*Section 2 of*

# Replica Wormholes and the Black Hole Interior

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Work with **Steve Shenker, Douglas Stanford and Zhenbin Yang**

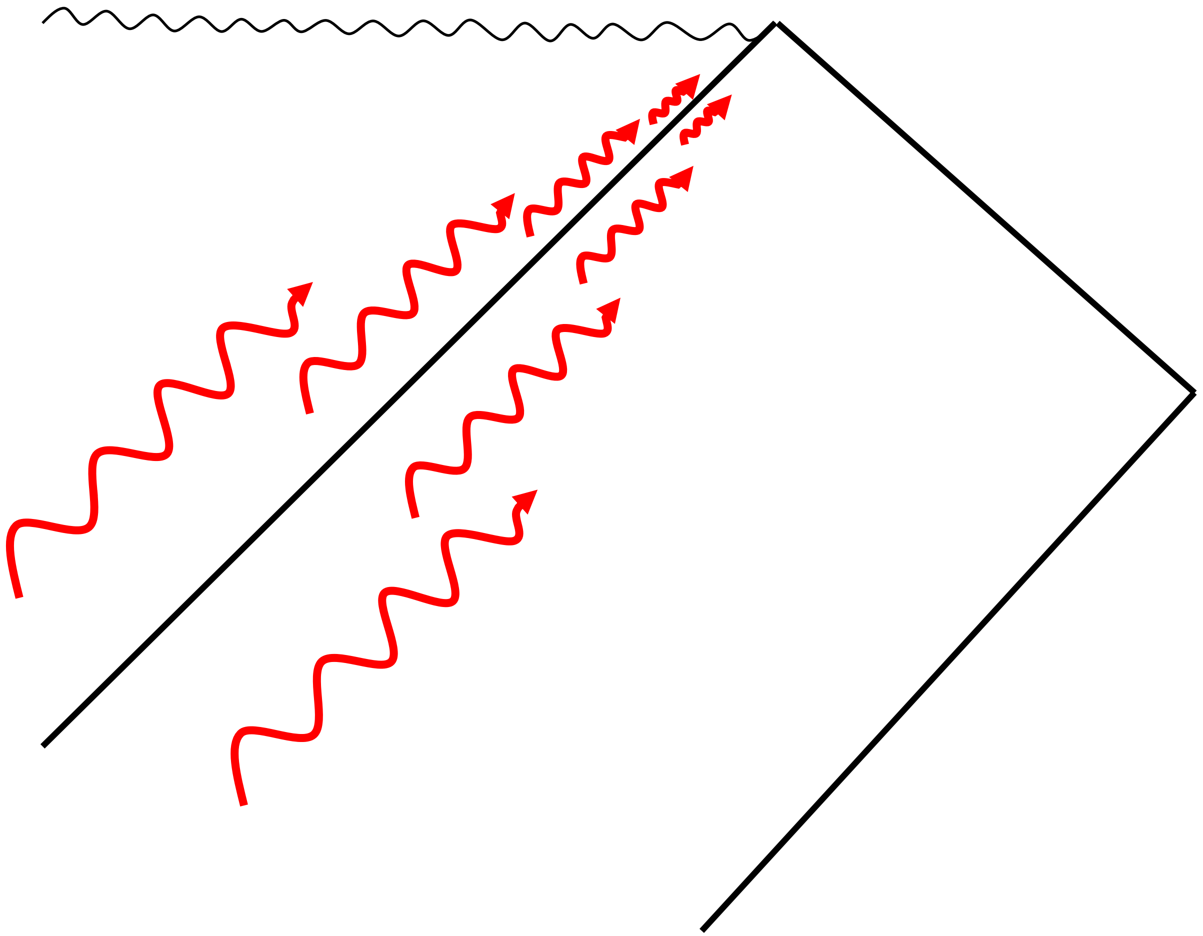
**See also:** Almheiri, Hartman, Maldacena, Shaghoulian, Tajdini

# The Claim

- We can successfully calculate the Page curve for an evaporating black hole just using a **gravitational path integral**
- In this talk, I will focus on a **highly simplified model**, where we can do the most precise calculations, but the basic story seems to be very general
- The arguments came out of ideas (QES prescription, entanglement wedge reconstruction) from AdS/CFT. **However**, don't need a CFT, string theory, anti-de Sitter space, etc.
- All we need is an effective low energy gravitational path integral, where you sum over arbitrary topologies, including '**spacetime wormholes**'.
- Very similar arguments can be used to show that information **escapes** from the interior of the black hole into the Hawking radiation (see Douglas' talk)

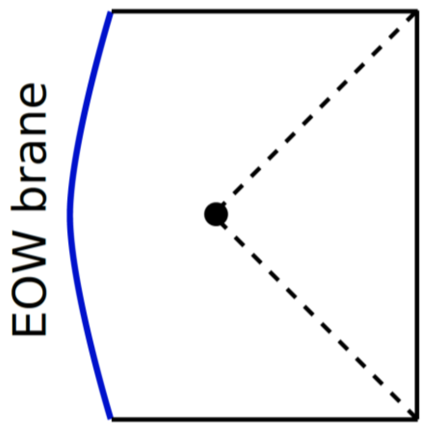
# Part 1: A Very Simple Information Paradox

# The Information Paradox in Evaporating Black Holes

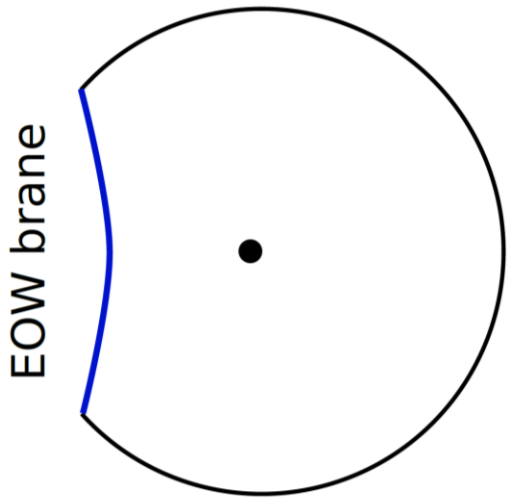


- Hawking radiation is entangled with interior partner modes **inside the black hole**
- Eventually the entanglement entropy seems to become larger than the **Bekenstein-Hawking** entropy of the black hole (the **Page time**)
- If the BH entropy is truly the **statistical entropy** of black hole microstates (true in string theory, AdS/CFT), this is a **paradox**: not enough BH degrees of freedom to be able to purify the Hawking radiation
- Possible resolutions: a) **information loss** or b) entanglement entropy starts **decreasing** at/before Page time (**Page curve**)

# A Very Simple Model: Pure JT gravity plus EOW Branes




Lorentzian



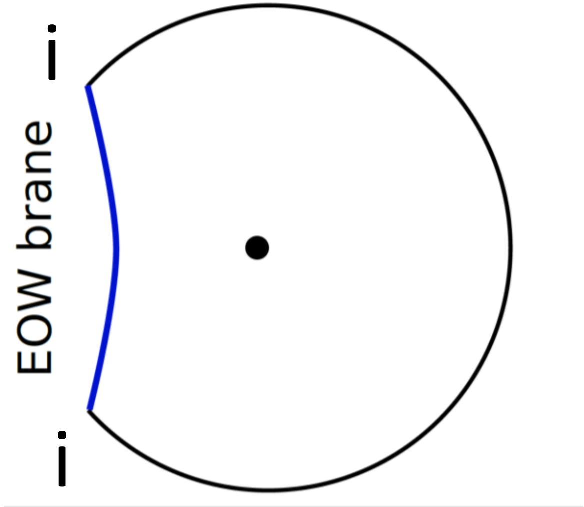

Euclidean

1+1-dimensional, one-sided eternal black hole, where the spacetime ends on an **‘end-of-the-world’ brane** (with mass $\mu$) in the BH interior

**Analogue for Hawking radiation:** add internal degrees of freedom to the EOW brane **(interior modes)** that are maximally entangled with a reference system

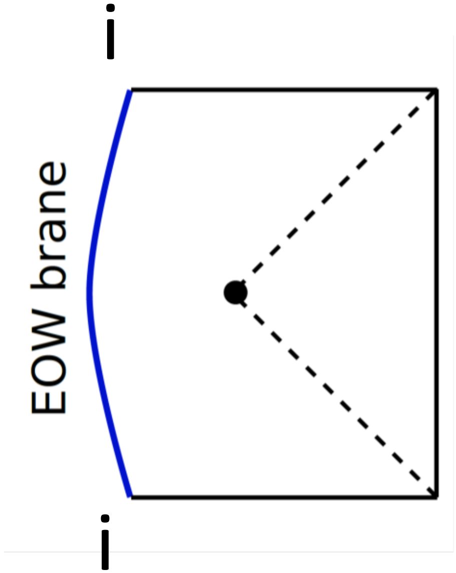
# A Very Simple Information Paradox

$|\psi_i\rangle_B$ =




Euclidean

=




Lorentzian

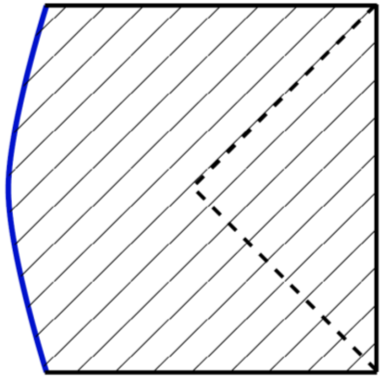
**Orthogonal basis** for **internal state** of EOW brane

$$|\Psi\rangle = \frac{1}{\sqrt{k}} \sum_{i=1}^{k} |\psi_i\rangle_B |i\rangle_R.$$

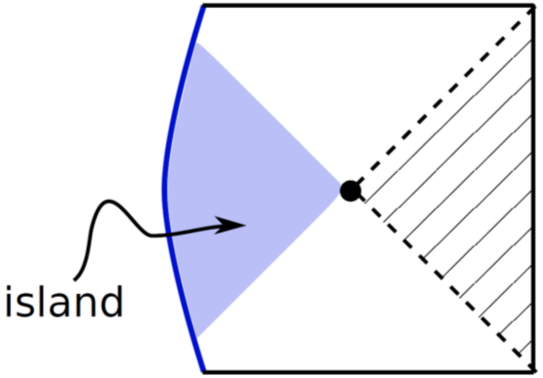
(Simple) **information paradox** when $k \gg e^{S_{BH}}$. Entanglement entropy seemingly becomes **larger** than the **Bekenstein-Hawking** entropy

# Part 2: Finding the Page Curve

# The Island Rule/QES Prescription



$k \ll e^{S_{BH}}$




$k \gg e^{S_{BH}}$

**Rule** for computing entanglement entropies in gravity:

1. Find all (quantum) extremal surfaces
2. Entanglement entropy is the **generalised entropy** of the **minimal QES**.

Two extremal surfaces: the **empty set** ($S_{gen} = \log k$) and the **bifurcation surface** ($S_{gen} = S_{BH}$)

$$S = \min(\log k, S_{BH})$$

# Deriving the Island Rule from the Gravitational Path Integral

This rule wasn't just **made up** to give the 'right' answer! It can be **derived** from the gravitational path integral (just like the **BH entropy** can be derived using a Euclidean path integral)

**Rest of this talk:** derive the Page curve, directly from the gravitational path integral at three increasing levels of precision.

| Level of Precision | Formula for Page Curve |
|---|---|
| Topology | $S = \min(\log k, S_0)$ |
| Geometry of leading classical saddle | $S = \min(\log k, S_{BH})$ |
| Full path integral | $S = -\int d\lambda\, \lambda \log \lambda\, Im[R(\lambda)]/\pi$ where $\lambda R = k + e^{S_0} \int ds\, \rho(s) \frac{y(s)R}{k\, Z_1 - y(s)R}$ |

# The Replica Trick

- How do you calculate **von Neumann entropies** using a path integral?
- **Answer**: the **integer n Renyi entropies**

$$\frac{1}{1-n} \log \mathrm{Tr} \rho_R^n$$

are proportional to the logarithm of an observable on **n copies of the system**.

- We can calculate the von Neumann entropy by **analytically continuing the Renyi entropies** to **n=1**.
- **The key idea**: the gravitational path integral includes topologies that connect the different replicas via **spacetime wormholes**.

# Calculating the Purity

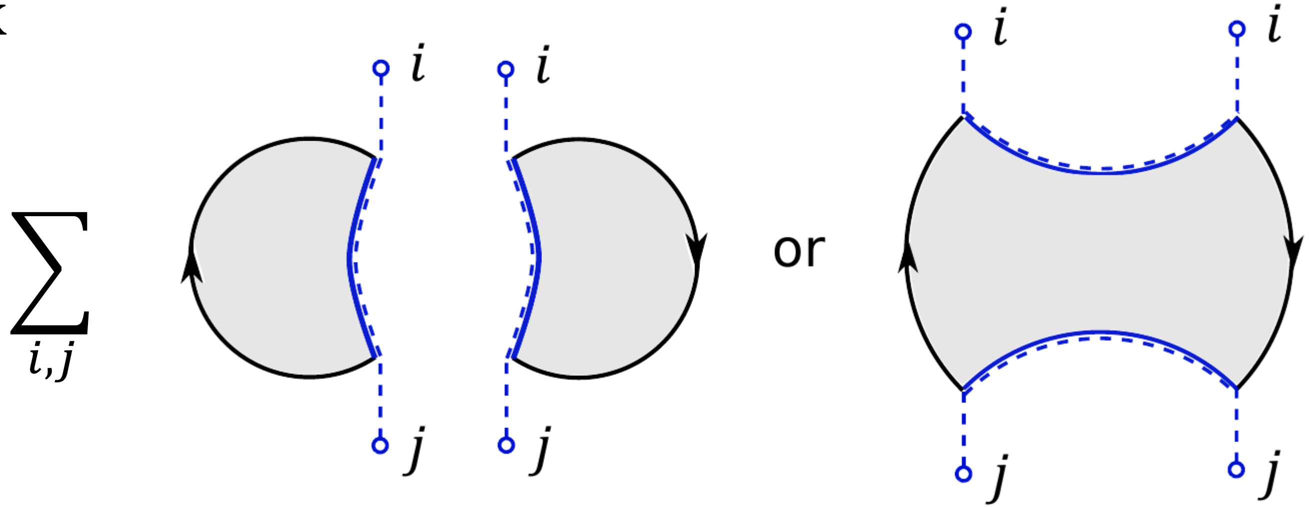
Einstein-Hilbert term
~ Euler characteristic

Calculate the purity $Tr(\rho_R^2)$ using a **Euclidean path integral**, where we sum over **all topologies** with the correct boundary conditions:

$$\mathrm{Tr}(\rho_{\mathrm{R}}^2) = \frac{1}{k^2} \sum_{i,j=1}^{k} |\langle \psi_i | \psi_j \rangle|^2 .$$

Very small but doesn't decay at large k

$$= \frac{1}{k}$$

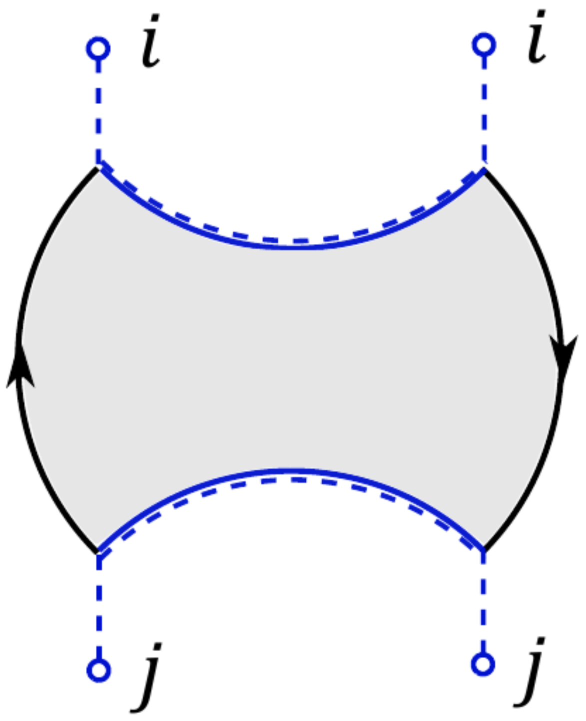



$\sim e^{-S_0}$

Two copies of the original black hole

Two black holes connected by a wormhole

# Factorisation and Averaging




We didn't have to sum over k! The saddles exist even without the sum

Wormhole topology means $|\langle\psi_i|\psi_j\rangle|^2$ is **non-zero** even when $i \neq j$

However, we do have $\langle\psi_i|\psi_j\rangle = 0$ when $i \neq j$.

The theory has a **factorisation problem**

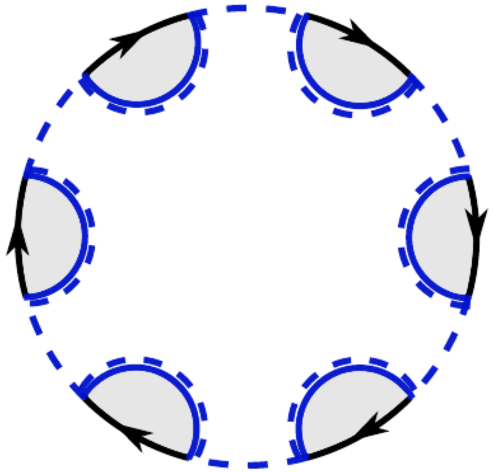
**Explanation**: the gravity calculation (at least in this simple case) is not computing the inner product in a single quantum theory, it is computing an **average** inner product, across an **ensemble of theories**

# Calculating the von Neumann Entropy I: Topology

In general, there are a lot of topologies that can contribute to $\text{Tr}(\rho_R^n)$. However, in the limit where k is **very large/small** one of **two families of topologies** dominates

Small k

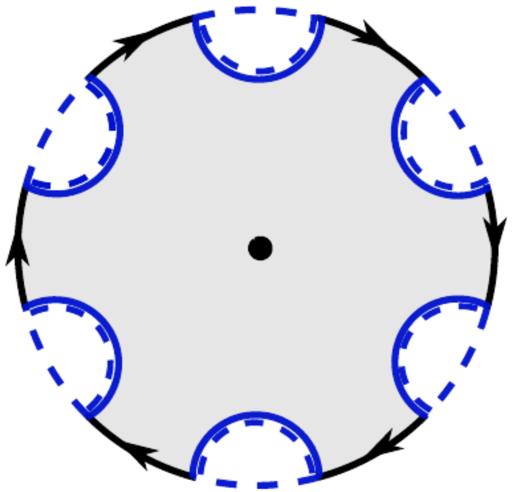
$$\frac{1}{k^{n-1}} =$$



Disconnected topology
Gives S = $\log k$

or

Large k

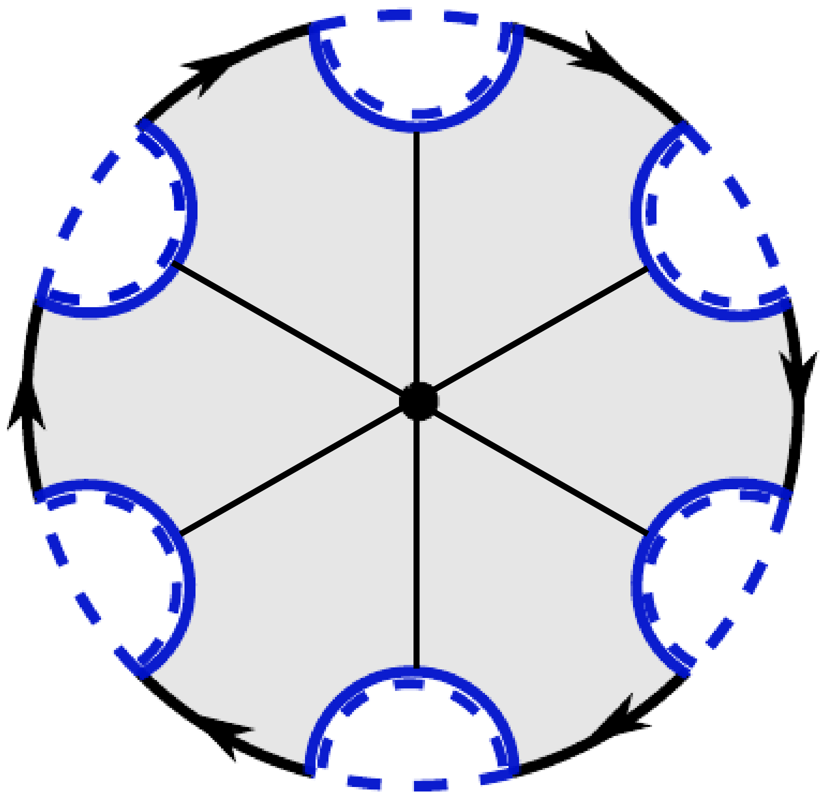


$$\sim e^{-(n-1)S_0}$$

Connected topology
Gives $S \sim S_0$

Dotted lines = summed-over indices

# Calculating the von Neumann Entropy II: The Leading Classical Saddle



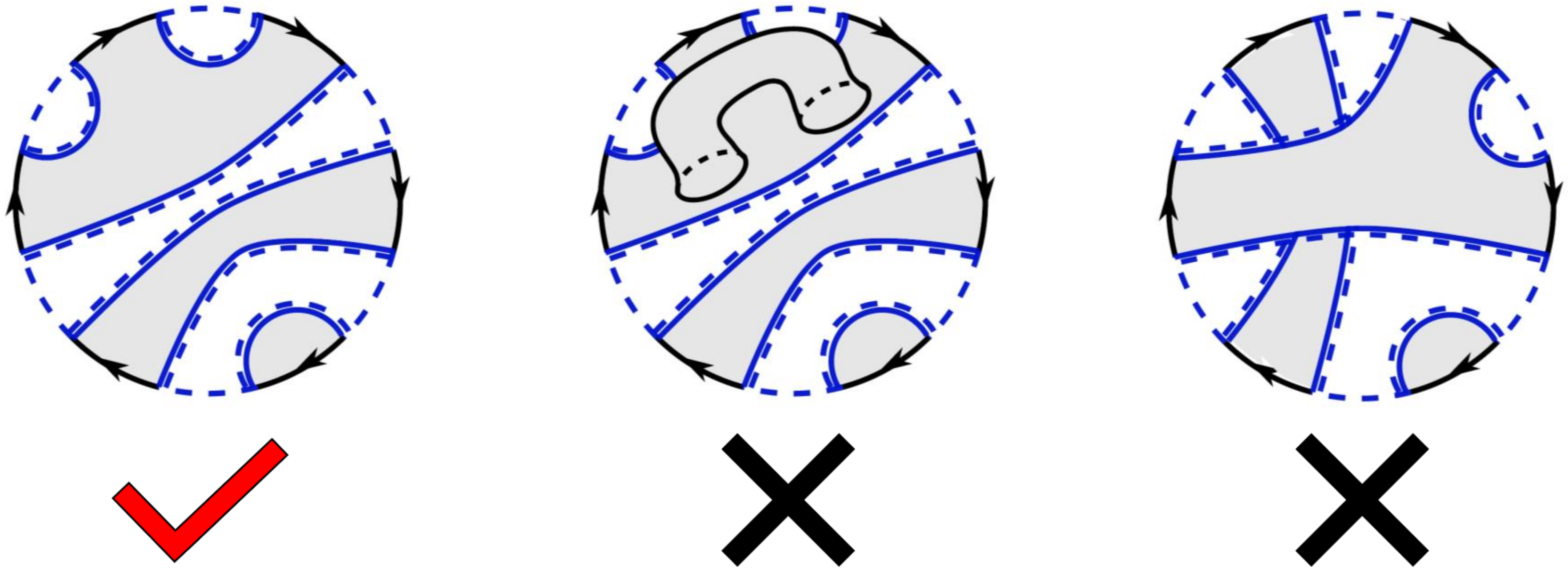
- Connected topology has a $Z_n$ replica symmetry.
- After **quotienting** by this symmetry, we get *roughly* the original black hole geometry, except that there is a **conical singularity** at the fixed point of the replica symmetry
- In the limit $n \to 1$, the singularity **vanishes** (get original unbackreacted geometry)
- Von Neumann entropy given by the “area” of replica fixed point (in this case the **bifurcation surface**)

# Calculating the von Neumann Entropy III: The Full Path Integral

Assume $k, e^{S_0} \to \infty$, but ratio arbitrary,

Suppressed by $e^{-2S_0}$

Suppressed by $1/k^2$



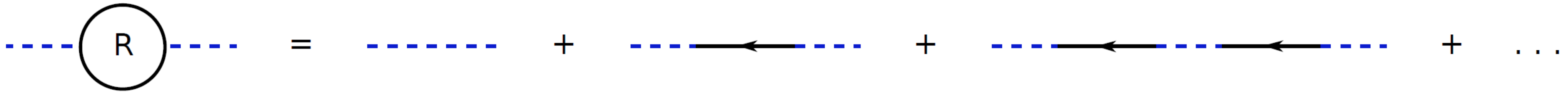
Only **planar topologies** survive

**Problem**: we need to not only evaluate this sum over topologies for finite integer n (**doable**) but also **analytically continue** the formula to **non-integer n** (**hard**)

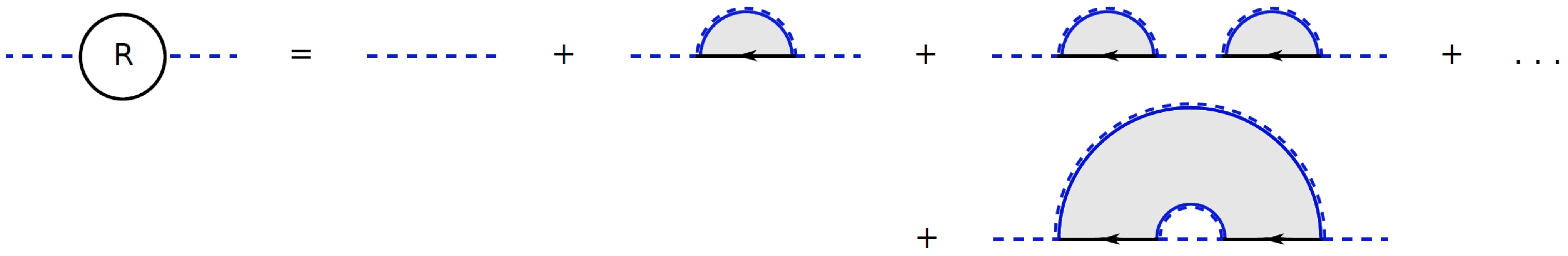
# Calculating the von Neumann Entropy III: Resolvents to the Rescue

$$R_{ij}(\lambda) = \left(\frac{1}{\lambda\mathbb{1} - \rho_{\mathrm{R}}}\right)_{ij} = \frac{1}{\lambda}\delta_{ij} + \sum_{n=1}^{\infty}\frac{1}{\lambda^{n+1}}(\rho_R^n)_{ij}.$$

Graphically,




Summing over bulk planar topologies,



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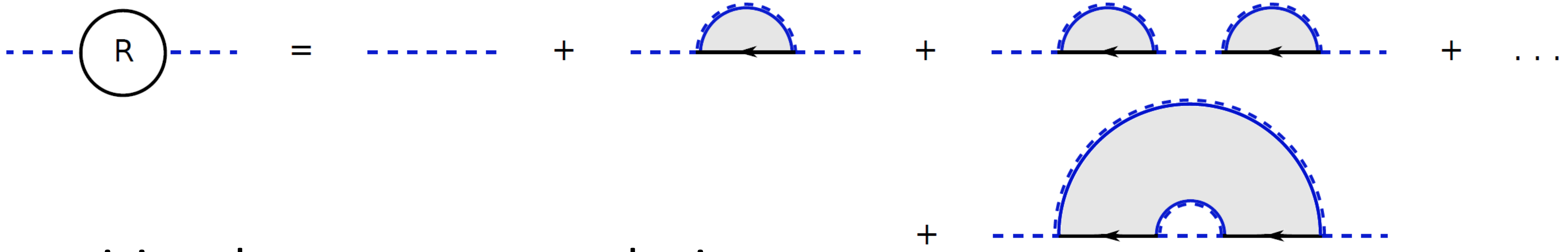
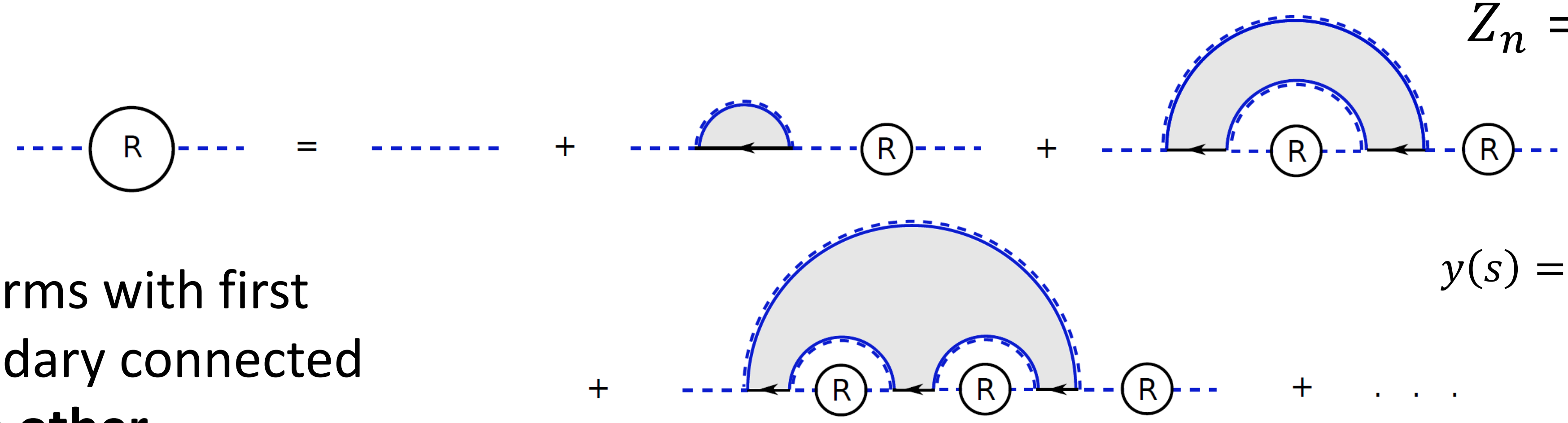
Summing over bulk planar topologies,




Reorganising the sum over topologies,

$$Z_n = e^{S_0} \int ds \;\; \rho(s) y(s)^n$$




$$\rho(s) = \frac{s}{2\pi^2} \sinh 2\pi s$$

$$y(s) = \mathrm{e}^{-\frac{\beta s^2}{2}} 2^{1-2\mu} \left| \Gamma\left(\mu - \frac{1}{2} + is\right)\right|^2$$

All terms with first boundary connected to **no other boundaries**

# Calculating the von Neumann Entropy III: The Density of States

Evaluating this infinite sum (as a **geometric series**), gives the equation

$$\lambda\, R = k + e^{S_0} \int ds \, \frac{y(s)R}{k\, Z_1 - y(s)R}$$

$$\sum_i R_{ii}(\lambda)$$

The resolvent $R$ encodes not just the **entanglement entropy**, but the entire **entanglement spectrum** of the state, since the density of states

$$D(\lambda) = \frac{1}{\pi} Im(\mathrm{R}(\lambda))$$

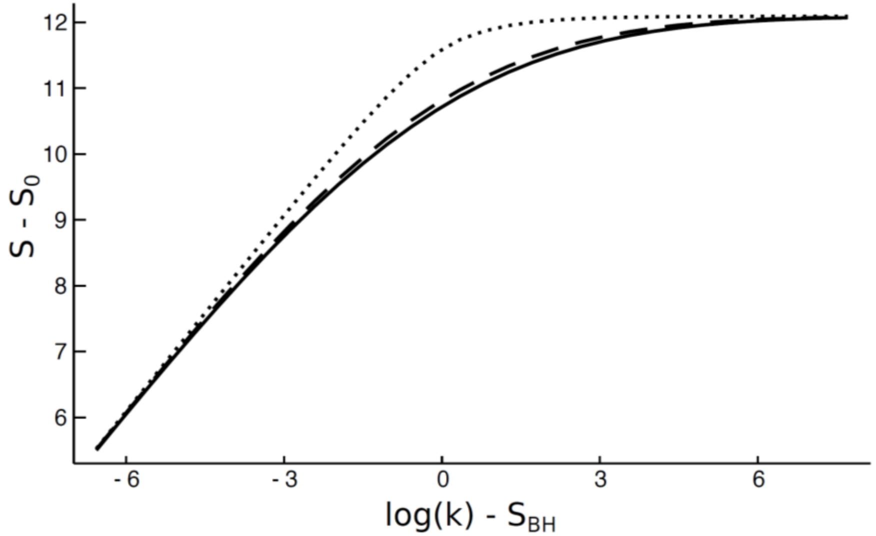
We don't have an **explicit solution** to the resolvent equation, but it can be solved **numerically**, and is also **well approximated** by a **shifted, cut-off thermal spectrum**

# Calculating the von Neumann Entropy III: The Final Answer

Using this approximate solution, one can calculate the Page curve up to **at most $\boldsymbol{O(G_N)}$ corrections** (and much more accurately far away from the phase transition)

Far away from the transition, the corrections to the naïve Page curve we found before are **exponentially small** (although they can still be computed)

The transition is **smoothed out** by quite large $O\left(\sqrt{1/G_N}\right)$ corrections. These corrections are much larger in gravity than the $O(1)$ corrections in toy models such as **Haar random states**, with the enhancement coming from the **fluctuations** in the energy of the black hole.



# Why bother with the more complicated calculation?

- We already got (basically) the right answer for the Page curve, just by looking at a single family of classical saddles. Why **bother** with all this more complicated technology when it's just a **stupid toy model**?
- Because we can. And it's fun
- The **enhanced correction** from the energy fluctuations at the **transition** is interesting (and universal?) physics [Marolf, Wang, Wang; Dong, Wang]
- The **nonperturbative corrections** far from the transition are crucial to make the result consistent [Hayden, GP]
- It suggests the **full gravitational path integral** should be taken seriously [Saad, Shenker, Stanford]
- Very close correspondence between the gravity calculations and calculations in **random tensor networks** [Hayden, Nezami, Qi, Thomas, Walter, Yang]

Thank you!