\frac{1}{\gamma_{13}-\Delta+\frac{\tau}{2}}\right]$$

Crossing symmetry:

$\gamma_{12} \leftrightarrow \gamma_{13}$ manifest

$\gamma_{12} \leftrightarrow \gamma_{14} \implies$ sum rules

For example:

$$\frac{\partial}{\partial y} M\left(\gamma_{12}=\frac{\Delta}{3}+y, \gamma_{14}=\frac{\Delta}{3}-y\right)\Bigg|_{y=0} = 0$$

$$\implies \boxed{\sum_{\tau,J} C^2_{\tau,J}\,\alpha_{\tau,J} = 0}$$

$$\alpha_{\tau,J} = -\frac{(\tau-\Delta)Q_{\tau,J}\left(\frac{\Delta}{3}\right)}{\left(\tau-\frac{4\Delta}{3}\right)^2\left(\tau-\frac{2\Delta}{3}\right)^2}+\frac{\Delta}{3}\frac{Q'_{\tau,J}\left(\frac{\Delta}{3}\right)}{\left(\tau-\frac{4\Delta}{3}\right)\left(\tau-\frac{2\Delta}{3}\right)(\tau-2\Delta)}$$

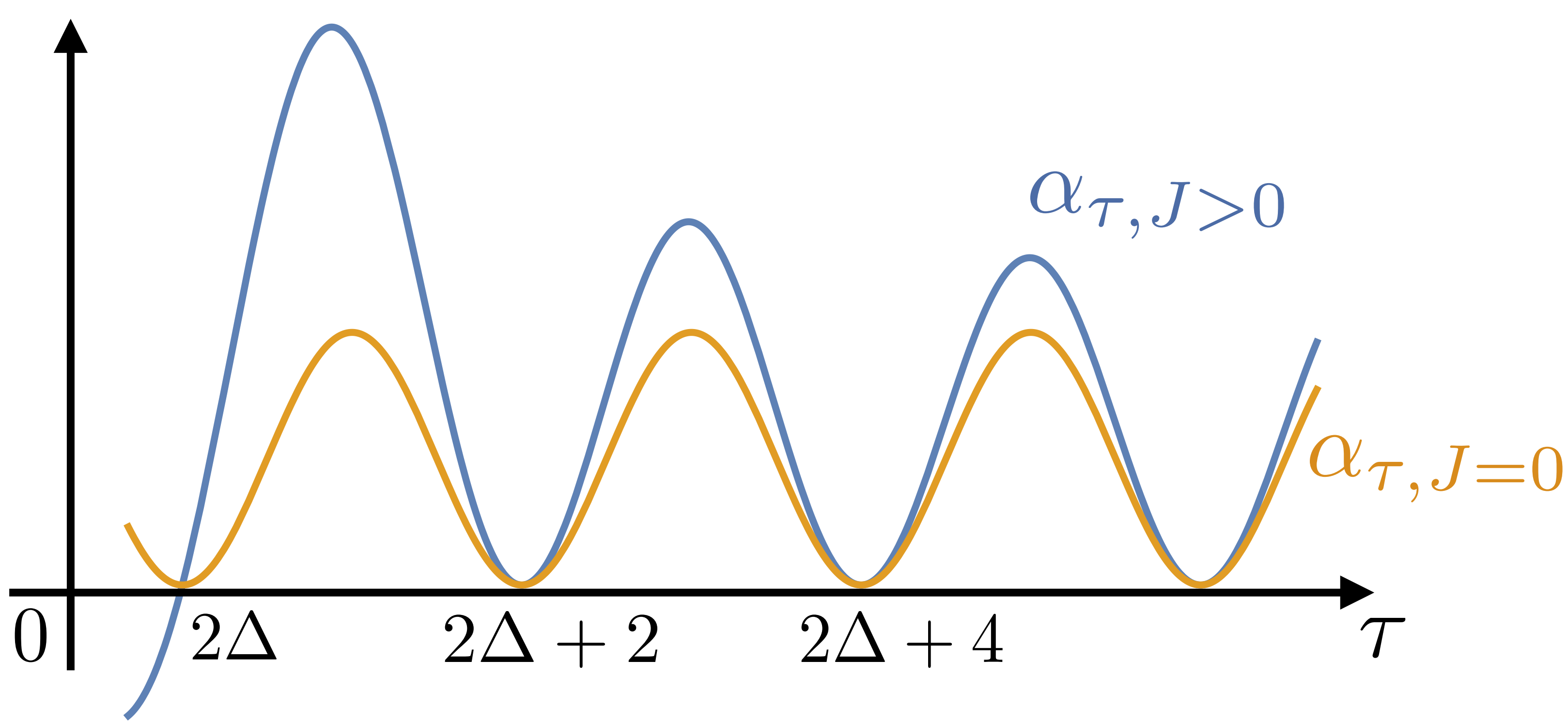
Extremal functional

$$\boxed{\sum_{\tau,J} C^2_{\tau,J}\,\alpha_{\tau,J} = 0}$$

Conjecture: α is an extremal functional for the question $\mathrm{Max}\,\tau_{gap} = ?$

If $\alpha_{\tau,J} \geq 0$ for $\tau \geq 2\Delta$ $\Rightarrow$ $\mathrm{Max}\,\tau_{gap} = 2\Delta$

Mean Field Theory



Double zeros from

$$Q_{\tau,J} \propto \frac{1}{\Gamma^2\left(\Delta - \frac{\tau}{2}\right)}$$

Check: 3D Ising

Sum rule applied to $\langle \sigma\sigma\sigma\sigma \rangle$

leading Regge trajectory **other operators**

$$-\sum_{J=2}^{\infty} C^2_{\tau,J}\alpha_{\tau,J} = \sum_{\text{rest}} C^2_{\tau,J}\alpha_{\tau,J}$$

['16 Simmons-Duffin]

$0.028968_{T_{\mu\nu}}$	0.084569_{ϵ}
$0.012122_{J=4}$	$0.0018_{[\sigma,\sigma]_1^{0\leq J\leq 30}}$
$0.029107_{6\leq J\leq 30}$	$0.0016_{[\epsilon,\epsilon]_0^{4\leq J\leq 30}}$
$0.0222_{J>30}$	$0.0014_{[\epsilon,\epsilon]_0^{J\geq 32}}$
	$\cdots$
0.0924	$0.0894 + \ldots$

Two sides match up to 3% error!

3.
Applications

EFTs in AdS

Constraining EFTs in AdS

Large N expansion in CFT $\Leftrightarrow$ Perturbative expansion in AdS

Single-trace: $C^{\rm st}_{\tau,J} \sim \dfrac{1}{N}$

Double-trace: $\tau\left[:\mathcal{O}(\partial^2)^n\partial^J\mathcal{O}:\right] - 2\Delta - 2n = \gamma(n,J) \sim \dfrac{1}{N^2}$

Sum rule at large N:

$$\underbrace{\overbrace{\sum_{\tau,J}\left(C^{\rm st}_{\tau,J}\right)^2\alpha_{\tau,J}}^{\text{light single-trace}} + \overbrace{\sum_{J=2}^{\infty}\left(C^{\rm MFT}_{\tau=2\Delta,J}\right)^2\frac{\partial\alpha_{\tau,J}}{\partial\tau}\bigg|_{\tau=2\Delta}\gamma(0,J)}^{\text{leading double-trace trajectory}}}_{H\sim 1/N^2} + \underbrace{\overbrace{\sum_{\rm rest} C^2_{\tau,J}\alpha_{\tau,J}}^{\text{heavy operators}}}_{\geq 0} = O(1/N^4)$$

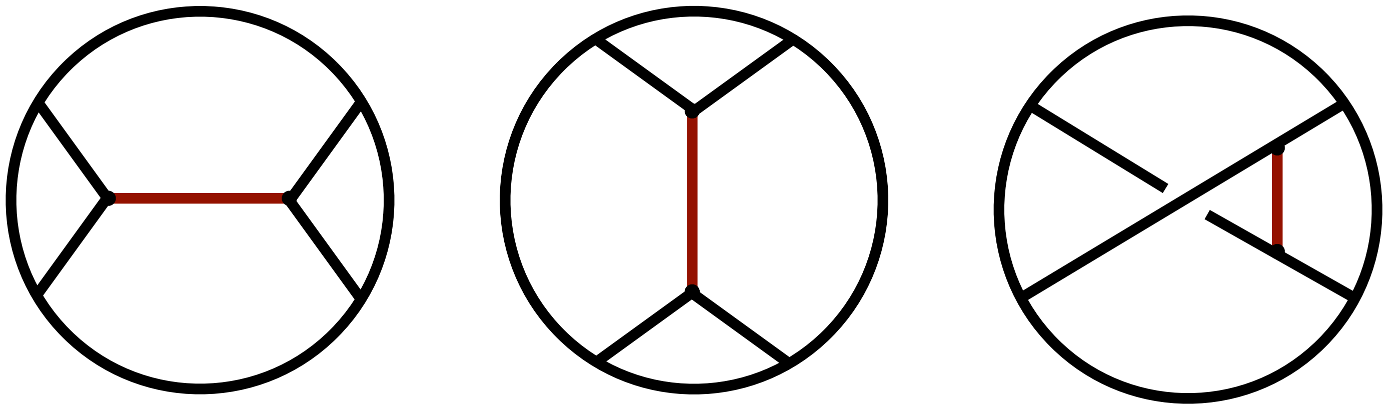
computable from EFT in AdS

$H > 0 \quad \Longrightarrow \quad$ EFT in the swampland

[related ongoing work by Caron-Huot, Mazac, Rastelli, Simmons-Duffin]

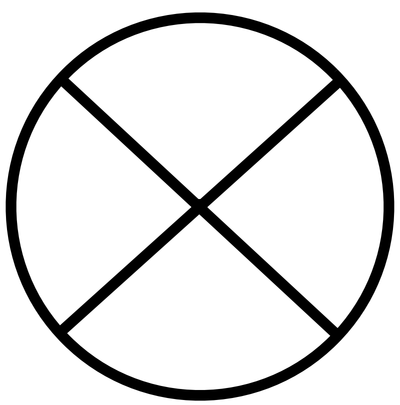
Constraining EFTs in AdS

Scalar cubic theory $\lambda\phi^2\chi$



$\Rightarrow$ $\boxed{H = 0}$

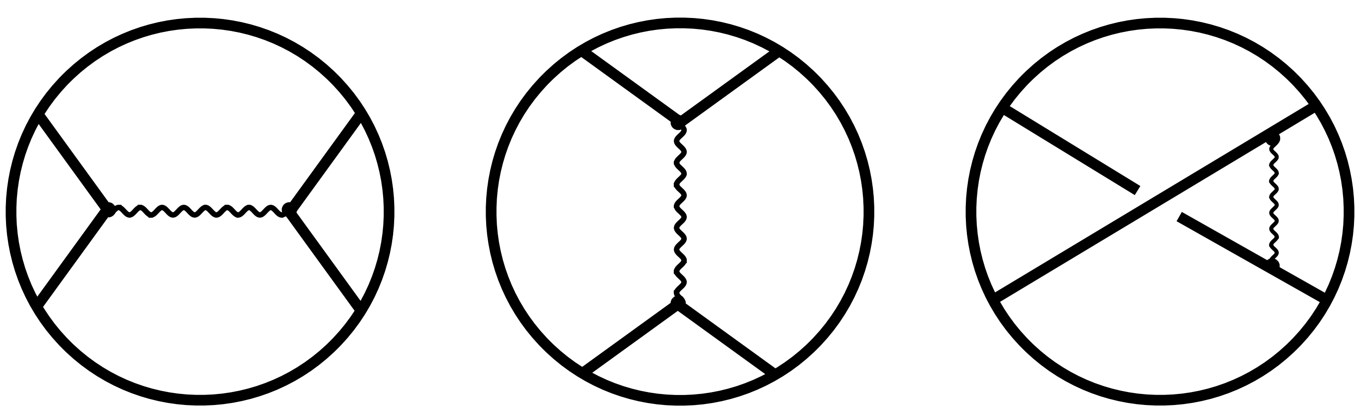
Scalar quartic theory $\lambda(\partial\phi)^4$



$\Rightarrow$ $\boxed{H = -\lambda \text{ (positive)}}$ $\Rightarrow$ $\boxed{\lambda > 0}$

['06 Adams, Arkani-Hamed, Dubovsky, Nicolis, Rattazzi]

Minimal coupling to graviton



$\Rightarrow$ $\boxed{H < 0}$

$\epsilon -$ expansion

$\epsilon-$expansion

$$d = 4 - \epsilon \qquad \Delta_\phi = \frac{d-2}{2} + \gamma_\phi \qquad \Delta_{:\phi\partial^J\phi:} = 2\Delta_\phi + J + \gamma_J$$

Mellin dispersion relation for $\langle\phi\phi\phi\phi\rangle$

$$M(\gamma_{12},\gamma_{14}) = \sum_{\tau,J} C^2_{\tau,J} Q_{\tau,J}(\gamma_{14}) \frac{\gamma_{12}\gamma_{13}}{(\Delta-\frac{\tau}{2})(\frac{\tau}{2}-\gamma_{14})}\left[\frac{1}{\gamma_{12}-\Delta+\frac{\tau}{2}} + \frac{1}{\gamma_{13}-\Delta+\frac{\tau}{2}}\right]$$

Impose next **Polyakov condition**: $\boxed{M(\gamma_{12}=-1,\gamma_{14})=0}$

$$\Longrightarrow \gamma_0 = -\frac{\epsilon}{3} + O(\epsilon^2) \qquad \gamma_J = -\frac{\epsilon^2}{9J(J+1)} + O(\epsilon^3) \qquad J \geq 2$$

$$\Delta_{T_{\mu\nu}} = d \quad \Longrightarrow \quad \gamma_\phi = \frac{\epsilon^2}{108} + O(\epsilon^3)$$

More Polyakov conditions give access to more CFT data.

['16 Gopakumar, Kaviraj, Sen, Sinha]

Future work

Future work

- **Prove** the Maximal Mellin Analyticity conjecture and the full Regge bound
- **Extend** our results to non-identical operators, spinning operators, higher point functions, BCFT, etc
- **Construct** a basis of efficient functionals ['19 Paulos] ['19 Mazac, Rastelli, Zhou] ['19 Carmi, Caron-Huot]
- **Implement** the numerical bootstrap using these functionals
- **Find** a CFT sum rule that places a seemingly healthy EFT in the swampland?
- **Use** Mellin Amplitudes in de Sitter space [see next talk by C. Sleight]