IPMU 14-0???
UT-14-??

On some conjectures extracted from supersymmetric quantum field theories

Yuji Tachikawa[♯,♭]

[♭] Department of Physics, Faculty of Science,
University of Tokyo, Bunkyo-ku, Tokyo 133-0022, Japan
[♯] Kavli Institute for the Physics and Mathematics of the Universe (WPI),
University of Tokyo, Kashiwa, Chiba 277-8583, Japan

abstract

We present several conjectures which we extracted from the study of $\mathcal{N}=2$ supersymmetric quantum field theories, hopefully in a way understandable to mathematicians. Conjectures involve topics such as holomorphic symplectic varieties, Hall-Littlewood and other symmetric polynomials, vertex operator algebras and nilpotent orbits.

Contents

1 Introduction

The aim of this short note is to present various mathematical conjectures which arose in the last few years from the consideration of $\mathcal{N}$=2 supersymmetric quantum field theories in four dimensions (4d $\mathcal{N}$=2 SUSY QFTs), in a way hopefully understandable to mathematicians. The author has already prepared a review article [1] with a similar objective. It turned out, unfortunately, that the pseudo-mathematical discussions on what exactly are the 4d $\mathcal{N}$=2 SUSY QFTs tend to distract the mathematical readers too much and that not many readers of that article come to the point of studying the precisely formulated mathematically conjectures themselves. In this note, the discussion on the nature of the supersymmetric field theories is cut down to the bare minimum, and the mathematical conjectures are stated explicitly as such, so that an interested mathematician can directly grasp what kind of mathematical conjectures have arisen in this type of study by physicists.

Before proceeding, it is to be noted that conjectures are gathered from various sources, mainly from [2, 3, 4, 5] and many of them are not by the author himself.

The rest of the note is organized as follows. We start by introducing in Sec. 2 the properties of 4d $\mathcal{N}$=2 SUSY QFTs. There we explain two main classes of 4d $\mathcal{N}$=2 SUSY QFTs, one associated to symplectic vector spaces and another associated to punctured Riemann surfaces. We also summarize various mathematical objects associated to $\mathcal{N}$=2 SUSY QFTs. Then sections 3, 4, 5, 6, 7 and 8 contain conjectures related to the section titles. Most of these sections can be read independently. The conjectures concerning the instanton moduli spaces and the W-algebras, proposed in [6] and recently partially proved in [7, 8], also follow from the consideration of 4d $\mathcal{N}$=2 SUSY QFTs, but its exposition requires an understanding of much more properties of SUSY QFTs. The readers are referred to [1].

Acknowledgments

The author thanks the organizers (A. Tsuchiya, K. Saito, H. Nakajima, T. Kohno, and Y. Saito) of various informal talks and lecture series, who gave the author the chance to present to mathematicians pseudo-mathematical reformulations of supersymmetric gauge theories. The original, unfinished lecture notes [1] grew out of these experiences, of which this note is an excerpt. He also thanks the participants on those talks and lecture series, who gave valuable feedbacks.

This work is supported in part by JSPS Grant-in-Aid for Scientific Research No. 25870159 and in part by WPI Initiative, MEXT, Japan at IPMU, the University of Tokyo.

2 4d $\mathcal{N}$=2 SUSY QFTs

2.1 Formal properties

A 4d $\mathcal{N}$=2 SUSY QFT Q is, first of all, a mathematical object in the category $\mathcal{Q}$ of 4d $\mathcal{N}$=2 SUSY QFTs[1]. As a mathematician, the reader should know how to deal with a mathematical object only using its formal properties, or axioms, without asking how it is constructed or realized in the sense of set theory. The only properties we use concerning 4d $\mathcal{N}$=2 SUSY QFTs are the following:

- Given a reductive group G over $\mathbb{C}$, there is a subcategory $\mathcal{Q}(G)$ of '4d $\mathcal{N}$=2 SUSY QFTs with G symmetry'.
- Given a homomorphism $\varphi : H \to G$, there is a functor from $\mathcal{Q}(G)$ to $\mathcal{Q}(H)$, satisfying expected properties.
- There is a canonical object $\mathrm{triv} \in \mathcal{Q}(G)$ for any G, which behaves naturally under the functors given above.
- Given $Q_1 \in \mathcal{Q}(G_1)$ and $Q_2 \in \mathcal{Q}(G_2)$, we have an operation $\times$ such that $Q_1 \times Q_2 \in \mathcal{Q}(G_1 \times G_2)$.
- In particular, using the diagonal embedding $G \subset G \times G$, we see that for $Q_{1,2} \in \mathcal{Q}(G)$ we have $Q_1 \times Q_2 \in \mathcal{Q}(G)$. triv is the unit under this product operation.
- Given $Q \in \mathcal{Q}(F \times G)$, there is $Q /\!\!/\!\!/ G \in \mathcal{Q}(F)$.

In a word, the categories $\mathcal{Q}(G)$ behave rather like the category of G-spaces, with the operation $/\!\!/\!\!/$ playing the role of the quotient operation.

Concrete mathematical conjectures arise when one associates already-well-defined mathematical objects to a SUSY QFT Q. Before discussing them, we need to describe two major known ways to construct $\mathcal{N}$=2 SUSY QFTs.

[1] A 4d $\mathcal{N}$=2 SUSY QFT Q here corresponds to, in the terminology in [1], a *largest family* containing a 4d $\mathcal{N}$=2 SUSY QFT. In [1], we used the letter G to refer to a compact Lie group and used $G_{\mathbb{C}}$ to denote the corresponding reductive group over $\mathbb{C}$. The author made this change so that there is less clutter in the note.

2.2 SUSY gauge theories

The first method is the following. Let V be a pseudoreal representation of G. Equivalently, let V be a $\mathbb{C}$-vector space with G action, together with a G-invariant holomorphic symplectic form ω. Then there is

$$\mathrm{Hyp}(V) \in \mathcal{Q}(G), \tag{2.1}$$

called a half-hypermultiplet based on V. The basic properties are

$$\mathrm{Hyp}(\{\mathrm{pt}\}) = \mathrm{triv}, \qquad \mathrm{Hyp}(V \oplus W) = \mathrm{Hyp}(V) \times \mathrm{Hyp}(W). \tag{2.2}$$

Given a pseudo-real representation V of $F \times G$, we have $\mathrm{Hyp}(V)/\!\!/\!\!/G \in \mathcal{Q}(F)$. A 4d $\mathcal{N}$=2 SUSY QFT of this form is called a 4d $\mathcal{N}$=2 SUSY gauge theory.

2.3 Class S theories

The second method is the following. Let G be a simply-laced, simply-connected reductive group over $\mathbb{C}$. Pick a closed 2d surface C with punctures p_i, labeled by nilpotent orbits O_i of $\mathfrak{g}$. Using the Jacobson-Morozov theorem, we can equally label the puncture p_i by a nilpotent element e_i up to conjugation. We mostly stick to this latter convention.

Then we have a 4d $\mathcal{N}$=2 SUSY QFT associated to this marked surface

$$S_G(C, \{(p_i, e_i)\}) \in \mathcal{Q}(\prod_i G(e_i)) \tag{2.3}$$

where the reductive group $G(e)$ for a nilpotent element e is defined as follows. There is a triple $\{e, h, f\} \subset \mathfrak{g}$, unique up to a conjugation, generating an $\mathfrak{su}(2)$ subalgebra: $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$. We then let $G(e) = G^{\{e,h,f\}}$, the subgroup fixed by the adjoint action of e, h and f. The three basic properties are the following:

- When there is a homeomorphism mapping $(C, \{p_i, e_i\})$ to $(C', \{p'_i, e'_i\})$, we have

$$S_G(C, \{(p_i, e_i)\}) \simeq S_G(C', \{(p'_i, e'_i)\}). \tag{2.4}$$

- Let e_{prin} be the principal nilpotent element. Then we have

$$S_G(C, \{(p_i, e_i)\} \sqcup \{(p, e_{\text{prin}})\}) \simeq S_G(C, \{(p_i, e_i)\}). \tag{2.5}$$

  In other words, having a puncture p marked with e_{prin} is equivalent to having no puncture p at all.

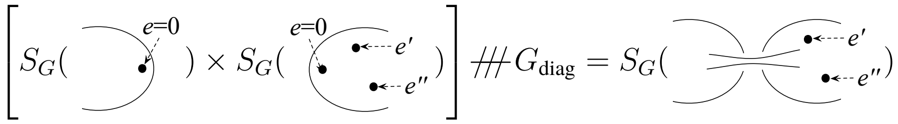
- Suppose we have two marked surfaces $(C, \{(p_i, e_i)\}_{i=0,1,\dots})$ and $(C', \{p'_i, e'_i\}_{i=0,1,\dots})$, such that $(p_0, O_0) = (p, 0)$ and $(p'_0, O'_0) = (p', 0)$. Then $G(e_0) = G(e'_0) = G$, and let

$$G_{\text{diag}} \subset G(e_0) \times G(e'_0) \tag{2.6}$$

be the diagonal subgroup.[2] The crucial property we have is that [9]

$$[S_G(C, \{(p_i, e_i)\}_{i=0,1,\ldots}) \times S_G(C', \{(p'_i, e'_i)\}_{i=0,1,\ldots})] /\!\!/\!\!/ G_{\text{diag}} \\ = S_G(C_{p_0} {+}_{p'_0} C', \{(p_i, e_i)\}_{i=1,\ldots} \sqcup \{(p'_i, e'_i)\}_{i=1,\ldots}). \quad (2.7)$$

Here, $C_{p_0} {+}_{p'_0} C'$ is the 2d surface obtained by smoothly connecting the point p_0 on C and the point p'_0 on C'. Note that both sides of the equation are in $\mathcal{Q}(\prod_{i=1,\ldots} G(e_i) \times \prod_{i=1,\ldots} G(e'_i))$. More pictorially, we denote this relation as



(2.8)

There is a similar relation where we connect two punctures labeled by $e = 0$ on one and the same surface. It should be emphasized that only punctures labeled by $e = 0$ can be connected this way.

2.4 Mathematical objects associated

Given an $\mathcal{N}{=}2$ SUSY QFT $Q \in \mathcal{Q}$, we can associate various mathematical objects which can be discussed in already-well-established fields of mathematics. Those discussed in this paper are summarized in Figure 1. In the figure, the solid black line from A to B means that the object B to should in principle be obtained from the object A, and the dashed red line from A to B says that the object B can be easily obtained from A within the understanding at present. The objects pointed to by the dashed red lines starting from class S theories $S_G(C, \{e_i\})$ and SUSY gauge theories $\text{Hyp}(V) /\!\!/\!\!/ G$ are rather complementary. Physics arguments tell us that $\text{Hyp}(V) /\!\!/\!\!/ G = S_G(C, \{e_i\})$ for certain nice choices of V, G, C and $\{e_i\}$, and this gives rise to a host of conjectures.

Let us list the objects appearing in the figure:

- $\mathcal{M}_{\text{Higgs}}(Q)$ is a hyperkähler manifold called the Higgs branch. We treat it as a holomorphic symplectic variety in this paper. We discuss it in Sec. 3.
- $DW(Q)$ is an holomorphic integrable system called the Donagi-Witten integrable system.
- $\mathcal{M}_{\text{Coulomb}}(Q)$ is an affine space which is the base of the integrable system $DW(Q)$. Both $DW(Q)$ and $\mathcal{M}_{\text{Coulomb}}(Q)$ are discussed in Sec. 5. An aspect of $DW(Q)$ will be dealt with in Sec. 8, too.
- $VOA(Q)$ is a vertex operator algebra associated to Q, recently introduced in [10]. This will be described in Sec. 7.
- For $Q \in \mathcal{Q}(G)$, $Z^{\text{SCI}}_{p,q,t}(Q)$ is an element in $K_G(\text{pt})[[p, q, t^{1/2}]]$. In the most generality this will be discussed in Sec. 6, but its limit when $p = q = 0$ will appear early in Sec. 4.

[2] More precisely, when G is a simply-laced group with order-2 outer automorphism, the embedding is defined by sending $g \in G_{\text{diag}}$ to $(g, o(g))$ where o is a non-inner automorphism of order 2.

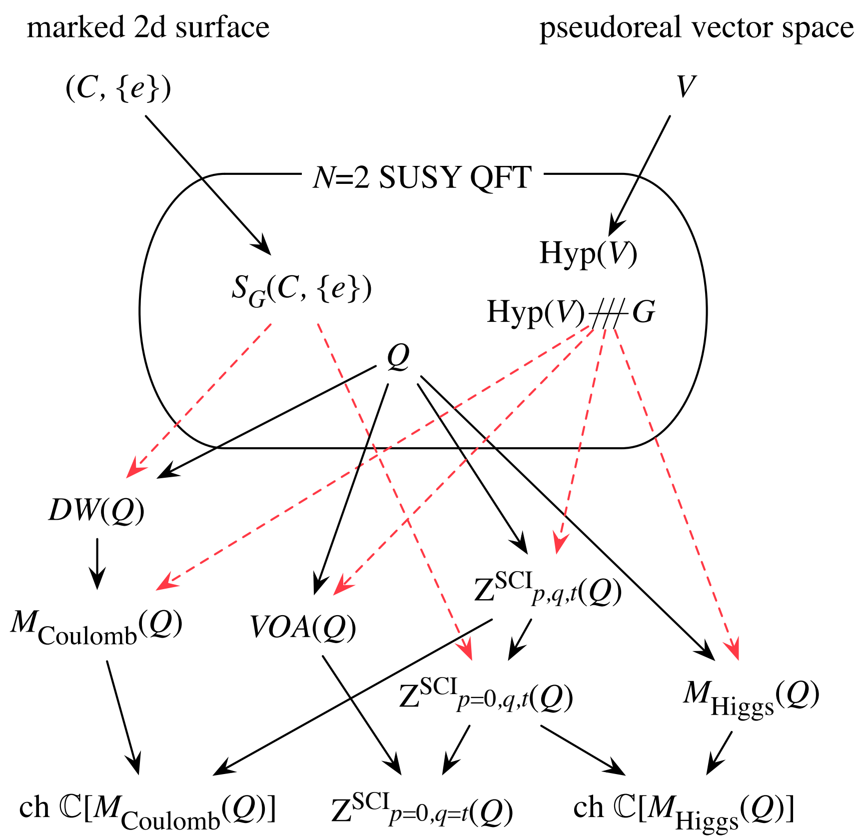


Figure 1: Mathematical objects associated to $\mathcal{N}$=2 SUSY QFTs. Solid black lines show logical relationships, while dashed red lines mean that the computational method is known.

3 Holomorphic symplectic varieties

There is a functor $\mathcal{M}_{\text{Higgs}}$ from the category $\mathcal{Q}(G)$ of 4d $\mathcal{N}$=2 SUSY QFTs to the category of holomorphic symplectic varieties with Hamiltonian G action together with an action of $\tau \in \mathbb{C}^\times$ sending $\omega \to \tau^2\omega$, where ω is the holomorphic symplectic form. The functor $\mathcal{M}_{\text{Higgs}}$ satisfies the relation

$$\mathcal{M}_{\text{Higgs}}(\text{triv}) = \{\text{pt}\}, \qquad \mathcal{M}_{\text{Higgs}}(Q_1 \times Q_2) = \mathcal{M}_{\text{Higgs}}(Q_1) \times \mathcal{M}_{\text{Higgs}}(Q_2) \tag{3.1}$$

and

$$\mathcal{M}_{\text{Higgs}}(Q /\!\!/\!\!/ G) = \mathcal{M}_{\text{Higgs}}(Q) /\!/\!/ G \tag{3.2}$$

where the operation on the right hand side is the holomorphic symplectic quotient, namely that on a holomorphic symplectic variety X with Hamiltonian G action given by the moment map $\mu : X \to \mathfrak{g}^*$, we have

$$X /\!/\!/ G = \mu^{-1}(0)/G. \tag{3.3}$$

We have a basic equality

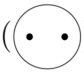
$$\mathcal{M}_{\text{Higgs}}(\text{Hyp}(V)) = V, \tag{3.4}$$

and therefore

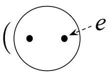
$$\mathcal{M}_{\text{Higgs}}(\text{Hyp}(V)/\!\!/\!\!/G) = V/\!\!/\!\!/G, \tag{3.5}$$

whose right hand side is a holomorphic symplectic quotient of a linear space. This includes Nakajima's quiver varieties.

Mathematical conjectures arise by considering the composition $\mathcal{M}_{\text{Higgs}} \circ S_G$. First, we have the basic property

$\mathcal{M}_{\text{Higgs}}(S_G($  $)) = T^*G.$ (3.6)

Here and in the following, it is to be understood that unmarked punctures in the figures are marked by the nilpotent element $e = 0$. More generally, we have

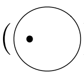
$\mathcal{M}_{\text{Higgs}}(S_G($  $)) = \{(g, x) \subset G \times \mathfrak{g} \simeq T^*G \mid x \in e + S_e\} \subset T^*G$ (3.7)

where $e + S_e$ is the Slodowy slice at e:

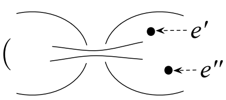
$$e + S_e = \{e + x \mid [f, x] = 0,\ x \in \mathfrak{g}\}. \tag{3.8}$$

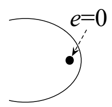
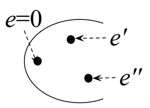
Holomorphic symplectic structures on these varieties were constructed in [11].

Recalling that a puncture marked with the principal nilpotent element is equivalent with not having a puncture, we have

$\mathcal{M}_{\text{Higgs}}(S_G($  $)) = \{(g, x) \subset G \times \mathfrak{g} \simeq T^*G \mid x \in e_{\text{prin}} + S_{e_{\text{prin}}}\} \subset T^*G$ (3.9)

The property (2.8) becomes, after applying $\mathcal{M}_{\text{Higgs}}$:

$\mathcal{M}_{\text{Higgs}}(S_G($  $)) =$

$\Big[\mathcal{M}_{\text{Higgs}}(S_G($  $)) \times \mathcal{M}_{\text{Higgs}}(S_G($  $))\Big] /\!\!/\!\!/G_{\text{diag}}.$ (3.10)

More explicitly, the right hand side is given as

$$\{(x, y) \in X \times Y \mid \mu_X(x) = \mu_Y(y)\}/G_{\text{diag}} \tag{3.11}$$

where X, Y are two holomorphic symplectic varieties in the numerator, and μ_X, μ_Y are the moment maps.[3] Note that if $\mathcal{M}_{\text{Higgs}}(S_G(C, \{(p_i, 0)\}))$ is known, $\mathcal{M}_{\text{Higgs}}(S_G(C, \{(p_i, e_i)\}))$ can be obtained by gluing the spaces (3.7) via the procedure (3.10).

Let us define

$$W_{G,n} := \mathcal{M}_{\text{Higgs}}(S_G(S^2, n \text{ points with } e = 0)) \tag{3.12}$$

[3]The outer automorphism in footnote 2 is inserted to have $\mu_X(x) = \mu_Y(y)$, instead of $\mu_X(x) + \mu_Y(y) = 0$ in this equation.

where C_g is a genus-g surface. This is a holomorphic symplectic variety with Hamiltonian action of

$$S_n \wr G = S_n \ltimes [\underbrace{G \times G \times \cdots \times G}_{n \text{ times}}] \tag{3.13}$$

where the permutation group S_n acts on G^n by permuting them; the action of S_n arises from the property (2.4). Then we have the following conjecture

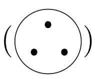
Conjecture 1. *Let G be a simply-connected simply-laced reductive group over $\mathbb{C}$. Let U_G be the holomorphic symplectic variety* (3.7)*. Then there are holomorphic symplectic varieties $W_{G,n}$ with a Hamiltonian action of $S_n \wr G$ such that*

- *$W_{G,1}$ is the space given in* (3.9) *and $W_{G,2} = T^*G$, and*
- *$(W_{G,m} \times W_{G,n})///G_{diag} = W_{G,m+n-2}$.*

Some mathematicians might find a different formulation of the conjecture given in [2] in terms of a 2d topological quantum field theory taking values in the category of holomorphic symplectic varieties more appealing. Ginzburg and Kazhdan had informed the author that they constructed $W_{G,n}$ which satisfy these two conditions.

For $G = A_1$ these varieties are easy to describe. We start from the known relation that

$$S_{A_1}(\ \cdot\ ) = \mathrm{Hyp}(V_1 \otimes_{\mathbb{C}} V_2 \otimes_{\mathbb{C}} V_3) \tag{3.14}$$

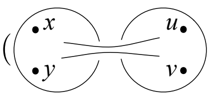


where $V_i \simeq \mathbb{C}^2$ so that V_i is acted naturally by $\mathrm{SL}(2)$. Therefore we have

$$W_{A_1,3} = V_1 \otimes_{\mathbb{C}} V_2 \otimes_{\mathbb{C}} V_3. \tag{3.15}$$

It is instructive to check that the action of $S_3 \wr \mathrm{SL}(2)$ preserves the natural holomorphic symplectic structure. By the gluing property, we have

$$W_{A_1,4} = \mathcal{M}_{\text{Higgs}}(S_{A_1}(\ \cdot\ )) = [V_x \otimes V_y \otimes V \oplus V \otimes V_u \otimes V_v]/// \,\mathrm{SL}(V). \tag{3.16}$$



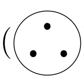
Here, the subscripts x, y, u, v are put in the figure to distinguish distinct copies of $\mathrm{SL}(2)$ action associated to the punctures. The right hand side should be invariant under the exchange $V_y \leftrightarrow V_u$ but this is not obvious in this notation. The right hand side, when written as

$$V \otimes_{\mathbb{R}} \mathbb{R}^8 /// \,\mathrm{SL}(V), \tag{3.17}$$

is the ADHM construction of the minimal nilpotent orbit of $\mathrm{SO}(8) \supset \mathrm{SL}(V_x) \times \mathrm{SL}(V_y) \times \mathrm{SL}(V_u) \times \mathrm{SL}(V_v)$, and the exchange $V_y \leftrightarrow V_u$ is given by an outer automorphism of $\mathrm{SO}(8)$.

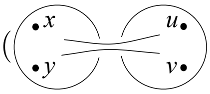
For $G = A_2$, it is conjectured that

$$W_{A_2,3} = \mathcal{M}_{\text{Higgs}}(S_{A_2}(\ \cdot\ )) = \overline{\text{minimal nilpotent orbit of } E_6} \tag{3.18}$$



where the overline denotes the closure. This has $S_3 \wr \mathrm{SL}(3) \subset E_6$ Hamiltonian action. Then

$$\mathcal{M}_{\text{Higgs}}(S_{A_2}(\;\;\text{[diagram]}\;\;)) = (W_{A_2,3} \times W_{A_2,3}) /\!/\!/ \,\mathrm{SL}(3). \tag{3.19}$$



The action of $S_4 \wr \mathrm{SL}(3)$ is not manifest.

As a natural generalization of (3.16) and (3.18), we have the following conjectures. On a flat $\mathbb{C}^2 \simeq \mathbb{R}^4$, we consider the anti-self-dual equation of G_{cpt}-connection $F + \star F = 0$ with the condition that $\int_{\mathbb{C}^2} \operatorname{tr} F \wedge F = -16\pi^2 h^\vee(G) n$, where n is a positive integer. Here G_{cpt} is the compact Lie group associated to the reductive simply-laced simple group G over $\mathbb{C}$, and $h^\vee(G)$ is the dual Coxeter number. The moduli space $\mathcal{M}_{\text{G,n}}$ of solutions of this equation, up to the gauge transformation trivial at infinity, is called the non-centered framed n-instanton moduli space of group G. This is a holomorphic symplectic variety of dimension $2h^\vee(G)n$, with a natural Hamiltonian action of $G \times \mathrm{SL}(2)$. The action of G comes from the action of G_{cpt} at the asymptotic infinity of $\mathbb{C}^2 \simeq \mathbb{R}^4$, and the action of $\mathrm{SL}(2)$ comes from its natural action on $\mathbb{C}^2$. The action of $\mathbb{C}^\times$ on $\mathbb{C}^2$ becomes an action of $\tau \in \mathbb{C}^\times$ on $\mathcal{M}_{\text{G,n}}$ sending the holomorphic symplectic form ω by $\omega \mapsto \tau^2\omega$. The non-centered framed moduli space naturally is a product $\mathcal{M}_{G,n} = \mathbb{C}^2 \times \tilde{\mathcal{M}}_{G,n}$, where $\tilde{\mathcal{M}}_{G,n}$ is called the centered framed moduli space. It is known that $\tilde{\mathcal{M}}_{G,1}$ is the minimal nilpotent orbit of G.

For the centered moduli spaces, we have the following:

Conjecture 2. *Let us denote nilpotent elements of A_{n-1} by a partition of n, which we write as $[n_1, n_2, \ldots]$ with $\sum n_i = n$. We also use the standard abbreviations such as $[3,2,2] = [3,2^2]$. Then we have:*

$$\mathcal{M}_{\text{Higgs}}(S_{A_{3n-1}}(S^2; [n^3], [n^3], [n^3])) = \overline{\tilde{\mathcal{M}}_{E_6,n}}, \tag{3.20}$$
$$\mathcal{M}_{\text{Higgs}}(S_{A_{4n-1}}(S^2; [2n, 2n], [n^4], [n^4])) = \overline{\tilde{\mathcal{M}}_{E_7,n}}, \tag{3.21}$$
$$\mathcal{M}_{\text{Higgs}}(S_{A_{6n-1}}(S^2; [3n, 3n], [2n, 2n, 2n], [n^6])) = \overline{\tilde{\mathcal{M}}_{E_8,n}}. \tag{3.22}$$

Here, the left hand side is to be defined using $W_{G,n}$ constructed in Conjecture 1 and the spaces (3.7) *in terms of the holomorphic symplectic quotient* (3.10). *On the right hand side, the overline denotes Uhlenbeck compactification. On the left hand side, we have a manifest action of $\prod_i G(e_i) = \prod_i \mathrm{SL}(k_i)$ where $(k_1, k_2, k_3) = (3,3,3)$, $= (2,4,4)$, $= (2,3,6)$ for $E_{6,7,8}$, respectively, and a suitable finite quotient of $\prod_i \mathrm{SL}(k_i)$ is a natural maximal subgroup of E_r.*

For non-centered moduli spaces $\mathcal{M}_{E_r,n}$, we have the following:

Conjecture 3. *For $n > 1$, there are the equivalences*

$$\mathcal{M}_{\text{Higgs}}(S_{A_{3n-1}}(S^2; [n^2, n-1, 1], [n^3], [n^3]) = \overline{\mathcal{M}_{E_6,n}}, \tag{3.23}$$
$$\mathcal{M}_{\text{Higgs}}(S_{A_{4n-1}}(S^2; [2n, 2n-1, 1], [n^4], [n^4]) = \overline{\mathcal{M}_{E_7,n}}, \tag{3.24}$$
$$\mathcal{M}_{\text{Higgs}}(S_{A_{6n-1}}(S^2; [3n, 3n-1, 1], [2n, 2n, 2n], [n^6]) = \overline{\mathcal{M}_{E_8,n}}. \tag{3.25}$$

On the left hand side, we have spaces defined by starting from $W_{G,3}$ in Conjecture 1 and gluing spaces (3.7) *via the holomorphic symplectic quotient* (3.10). *On the right hand side, we have the non-centered instanton moduli space $\mathcal{M}_{E_r,n} = \mathbb{C}^2 \times \tilde{\mathcal{M}}_{E_r,n}$, and the overline denotes Uhlenbeck compactifications. There is a natural action of $\mathrm{SL}(2)$ coming from its action on $\mathbb{C}^2$ preserving its holomorphic symplectic structure. The space on the left hand side has a manifest Hamiltonian action of $\mathbb{C}^\times \times \mathbb{C}^\times \times \mathrm{SL}(k_2) \times \mathrm{SL}(k_3)$ where $(k_2, k_3) = (3,3)$, $= (4,4)$ and $= (3,6)$ for $E_{6,7,8}$ respectively, which is a natural subgroup of $\mathrm{SL}(2) \times E_r$.*

4 Hall-Littlewood polynomials

Let us study the ring of functions of $W_{G,n}$. If the readers are uncomfortable with working with spaces not rigorously constructed yet, take $G = A_1$, for which we know $W_{A_1,3}$ (3.15). They are holomorphic symplectic varieties with Hamiltonian G^3 action, together with an action of $\tau \in \mathbb{C}^\times$ sending $\omega \to \tau^2\omega$ where ω is the holomorphic symplectic form. The action of $\tau \in \mathbb{C}^\times$ on $W_{A_1,3} = V_1 \otimes V_2 \otimes V_3$ is by a constant multiplication. Consider the character of $\mathbb{C}[W_{A_1,3}]$ under $\mathrm{SL}(2)^3 \times \mathbb{C}^\times$, which is

$$\mathrm{tr}_{\mathbb{C}[W_{A_1,3}]}\, abc\tau = \prod_{\pm\pm\pm} \frac{1}{1 - a_1{}^{\pm 1} b_1{}^{\pm 1} c_1{}^{\pm 1} \tau} \tag{4.1}$$

where we used a notation $a = \mathrm{diag}(a_1, 1/a_1) \in \mathrm{SL}(2)$. It is possible to rewrite it in terms of Hall-Littlewood polynomials of type A_1. To write it down, it is more convenient to consider arbitrary G, but then the relation has to be stated as a conjecture [4]:

Conjecture 4.

$$\mathrm{tr}_{\mathbb{C}[W_{G,3}]}\, abc\tau = \frac{K_0(a)K_0(b)K_0(c)}{K_{e_{prin}}} \sum_\lambda \frac{\underline{P_\lambda}(a)\underline{P_\lambda}(b)\underline{P_\lambda}(c)}{\underline{P_\lambda}(\tau^{2\rho})}. \tag{4.2}$$

Here, $\underline{P_\lambda}(z)$ is defined to be $= N_\lambda P_\lambda(z)$ where $P_\lambda(z)$ is the standard Hall-Littlewood polynomial of type G when $z \in G$ is regarded as an element $z = (z_1, \ldots, z_r) \in T^r \subset G$ where T^r is the Cartan torus of G. N_λ is a normalization factor so that $\underline{P_\lambda}$ is orthonormal under the following measure:

$$\delta_{\mu\nu} = \frac{1}{|W_G|} \int_{T^r} \underline{P_\lambda}(z)\underline{P_\mu}(1/z) \frac{1}{(1-\tau^2)^r} \prod_{\alpha:\text{roots of } G} \frac{1 - z^\alpha}{1 - \tau^2 z^\alpha} \prod_{i=1}^r \frac{dz_i}{2\pi\sqrt{-1} z_i} \tag{4.3}$$

where $|W_G|$ is the order of the Weyl group, and we used the standard abbreviation $z^\alpha = \prod_i z_i^{\alpha_i}$ for a weight $\alpha = (\alpha_1, \ldots, \alpha_r)$ of G. Then $\tau^{2\rho}$ is an element of G given by τ and the Weyl vector ρ. For example, $\tau^{2\rho} = \mathrm{diag}(\tau, 1/\tau)$ for $G = A_1$.

The prefactors K are given by

$$K_0(z) = \frac{1}{(1-\tau^2)^r} \prod_{\alpha:\text{roots of } G} \frac{1}{1 - \tau^2 z^\alpha}, \tag{4.4}$$

$$K_{e_{prin}} = \prod_{i=1}^r \frac{1}{1 - \tau^{2d_i}} \tag{4.5}$$

where $d_1, \ldots, d_r$ are the exponents plus one of G. For example, $(d_1, d_2, \ldots, d_{n-1}) = (2, 3, \ldots, n)$ for $G = A_{n-1}$.

Assuming the validity of the conjecture above, we can easily compute the character of the ring of functions of $W_{G,n}$, due to the following relation between the holomorphic symplectic quotient and the Hall-Littlewood polynomials. Suppose X is a holomorphic symplectic variety with a Hamiltonian action of $G \times F$, such that the stabilizer at generic points of the action of G is trivial. Suppose furthermore that there is an action of $\tau \in \mathbb{C}^\times$ such that the holomorphic symplectic form ω is acted as $\omega \mapsto \tau^2\omega$. Recall the definition of the holomorphic symplectic quotient (3.3) and note that the $\mathbb{C}^\times$ action on the moment map is given by $\mu \mapsto \tau^2\mu$. Then, for an element $y \in F$, we have

$$\mathrm{tr}_{\mathbb{C}[X///G]}\, y\tau = \frac{1}{|W_G|}\int_{T^r} (\mathrm{tr}_{\mathbb{C}[X]}\, yz\tau) \left[(1-\tau^2)^r \prod_\alpha (1-\tau^2 z^\alpha)\right] \prod_\alpha (1-z^\alpha) \prod_{i=1}^r \frac{dz_i}{2\pi\sqrt{-1}z_i} \tag{4.6}$$

$$= \frac{1}{|W_G|}\int_{T^r} (\mathrm{tr}_{\mathbb{C}[X]}\, yz\tau) K_0(z)^{-2} \frac{1}{(1-\tau^2)^r} \prod_\alpha \frac{1-z^\alpha}{1-\tau^2 z^\alpha} \prod_{i=1}^r \frac{dz_i}{2\pi\sqrt{-1}z_i}. \tag{4.7}$$

Then, we find

$$\mathrm{tr}_{\mathbb{C}[W_{G,n}]}\, a_1 a_2 \cdots a_n \tau = \frac{\prod K_0(a_i)}{K_{e_{\text{prin}}}^{n-2}} \sum_\lambda \frac{\prod_i \underline{P_\lambda}(a_i)}{\underline{P_\lambda}(\tau^{2\rho})^{n-2}}. \tag{4.8}$$

So far we have been considering the function rings of $W_{G,n}$ (3.12), with all points marked by the zero orbit. In favorable circumstances, it is possible to write down the character of the function rings when the points are marked by non-zero nilpotent elements, since a point marked by 0 can be converted to a point marked by other nilpotent orbits e by gluing (3.7) using (3.10). At the level of function rings, this just means to set the moment map μ associated to the point to lie on the Slodowy slice $\mu \in e + S_e$. Assuming that this equation determines a subvariety which is nice in some sense, we have the following:

Conjecture 5. *In favorable circumstances, the character of the function ring of*

$$X = \mathcal{M}_{\text{Higgs}}(S_G(S^2, \{p_i, e_i\}_{i=1}^n)) \tag{4.9}$$

is given by

$$\mathrm{tr}_{\mathbb{C}[X]}\, a_1 a_2 \cdots a_n \tau = \frac{\prod K_{e_i}(a_i)}{K_{e_{\text{prin}}}^{n-2}} \sum_\lambda \frac{\prod_i \underline{P_\lambda}(a_i \tau^{2h_i})}{\underline{P_\lambda}(\tau^{2\rho})^{n-2}}. \tag{4.10}$$

where $a_i \in G(e_i)$ and h_i is a semisimple element such that (e_i, h_i, f_i) is an $\mathfrak{sl}(2)$ triple. The prefactor $K_e(a)$ for $a \in G(e)$ is given as follows. Using the $\mathfrak{sl}(2)$ triple (e, h, f), we can decompose $\mathfrak{g}$ according to $\mathfrak{sl}(2) \times G(e)$ as

$$\mathfrak{g} = \oplus_d V_d \otimes R_d, \tag{4.11}$$

where V_d is the irreducible representation of dimension d of $\mathfrak{sl}(2)$. Then

$$K_e(a) = \prod_d \prod_{w:\text{weights of } R_d} \frac{1}{1-\tau^{d+1}a^w}. \tag{4.12}$$

Note that $K_{e_{prin}}$ defined in (4.5) *is a special case of this construction.*

It is not clear to the author how to state concisely which combination of e_i make the configuration 'favorable'.

This conjecture, when combined with a partial knowledge of $\mathcal{M}_{\text{Higgs}}(S_G(S^2, \{p_i, e_i\})$ as concrete holomorphic symplectic varieties, gives rise to a number of conjectural identities. First of all, for $G = A_1$, we have an equality between an infinite product (4.1) and an infinite sum (4.2). Second, $W_{A_2,3}$ is the minimal nilpotent orbit of E_6. The function ring of the minimal nilpotent orbit of arbitrary $\mathfrak{g}$ is known to be of the form

$$\mathbb{C}[\text{min. nilp. orbit of } \mathfrak{g}] = \mathbb{C} \oplus \tau^2 V_\lambda \oplus \tau^4 V_{2\lambda} \oplus \cdots \tag{4.13}$$

where V_μ is an irreducible representation with the highest weight μ and λ is such that $\mathfrak{g} = V_\lambda$; τ^n in each term specifies the action of $\tau \in \mathbb{C}^\times$. Then the infinite sum (4.2) for $\mathfrak{g} = A_2$ should give the character of the function ring of the minimal nilpotent orbit of E_6, given as (4.13). The same argument should apply to the cases listed in Conjecture 2 when $n = 1$.

For $n > 1$, the choices of e_i listed in Conjecture 2 are not in the favorable conditions where the equality (4.10) is applicable, but the choices in Conjecture 3 are. Then we have:

Conjecture 6. *The character of the function ring of the framed, non-centered n-instanton moduli space $\overline{\mathcal{M}_{E_r,n}}$ with $n > 1$ under $\mathrm{SL}(2) \times E_r \times \mathbb{C}^\times$ is given in terms of Hall-Littlewood polynomials as*

$$\mathrm{tr}_{\mathbb{C}[\overline{\mathcal{M}_{E_r,n}}]}\, abc\tau = \frac{K_{e_1}(a)K_{e_2}(b)K_{e_3}(c)}{K_{e_{prin}}} \sum_\lambda \frac{\underline{P_\lambda}(a\tau^{2h_1})\underline{P_\lambda}(b\tau^{2h_2})\underline{P_\lambda}(c\tau^{2h_3})}{\underline{P_\lambda}(\tau^{2\rho})} \tag{4.14}$$

where

- *For E_6, the Hall-Littlewood polynomials are of type A_{3n-1}, with $e_1 = [n^2, n-1, 1]$, $e_{2,3} = [n^3]$. We then have $a = (t, a') \in G(e_1) = \mathbb{C}^\times \times \mathbb{C}^\times$, $b, c \in G(e_{2,3}) \simeq \mathrm{SL}(3)$, and $t \in \mathbb{C}^\times \subset \mathrm{SL}(2)$, $(a', b, c) \in \mathbb{C}^\times \times \mathrm{SL}(3)^2 \subset E_6$.*
- *For E_7, the Hall-Littlewood polynomials are of type A_{4n-1}, with $e_1 = [2n, n-1, 1]$, $e_{2,3} = [n^4]$. We then have $a = (t, a') \in G(e_1) = \mathbb{C}^\times \times \mathbb{C}^\times$, $b, c \in G(e_{2,3}) \simeq \mathrm{SL}(4)$, and $t \in \mathbb{C}^\times \subset \mathrm{SL}(2)$, $(a', b, c) \in \mathbb{C}^\times \times \mathrm{SL}(4)^2 \subset E_7$.*
- *For E_8, the Hall-Littlewood polynomials are of type A_{6n-1}, with $e_1 = [3n, 3n-1, 1]$, $e_2 = [2n, 2n, 2n]$, $e_3 = [n^6]$. We then have $a = (t, a') \in G(e_1) = \mathbb{C}^\times \times \mathbb{C}^\times$, $b \in G(e_2) \simeq \mathrm{SL}(3)$, $c \in G(e_3) \simeq \mathrm{SL}(6)$, and $t \in \mathbb{C}^\times \subset \mathrm{SL}(2)$, $(a', b, c) \in \mathbb{C}^\times \times \mathrm{SL}(3) \times \mathrm{SL}(6) \subset E_8$.*

and semisimple elements h_i are chosen so that (e_i, h_i, f_i) are $\mathfrak{sl}(2)$ triples.

These relations have been put to some test in [12, 13, 14].

If the reader finds the function rings of instanton moduli spaces slightly too daunting, we can also consider the cases when $S_G(S^2, \{e_1, e_2, e_3\}) = \mathrm{Hyp}(X)$ for a symplectic vector space X. Applying $\mathcal{M}_{\text{Higgs}}$ to both sides, we have

$$X = \mathcal{M}_{\text{Higgs}}(S_G(S^2, \{e_1, e_2, e_3\})). \tag{4.15}$$

Taking the character, we have the following:

Conjecture 7. *For suitable choices we list below of G, $e_{1,2,3}$ and a pseudoreal representation X of $\prod_i G(e_i)$, we have the equality*

$$\prod_{u,v,w} \frac{1}{1-a^u b^v c^w \tau} = \frac{K_{e_1}(a)K_{e_2}(b)K_{e_3}(c)}{K_{e_{prin}}} \sum_\lambda \frac{\underline{P_\lambda}(a\tau^{2h_1})\underline{P_\lambda}(b\tau^{2h_2})\underline{P_\lambda}(c\tau^{2h_3})}{\underline{P_\lambda}(\tau^{2\rho})}. \tag{4.16}$$

where (u, v, w) runs over the weights of X as a representation of $G(e_1) \times G(e_2) \times G(e_3)$, and as always, (e_i, h_i, f_i) are chosen so that they form $\mathfrak{sl}(2)$ triples. Some of the choices of G, e_i, X are as follows:

- *$G = A_1$, $e_{1,2,3} = 0$, $X = V_1 \otimes V_2 \otimes V_3$ where $V_i \simeq \mathbb{C}^2$, where we identify $G(e_i) = \mathrm{SL}(V_i) \simeq G$.*
- *$G = A_{n-1}$, $e_1 = e_3 = [1^n] = 0$, $e_3 = [n-1, 1]$, and $X = W \otimes V_1 \otimes V_3^* \oplus W^* \otimes V_1^* \otimes V_3$. Here, $V_i \simeq \mathbb{C}^n$ where we identify $G(e_i) = \mathrm{SL}(V_i) \simeq G$, and $G(e_2) = \mathbb{C}^\times$ with W its natural one-dimensional representation.*
- *$G = E_6$, $e_1 = E_6(a_1)$, $e_2 = A_2 + 2A_1$, $e_3 = 0$, and $X = V_{min} \otimes F \oplus \bar{V}_{min} \otimes \bar{F}$. Here, $G(e_1) = 1$, $G(e_2) = \mathrm{SL}(2) \times \mathbb{C}^\times$ with F its natural two dimensional representation, and $G(e_3) = E_6$ with V_{min} its minuscule representation of dimension 27.*
- *$G = E_7$, $e_1 = E_7(a_1)$, $e_2 = A_3 + A_2 + A_1$, $e_3 = 0$ and $X = V_{min} \otimes_{\mathbb{R}} \mathbb{R}^3$. Here, $G(e_1) = 1$, $G(e_2) = \mathrm{SO}(3)$ with $\mathbb{R}^3$ its natural three dimensional real representation, and $G(e_3) = E_7$ with V_{min} its minuscule representation of dimension 56.*

Note that together with this conjecture, there is a conjectural equality (4.15) where the right hand side is computed by starting from $W_{G,3}$ in Conjecture 1 and gluing spaces (3.7) via the holomorphic symplectic quotient (3.10).

5 Hitchin systems and Lusztig-Spaltenstein duality

A reader would surely ask how physicists know for which choices of $e_{1,2,3}$ we have a symplectic vector space as $\mathcal{M}_{\text{Higgs}} \circ S_G$ as in (4.15). One necessary condition is that

$$2 \dim G = \sum_{i=1}^{3} \dim d_{LS}(O_{e_i}) \tag{5.1}$$

where O_{e_i} is the G-orbit of e_i, and d_{LS} is the duality operation of Lusztig and Spaltenstein. This is an order-reversing map on the set of nilpotent orbits of G, where the partial order is defined by the relation $O_1 \subset \bar{O}_2$, where the bar stands for the closure. The duality operation satisfies

$$d_{LS}^2 = \mathrm{id} \tag{5.2}$$

when $G = A_{n-1}$, and is given by the transpose of the partition of n. In general it only satisfies

$$d_{LS}{}^3 = d_{LS}. \tag{5.3}$$

For more, see e.g. [15].

To state where the condition (5.1) comes from, we need to introduce two new concepts. For a 4d $\mathcal{N}$=2 SUSY QFT $Q \in \mathcal{Q}$, we have a family $DW(Q)$ of holomorphic integrable systems [16]. The base of this integrable system is $\mathcal{M}_{\text{Coulomb}}(Q)$, which is an affine space with an action of $\mathbb{C}^\times$. In particular, its function ring is a free polynomial ring. There is a projection $DW(Q) \to \mathcal{M}_{\text{Coulomb}}(Q)$. In general, we have

$$\mathbb{C}[\mathcal{M}_{\text{Coulomb}}(Q_1 \times Q_2)] = \mathbb{C}[\mathcal{M}_{\text{Coulomb}}(Q_1)] \otimes \mathbb{C}[\mathcal{M}_{\text{Coulomb}}(Q_2)] \tag{5.4}$$

and

$$\mathbb{C}[\mathcal{M}_{\text{Coulomb}}(Q /\!\!/\!\!/ G)] = \mathbb{C}[\mathcal{M}_{\text{Coulomb}}(Q)] \otimes \mathbb{C}[\mathfrak{g}]^G \tag{5.5}$$

where we assign degree 1 to a linear functional on $\mathfrak{g}$. Thus $\mathbb{C}[\mathcal{M}_{\text{Coulomb}}(Q /\!\!/\!\!/ G)]$ is guaranteed to be a free polynomial ring assuming $\mathbb{C}[\mathcal{M}_{\text{Coulomb}}(Q)]$ is. For $Q = \text{Hyp}(V)$, we have

$$\mathbb{C}[\mathcal{M}_{\text{Coulomb}}(\text{Hyp}(V))] = \mathbb{C}. \tag{5.6}$$

The properties satisfied by $DW(Q)$ for $Q = \text{Hyp}(V) /\!\!/\!\!/ G$ are given in detail in [1]. Determining $DW(Q)$ explicitly for these cases using these properties is usually called the Seiberg-Witten theory in the physics literature. Given a $Q \in \mathcal{Q}$ and a four-manifold M, there is a generalized Donaldson polynomial $Z(Q, M)$. For $Q = \text{triv} /\!\!/\!\!/ \text{SO}(3)$, $Z(Q, M)$ is literally the original Donaldson polynomial. A physics argument tells us that $Z(Q, M)$ can also be determined from the knowledge of $DW(Q)$; this leads to the Seiberg-Witten invariants of M. These are of course a very important mathematical conjecture arising from the study of 4d $\mathcal{N}$=2 SUSY QFTs, but are too huge to be considered in this note.

Instead, we concentrate on $DW(Q)$ and $\mathcal{M}_{\text{Coulomb}}(Q)$ for $Q = S_G(C, \{(p_i, e_i\})$ here. Essentially, $DW(Q)$ is given by the family of the G-Hitchin system on C with singularities given by e_i, where the family is over the complex structure on $(C, \{p_i\})$ regarded as a punctured Riemann surface. Then $DW(Q) \to \mathcal{M}_{\text{Coulomb}}(Q)$ is given by the associated Hitchin fibration, with

$$\dim \mathcal{M}_{\text{Coulomb}}(Q) = (g-1)\dim G + \sum_i \frac{1}{2} \dim_{\mathbb{C}} d_{LS}(O_{e_i}). \tag{5.7}$$

When $Q = \text{Hyp}(V)$, the Coulomb branch is a point, therefore the right hand side should be zero. This gives the condition (5.1). Let us discuss $DW(Q)$ more carefully following [5], which lead us to a few conjectures involving Lusztig-Spaltenstein duality.

Let C be a Riemann surface with punctures $p_1, \ldots p_k$ with labels which we describe later. Let $P \to C \setminus \{p_i\}$ be a G-bundle with a reference connection d''. We take

$$\phi \in \Omega^{1,0}(C \setminus \{p_i\}, P \times_G \mathfrak{g}) A'' \in \Omega^{0,1}(C \setminus \{p_i\}, P \times_G \mathfrak{g}). \tag{5.8}$$

Note that $D'' = d'' + A$ is also a connection. We demand that ϕ has a singularity of the form

$$\phi \simeq \alpha_i \frac{dz_i}{z_i} + \text{regular} + \cdots \tag{5.9}$$

where z_i is a local coordinate such that the puncture p_i is at $z_i = 0$ and

$$\alpha_i \in d_{LS}(O_{e_i}). \tag{5.10}$$

Not all of the group of gauge transformation

$$\mathcal{G} = \{f : C \to G\} \tag{5.11}$$

preserves the boundary condition. We let

$$\mathcal{G}_0 = \{f : C \to G \mid f(x_i) \in G^{\alpha_i}\}. \tag{5.12}$$

Then we can consider the Hitchin map

$$h : \{D''\phi = 0\}/\mathcal{G}_0 \to \bigoplus_a H^0(d_a K_C + (d_a - 1)\sum p_i). \tag{5.13}$$

Here, K_C is the canonical divisor and the Hitchin map h is given by

$$h : \phi \mapsto u_1(\phi) \oplus \cdots \oplus u_r(\phi) \tag{5.14}$$

where we fixed the isomorphism

$$\mathbb{C}[\mathfrak{g}]^G \simeq \mathbb{C}[u_1, \ldots, u_r] \tag{5.15}$$

so that u_a has degree d_a.

This fibration is almost $DW(Q) \to \mathcal{M}_{\text{Coulomb}}(Q)$, but not quite. First, let us describe the situation for type A_{n-1}. A label e is given by a nilpotent orbit, or equivalently a partition $[n_i]$ of n. The Lusztig-Spaltenstein dual α is given by the transpose partition $[a_i]$. From this we define integers $p_d(\alpha) = d - \nu_d(\alpha)$ where

$$(\nu_1(\alpha), \nu_2(\alpha), \ldots, \nu_N(\alpha)) = (\underbrace{1, \ldots, 1}_{a_1}, \underbrace{2, \ldots, 2}_{a_2}, \ldots,). \tag{5.16}$$

Then we find that the image of the Hitchin map h is in fact onto

$$h : \{D''\phi = 0\}/\mathcal{G}_0 \to \bigoplus_{d=2}^{n} H^0(dK_C + \sum_i p_d(\alpha_i)x_i) \tag{5.17}$$

when the choice of the genus and the labels of the punctures is generic enough. The right hand side is an affine space whose dimension is given by (5.7), and we identify thus:

$$\mathcal{M}_{\text{Coulomb}}(S_{A_{n-1}}(C, \{e_i\})) = \bigoplus_{d=2}^{n} H^0(dK_C + \sum_i p_d(\alpha_i)x_i). \tag{5.18}$$

Note that we have an equality

$$\mathcal{M}_{\text{Coulomb}}(S_{A_{n-1}}(C, \{e_i\})) = (g-1)(\sum_{d=2}^{n}(2d-1)L^d) + \sum_i V(e_i) \tag{5.19}$$

as elements of $K_{\mathbb{C}^\times}(\mathrm{pt})$, where $L \simeq \mathbb{C}$ is the one-dimensional standard representation of $\mathbb{C}^\times$ and

$$V(e_i) = \sum_{d=2}^{n} p_d(\alpha_i) L^d. \tag{5.20}$$

For other G (which we assume to be simply-laced), the image of the Hitchin projection (5.13) is not in general an affine space; however, we have the following conjecture. Due to the complexity of the full formulation, we first state the conjecture when all e is special, in the sense that O_e is in the image of d_{LS}:

Conjecture 8. *Fix a simply-laced G. Consider the G-Hitchin system on punctured Riemann surfaces C of genus g with punctures p_i labeled by special nilpotent elements e_i as defined from* (5.8) *to* (5.13)*. Assume that the choice of the genus and the labels of the punctures is generic enough. Then the Hitchin projection* (5.13) *factors through an affine space $\mathcal{M}_{\text{Coulomb}}(Q)$:*

$$h : DW(Q) \to \mathcal{M}_{\text{Coulomb}}(Q) \xrightarrow{\textit{finite}} h(DW(Q)) \tag{5.21}$$

where $DW(Q) := \{D''\phi = 0\}/\mathcal{G}_0$, the second map is a finite-to-one map, and the maps preserves the $\mathbb{C}^\times$ action. Moreover, there is a natural equality

$$\mathcal{M}_{\text{Coulomb}}(Q) = (g-1)(\sum_{a=1}^{r}(2d_a - 1)L^{d_a}) + \sum_i V(e_i) \tag{5.22}$$

as elements of $K_{\mathbb{C}^\times}(\mathrm{pt})$. Here, $r = \operatorname{rank} G$, and $(d_1, \ldots, d_r)$ are the exponents plus one. Furthermore, the affine spaces $V(e)$ attached to nilpotent elements satisfy the following properties: First, the dimension is given by

$$\dim V(e) = \frac{1}{2} \dim d_{LS}(O_e). \tag{5.23}$$

Second, writing $\mathbb{C}[V(e)] = \mathbb{C}[t_1, \ldots, t_s]$ where $s = \dim V(e)$, we have

$$\sum_i (2\deg(t_i) - 1) = 8\langle \rho, \rho - \frac{h}{2} \rangle + \frac{1}{2}(\operatorname{rank} G - \dim \mathfrak{g}^h). \tag{5.24}$$

Here, h is the element in the Cartan subalgebra such that (e, h, f) is an $SL(2)$ triple, ρ is the Weyl vector, $\langle , \rangle$ is the standard inner product on the Cartan subalgebra, and $\mathfrak{g}^h$ is the subalgebra of $\mathfrak{g}$ commuting with h. We define the right hand side of (5.24) *to be $n_v(e)$.*

It is not clear to the author how to state the genericity assumption in the conjecture explicitly. In practice it suffices if $g \geq 2$, or $g = 1$ with at least one nontrivial puncture, or $g = 0$ with at least two punctures with $e = 0$. But these are not necessary conditions.

A few comments are in order.

- When there is no puncture, the equality (5.22) is just that

$$\bigoplus_{a=1}^{r} H^0(K_C^{\otimes d_a}) = (g-1)(\sum_{a=1}^{r}(2d_a - 1)L^{d_a}) \tag{5.25}$$

  when $g > 1$.

- The relations (5.23) and (5.24) are rather strange, in that the property of the affine space $V(e)$ is given in terms of both the original orbit O_e and the Lusztig-Spaltenstein dual orbit O_α.
- By a direct computation, we see that $V(e)$ for a nilpotent element e of $G = A_{n-1}$ as defined in (5.20) satisfies both (5.23) and (5.24).
- For $e = 0$, we have
$$V(0) = \sum_{a=1}^{r}(d_a - 1)L^{d_a}, \tag{5.26}$$
which satisfies both (5.23) and (5.24), due to equalities expressing $\sum_a (d_a)^k$ in terms of r and $h^\vee(G)$.
- The relations (2.8), (5.4), (5.5) and (5.22) are compatible because
$$\sum_{a=1}^{r}(2d_a - 1)L^{d_a} = 2V(0) + (\mathfrak{g}/G) \tag{5.27}$$
as elements of $K_{\mathbb{C}^\times}(\mathrm{pt})$.
- The conjecture states only the dimension of $V(e)$ and the sum of the degrees of the generators of $V(e)$. Of course it is more desirable to describe $V(e)$ as an element of $K_{\mathbb{C}^\times}(\mathrm{pt})$ in terms of the nilpotent element e.

For an element $V \in K_{\mathbb{C}^\times}(\mathrm{pt})$, let us define
$$n_v(V) = \sum_i (2\deg(t_i) - 1) \tag{5.28}$$
where t_i are the generators of $\mathbb{C}[V]$ with fixed degrees
$$\mathbb{C}[V] = \mathbb{C}[t_1, \ldots, t_{\dim V}]. \tag{5.29}$$
We define the right hand side of (5.24) to be $n_v(e)$, then the equality (5.24) itself can be written as $n_v(V(e)) = n_v(e)$. Note also that
$$n_v(\mathfrak{g}/G) = \sum_{a=1}^{r} d_a = \dim G. \tag{5.30}$$
When $Q = \mathrm{Hyp}(V)$, $\mathcal{M}_{\text{Coulomb}}(Q) = \{\mathrm{pt}\}$, therefore $n_v(\mathcal{M}_{\text{Coulomb}}(Q)) = 0$. When $Q = S_G(S^2, \{e_1, e_2, e_3\})$, $\mathcal{M}_{\text{Coulomb}}(Q)$ is given in (5.22), and applying n_v to both sides of (5.22) we have
$$n_v(\mathcal{M}_{\text{Coulomb}}(Q)) = -(\frac{4}{3}h^\vee(G)\dim G + \operatorname{rank} G) + \sum_i n_v(e_i). \tag{5.31}$$

Then, when $Q = S_G(S^2, \{e_1, e_2, e_3\}) = \mathrm{Hyp}(V)$ as listed in Conjecture 7, we should not only have (5.1) but also have

$$\frac{4}{3}h^\vee(G)\dim G + \mathrm{rank}\, G = \sum_{i=1}^{3} n_v(e_i). \tag{5.32}$$

Checking this equality against the cases listed in Conjecture 7 is a fun exercise. We can also state a conjecture

Conjecture 9. *Pick a simply-laced G and three nilpotent orbits $e_{1,2,3}$ such that the equality* (5.32) *is satisfied. Then we automatically have* (5.1)*, and the main equality of Conjecture 7 holds for a suitable symplectic vector space X.*

When the nilpotent elements e marking the punctures are not necessarily special, we need to use a finite group introduced by Sommers and Achar [17, 18, 19] to describe $V(e)$. Given a special orbit O_e, the set of nilpotent orbits $O_{e'}$ such that $d^2_{LS}(O_{e'}) = O_e$ is called the special piece of O_e. Within the special piece of O_e, O_e itself is the maximal element. The partial order among the special piece is encoded in a subgroup $\mathcal{C}(O'_e) \subset \overline{A}(O_e)$ where $\overline{A}(O_e)$ is a reflection group introduced by Lusztig, defined as a certain quotient of the component group $A(O_e) = G^e/(G^e)^\circ$. Then for two orbits $O_{e'}$ and $O_{e''}$ in the special piece of O_e, we have

$$O_{e'} \le O_{e''} \longleftrightarrow \mathcal{C}(O_{e'}) \supset \mathcal{C}(O_{e''}). \tag{5.33}$$

In particular $\mathcal{C}(O_e) = \{\mathrm{id}\}$. Now we can state the conjecture when e can be allowed to be non-special:

Conjecture 10. *Fix a simply-laced G. Consider the G-Hitchin system on punctured Riemann surfaces C of genus g with punctures p_i labeled by nilpotent elements e_i as defined from* (5.8) *to* (5.13)*. There is a natural projection*

$$\pi : \mathcal{G}_0 \to \prod_i A(\alpha_i) \to \prod_i \overline{A}(\alpha_i) \tag{5.34}$$

where $A(\alpha) = G^\alpha/G^{\alpha\circ}$ is the component group of the stabilizer of α, and $\overline{A}(\alpha)$ is the Lusztig's component group. Using the Sommers-Achar groups $\mathcal{C}(e_i) \subset \overline{A}(\alpha_i)$, let us define $\mathcal{G}'_0$ via

$$\mathcal{G}'_0 = \pi^{-1} \prod_i \mathcal{C}(e_i) \tag{5.35}$$

and the Hitchin map

$$h : \{D''\phi = 0\}/\mathcal{G}'_0 \to \bigoplus_a H^0(d_a K_C + (d_a - 1)\sum p_i). \tag{5.36}$$

Define the leftmost side to be $DW(Q)$. Then the Hitchin map (5.13) *factors through an affine space $\mathcal{M}_{\text{Coulomb}}(Q)$:*

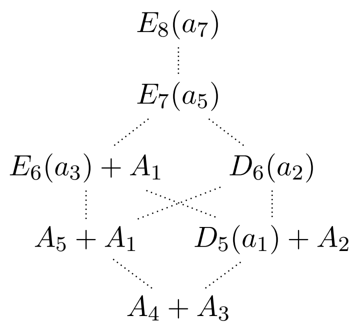
$$h : DW(Q) \to \mathcal{M}_{\text{Coulomb}}(Q) \xrightarrow{\text{finite}} h(DW(Q)) \tag{5.37}$$

where the second map is a finite-to-one map. Furthermore, $\mathcal{M}_{\text{Coulomb}}(Q)$ still has the decomposition (5.22)*. The spaces $V(e')$ for non-special e' satisfy both* (5.23) *and* (5.24)*, and in addition we have*

$$V(e') = V(e)/\mathcal{C}(e') \tag{5.38}$$

where $O_e = d^2_{LS}(O_{e'})$, $\mathcal{C}(e')$ is the Sommers-Achar group. Note that $\mathcal{C}(e')$ is a reflection group, and therefore both $V(e)$ and $V'(e)$ can be affine spaces.

A check of the latter conjecture is provided by the following heuristic study of a special piece of E_8. Take a special piece of $e_0 = E_8(a_7)$. Basic properties of each e in the special piece are displayed in Table 1. The Spaltenstein dual is e_0 for all e in the table. $\overline{A}(e_0)$ is S_5, and the subgroup of S_5 assigned to each of the 7 nilpotent orbits by Sommers is also shown in the table, in terms of the generating reflections $(i, i+1)$, which act on the set $\{1,2,3,4,5\}$. Using (5.24) one can compute $n_v(e)$ for each nilpotent orbit, as h for each e is known. Since $\dim_{\mathbb{C}} O_{e_0} = 208$, $\dim V(e) = 104$ for all e. The degrees of four of the bases can be determined as follows.



e	h	$\mathcal{C}(e)$	$n_v(e)$		known n_v
$E_8(a_7)$	0 0002000	$\emptyset$	4064	$6,\ 6,\ 6,\ 6$	44
$E_7(a_5)$	0 0010100	(12)	4076	$6,\ 6,\ 6, 12$	56
$D_6(a_2)$	1 0100010	(12), (34)	4088	$6,\ 6, 12, 12$	68
$E_6(a_3)+A_1$	0 0101001	(12), (23)	4100	$6,\ 6, 12, 18$	80
A_5+A_1	0 1000101	(12), (23), (45)	4112	$6, 12, 12, 18$	92
$D_5(a_1)+A_2$	0 1010010	(12), (23), (34)	4136	$6, 12, 18, 24$	116
A_4+A_3	0 0100100	(12), (23), (34), (45)	4184	$12, 18, 24, 30$	164

Table 1: A special piece in the set of nilpotent orbits of E_8, h given as the inner products of h with simple roots, the corresponding subgroups of $S_5 = \overline{A}(E_8(a_7))$, n_v and the degrees of generators of $V(e)$ governed by subgroups of S_5. The sixth column shows the contribution to n_v just from the known 4 generators.

Since $\overline{A}(E_8(a_7))$ is S_5, for the special nilpotent orbit e_0 we expect

$$V(e_0) = V \oplus V' \tag{5.39}$$

with $\dim V = 4$, $\dim V' = 100$ so that S_5 acts as the Weyl group of A_4 on V and acts trivially on V'. Let us say the degree of the bases of V is d. For Then, for $e = A_4 + A_3$ degrees of V are replaced by $\{2d, 3d, 4d, 5d\}$. These four numbers should be degrees of Casimir invariants of E_8, $\{2, 8, 12, 14, 18, 20, 24, 30\}$. The only possibility is $d = 6$. Then, for each of the 7 choices in the table, $\mathcal{C}(e)$ determines the degrees of these four generators , which are listed in the fourth column of Table 1, while the contribution to n_v from just these four generators is listed in the fifth column. The contribution from V' is not known but they should be completely the same for the 7 nilpotent elements. As a consistency check, the difference between $n_v(e)$ and the contribution to n_v from just the known 4 bases should be a constant. This is indeed so. The difference between entries on the same row in the third and fifth columns of Table 1 is always 4020.

6 Macdonald polynomials

So far we discussed the characters of $\mathbb{C}[\mathcal{M}_{\text{Higgs}}(Q)]$ and $\mathbb{C}[\mathcal{M}_{\text{Coulomb}}(Q)]$. There is in fact another functor $Z^{\text{SCI}}_{p,q,t}$ from the category $\mathcal{Q}(G)$ of 4d $\mathcal{N}$=2 SUSY QFTs to the category of representations of $G \times (\mathbb{C}^\times)^3$, so that its suitable limits give both types of characters. $Z^{\text{SCI}}_{p,q,t}(Q)$ is called the superconformal index of Q. The content of this section is based on a series of papers [20, 21, 22, 4, 23].

As in Sec. 4, we consider characters of G as Weyl-invariant functions on T^r. We use variables $z = (z_1, \ldots, z_r) \in T^r \subset G$ and $(p, q, t^2) \in (\mathbb{C}^\times)^3$, and standard abbreviations $z^w = \prod_i {z_i}^{w_i}$ for a weight $w = (w_1, \ldots, w_r)$ of G. The elliptic Gamma function $\Gamma_{p,q}(x)$ defined as follows will play an important role in this section:

$$\Gamma_{p,q}(x) = \prod_{m,n\geq 0} \frac{1 - x^{-1}p^{m+1}q^{n+1}}{1 - xp^m q^n} \tag{6.1}$$

The basic properties of $Z^{\text{SCI}}_{p,q,t}$ are $Z^{\text{SCI}}_{p,q,t}(\text{triv}) = 1$, $Z^{\text{SCI}}_{p,q,t}(Q_1 \times Q_2) = Z^{\text{SCI}}_{p,q,t}(Q_1)Z^{\text{SCI}}_{p,q,t}(Q_2)$, and for $Q \in \mathcal{Q}(F \times G)$ we have

$$Z^{\text{SCI}}_{p,q,t}(Q /\!\!/\!\!/ G) = (\frac{1}{\Gamma_{p,q}(t)\Gamma'_{p,q}(1)})^r \frac{1}{|W_G|} \int_{T^r} Z^{\text{SCI}}_{p,q,t}(Q) \times \prod_{\alpha:\text{roots of } G} \frac{1}{\Gamma_{p,q}(z^\alpha)\Gamma_{p,q}(tz^\alpha)} \prod_{i=1}^{r} \frac{dz_i}{2\pi\sqrt{-1}z_i} \tag{6.2}$$

where $z \in T^r \subset G$ and $|W_G|$ is the order of the Weyl group. At the level of the representation ring, the integration operation acts as

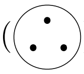
$$|W_G|^{-1} \int_{T^r} \cdots \prod_\alpha (1 - z^\alpha) \prod \frac{dz_i}{2\pi\sqrt{-1}z_i} : \text{Rep}(G \times F) \ni [V] \mapsto [V^G] \in \text{Rep}(F), \tag{6.3}$$

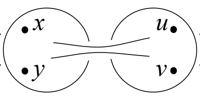
i.e. this extracts the invariant part under G.

We also have

$$Z^{\mathrm{SCI}}_{p,q,t}(\mathrm{Hyp}(V)) = \prod_{w:\text{weights of }V} \Gamma_{p,q}(t^{1/2}z^w). \tag{6.4}$$

Note that when $(p,q,t) = (0,0,\tau^2)$, the right hand side of (6.4) equals the character of $\mathbb{C}[V]$ and the right hand side of (6.2) equals the favorable case of the behavior of the function rings under the holomorphic symplectic quotient studied in (4.6).

Applying (6.4) and (6.2) to $V_1 \otimes V_2 \otimes V_3 = S_{A_1}($  $)$, we find that

$Z^{\mathrm{SCI}}_{p,q,t}(S_{A_1}[$  $])$

$$= \frac{1}{\Gamma_{p,q}(t)\Gamma'_{p,q}(1)}\frac{1}{2}\oint \frac{dz}{2\pi\sqrt{-1}z}\prod_{\pm}\frac{1}{\Gamma_{p,q}(z^{\pm 2})\Gamma_{p,q}(tz^{\pm 2})} \times \prod_{\pm\pm\pm}\Gamma_{p,q}(t^{1/2}u^{\pm}v^{\pm}z^{\pm})\prod_{\pm\pm\pm}\Gamma_{p,q}(t^{1/2}x^{\pm}y^{\pm}z^{\pm}). \tag{6.5}$$

It should be symmetric under the exchange $u \leftrightarrow x$, which is not apparent from the integral form on the right hand side. This symmetry was proved in [24].

The measure appearing in (6.2) is an elliptic generalization of the Macdonald inner product. When $p = 0$, it becomes

$$\Big(\prod_{n\geq 0}\frac{1-q^{n+1}}{1-tq^n}\Big)^r \frac{1}{|W_G|}\prod_{i=1}^{r}\frac{dz_i}{2\pi\sqrt{-1}z_i}\prod_{\alpha}\prod_{n\geq 0}\frac{1-q^n z^\alpha}{1-tq^n z^\alpha}K_0(z)^{-2} \tag{6.6}$$

where

$$K_0(z) = \Big(\prod_{n\geq 0}\frac{1}{1-tq^n}\Big)^r \prod_{\alpha}\prod_{n\geq 0}\frac{1}{1-tq^n z^\alpha}. \tag{6.7}$$

and

$$\frac{1}{|W_G|}\prod_{i=1}^{r}\frac{dz_i}{2\pi\sqrt{-1}z_i}\prod_{\alpha}\prod_{n\geq 0}\frac{1-q^n z^\alpha}{1-tq^n z^\alpha} \tag{6.8}$$

is the standard measure appearing in the theory of Macdonald polynomials. This means that the orthonormal polynomials under (6.6) are

$$K_0(z)\underline{P_\lambda}(z) \tag{6.9}$$

where

$$\underline{P_\lambda}(z) = \Big(\prod_{n\geq 0}\frac{1-q^{n+1}}{1-tq^n}\Big)^{-r/2} N_\lambda^{-1/2} P_\lambda(z). \tag{6.10}$$

Here, $P_\lambda(z)$ is the standard Macdonald polynomial and

$$N_\lambda = \frac{1}{|W_G|}\int_{T^r}\prod_{i=1}^{r}\frac{dz_i}{2\pi\sqrt{-1}z_i}\prod_{\alpha}\prod_{n\geq 0}\frac{1-q^n z^\alpha}{1-tq^n z^\alpha}P_\lambda(z)P_\lambda(z^{-1}) \tag{6.11}$$

is the norm of the Macdonald polynomial, which has an explicit infinite-product form.

We then have a generalization of Conjecture 7:

Conjecture 11. *For suitable choices of G, $e_{1,2,3}$ and a pseudoreal representation X of $\prod_i G(e_i)$ listed in Conjecture 7, we have the equality*

$$\prod_{u,v,w}\prod_{n\geq 0}\frac{1}{1-t^{1/2}q^n a^u b^v c^w}=\frac{K_{e_1}(a)K_{e_2}(b)K_{e_3}(c)}{K_{e_{prin}}}\sum_\lambda \frac{\underline{P_\lambda}(at^{h_1})\underline{P_\lambda}(bt^{h_2})\underline{P_\lambda}(ct^{h_3})}{\underline{P_\lambda}(t^\rho)}. \tag{6.12}$$

On the left hand side, the product runs over the weights of X as a representation of $G(e_1)\times G(e_2)\times G(e_3)$. On the right hand side, $\underline{P_\lambda}$ is defined above in (6.10)*, and the prefactors K_e are given by*

$$K_e(z)=\prod_d\prod_{n=0}^{\infty}\prod_{w:\textit{weights of } R_d}\frac{1}{1-t^{(d+1)/2}q^n z^w} \tag{6.13}$$

where R_d was defined in (4.11).

A slightly simpler version of the conjecture is obtained by taking $p=0$, $t=q$. Then $P_\lambda(a)$ reduces to the character $\chi_\lambda(a)$ of $a\in G$ in the irreducible representation with Dynkin label λ, or equivalently, it is given by the Schur polynomial of type G. This version will become relevant in Sec. 7.

A more general version of the conjecture is given by replacing the left hand side of (6.12) by (6.4) and Macdonald polynomials on the right hand side of (6.12) by elliptic Macdonald functions. As we know not much about elliptic Macdonald functions, it is not clear exactly how to phrase the conjecture. Some studies have been done in [25].

More generally, we have a conjectural formula about the $p=0$ version of the superconformal index:

$$Z^{\rm SCI}_{p=0,q,t}(S_G(C,\{p_i,e_i\}_{i=1}^n))=\frac{\prod K_{e_i}(a_i)}{K_{e_{\rm prin}}^{n-2}}\sum_\lambda\frac{\prod_i\underline{P_\lambda}(a_ie_i)}{\underline{P_\lambda}(t^\rho)^{n-2}}. \tag{6.14}$$

where $a_i\in\mathfrak{g}(e_i)$, $\underline{P_\lambda}$ is defined in (6.10), and K_e is defined in (6.13). However, it is hard to make this into a mathematical conjecture, since we do not know how to state the left hand side in a mathematically-defined way.

Another interesting limit of the superconformal index is when $u=pq/t$ is fixed and the limit $p,q\to 0$ is taken. We have

$$Z^{\rm SCI}_{u=pq/t,p\to0,q\to0}(\mathrm{Hyp}(V))=1 \tag{6.15}$$

and

$$Z^{\rm SCI}_{u=pq/t,p\to0,q\to0}(\mathrm{Hyp}(V)/\!\!/\!\!/G)=\prod_{i=1}^{\operatorname{rank}G}\frac{1}{1-u^{d_i}} \tag{6.16}$$

where d_a is one plus the a-th exponent of G; this follows from the explicit formulas (6.2), (6.4). In broad generality, it is believed that

$$Z^{\rm SCI}_{u=pq/t,p\to0,q\to0}(Q)=\mathrm{tr}_{\mathbb{C}[\mathcal{M}_{\rm Coulomb}(Q)]}\,u \tag{6.17}$$

where $u\in\mathbb{C}^\times$ is the $\mathrm{U}(1)$ action on the Coulomb branch of Q, discussed in Sec. 5. Once the superconformal index with general p, q and t is understood, we can apply this limit to (6.14) and compare it with (5.22) applied to (6.17). From this we obtain full information necessary to reconstruct $V(e)$ discussed in Sec. 5. Let us summarize the discussions above in a very vague conjecture:

Conjecture 12. *The Conjecture 11, depending on two parameters (q, t), can further be extended by additional parameter p, such that the left hand side is replaced by* (6.4)*, and the Macdonald polynomials on the right hand side are replaced by the elliptic Macdonald functions. The three-parameter version of the combination $K_e(a)\underline{P_\lambda}(ae)$ should have the following limit:*

$$\lim_{u=pq/t, p\to 0, q\to 0} K_e(a)\underline{P_\lambda}(ae) = \prod_i \frac{1}{1-u^{\deg t_i}} \tag{6.18}$$

where

$$\mathbb{C}[V(e)] = \mathbb{C}[t_1, \ldots, t_s], \qquad s = \dim V(e). \tag{6.19}$$

Recall that $V(e)$ was introduced in Conjecture 8.

7 Vertex operator algebras

Given a 4d $\mathcal{N} = 2$ SUSY QFT $Q \in \mathcal{Q}$, there is a canonically defined vertex operator superalgebra $\mathit{VOA}(Q)$, as shown in [10]. In this section we discuss the property of this functor VOA. We will denote a general vertex algebra by $\mathcal{V} = \bigoplus_n \mathcal{V}_n$ where n runs over non-negative integers and half-integers. We denote by L_0 the operator on $\mathcal{V}$ whose eigenvalue is given by n on $\mathcal{V}_n$. The vertex operator superalgebras we discuss will also be $\mathbb{Z}_2$ graded, and even / odd elements are called bosonic and fermionic. This $\mathbb{Z}_2$ grading does *not* coincide with the integrality of the index n in $\mathcal{V}_n$.

First, we have

$$\mathit{VOA}(Q_1 \times Q_2) = \mathit{VOA}(Q_1) \otimes \mathit{VOA}(Q_2). \tag{7.1}$$

Second, for $Q = \mathrm{Hyp}(V)$, $\mathit{VOA}(\mathrm{Hyp}(V))$ is given by the so-called symplectic boson vertex operator algebra $\mathit{SB}(V)$ defined as follows. Regard V as a holomorphic symplectic vector space, with the symplectic pairing $\langle \cdot, \cdot \rangle$. Then, $\mathit{SB}(V) = \mathit{VOA}(\mathrm{Hyp}(V))$ is such that $\mathcal{V}_{1/2} = V$, and we have a bosonic field $v(z)$ in the vertex operator algebra, with the operator product expansion given by

$$v_1(z)v_2(w) \sim \frac{1}{z-w}\langle v_1, v_2 \rangle, \tag{7.2}$$

with the energy-momentum tensor given by

$$T(z) = \frac{1}{2}\sum_i (v_i \partial w_i(z) - w_i \partial v_i(z)) \tag{7.3}$$

where the set (v_i, w_i) is a basis of V such that $\langle v_i, v_j \rangle = \langle w_i, w_j \rangle = 0$, $\langle v_i, w_j \rangle = \delta_{ij}$.

Third, when $Q \in \mathcal{Q}$, i.e. Q is G-symmetric, $\mathit{VOA}(Q)$ has an affine $\mathfrak{g}$ algebra as a vertex operator subalgebra. The vertex operator algebra $\mathit{VOA}(Q /\!\!/\!\!/ G)$ is obtained from $\mathit{VOA}(Q)$, but the method is understood only when the affine $\mathfrak{g}$ subalgebra of $\mathit{VOA}(Q)$ has the level $k = -2h^\vee(G)$. Given a vertex operator algebra $\mathcal{V}$ with the affine $\mathfrak{g}$ subalgebra of this particular level, we define a new vertex operator algebra $\mathcal{V}/G$ as follows.

First, let us denote by $J^A_{\mathcal{V}}$ for $A = 1, \ldots \dim G$ the affine $\mathfrak{g}$ currents of $\mathcal{V}$, with the standard operator product expansion

$$J^A_{\mathcal{V}}(z) J^B_{\mathcal{V}}(w) \sim \frac{k\eta^{AB}}{(z-w)^2} + \sum_C f^{AB}{}_C \frac{J^C_{\mathcal{V}}(w)}{z-w} \tag{7.4}$$

where $f^{AB}{}_C$ and η^{AB} are the structure constant and the invariant tensor of the Lie algebra $\mathfrak{g}$, respectively. We use the normalization of k such that the central charge is $k \dim G/(k + h^\vee(G))$.

Let us introduce another vertex operator algebra $\mathcal{V}_{bc}$, generated by pairs of fermionic fields $b^A(z)$, $c_A(z)$ for $A = 1, \ldots, \dim G$ with the operator product expansion

$$b^A(z) c_B(w) \sim \delta^A_B \frac{1}{z-w}, \tag{7.5}$$

with the energy momentum tensor

$$T_{bc}(z) = \sum_A c_A \partial b^A(z). \tag{7.6}$$

This bc system has an affine $\mathfrak{g}$ subalgebra generated by

$$J^A_{bc}(z) = f^{AB}{}_C c_B b^C(z). \tag{7.7}$$

This has level $k = +2h^\vee(G)$.

Define the BRST current on the product vertex operator algebra $\mathcal{V} \otimes \mathcal{V}_{bc}$ by

$$j_{\text{BRST}}(z) = \sum_A c_A (J^A_X + \frac{1}{2} J^A_{bc})(z) \tag{7.8}$$

where J^A_X is the affine $\mathfrak{g}$ currents of the vertex operator algebra X, and define the BRST operator

$$d_{\text{BRST}} \mathcal{O}(z) = \oint \frac{dw}{w} j_{\text{BRST}}(w) \mathcal{O}(z). \tag{7.9}$$

It can be checked that $d^2_{\text{BRST}} = 0$ only when the level of $J^A_{\mathcal{V}}$ is $-2h^\vee(G)$. Let us also define the operators

$$b^A_0 \mathcal{O}(z) = \oint \frac{dw}{w} b^A(w) \mathcal{O}(z), \qquad J^A_{\text{total},0} \mathcal{O}(z) = \oint \frac{dw}{w} J^A_{\text{tot}}(w) \mathcal{O}(z) \tag{7.10}$$

where $J^A_{\text{total},0}$ is the affine current $J^A_{\mathcal{V}} + J^A_{bc}$ of the total system. We consider the subspace $X \subset \mathcal{V} \otimes \mathcal{V}_{bc}$ by the condition

$$X = \cap_A (\text{Ker } b^A_0 \cap \text{Ker } J^A_{\text{total},0}). \tag{7.11}$$

We can show $d_{\text{BRST}} X \subset d_{\text{BRST}}$, and we define the vertex operator algebra $\mathcal{V}/G$ by

$$\mathcal{V}/G = H(X, d_{\text{BRST}}). \tag{7.12}$$

We can finally state the result of [10] concerning $\mathit{VOA}(Q /\!\!/\!\!/ G)$: When Q is such that $\mathit{VOA}(Q)$ has an affine $\mathfrak{g}$ subalgebra of level $-2h^\vee(G)$, we have

$$\mathit{VOA}(Q /\!\!/\!\!/ G) = \mathit{VOA}(Q)/G. \tag{7.13}$$

When a vertex operator algebra $\mathcal{V}$ has an affine $\mathfrak{g}$ subalgebra, each graded piece $\mathcal{V}_n$ is a representation of the finite-dimensional Lie algebra $\mathfrak{g}$, and we can consider its character

$$\operatorname{Str}\mathcal{V}(z) := \operatorname{Str}_{\mathcal{V}} q^{L_0} z = \sum_n q^n \operatorname{Str}_{\mathcal{V}_n} z \tag{7.14}$$

where $z \in G$ and Str is the super-trace with respect to the $\mathbb{Z}_2$ grading of $\mathcal{V}$. From the physics argument it is known that

$$\operatorname{Str}\mathit{VOA}(Q)(z) = Z_{p=0,q=t}(Q)(z). \tag{7.15}$$

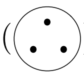
We can show that the operation (7.13) is consistent with the relation (6.2) when $p = 0$, $q = t$, which is given by

$$Z^{\mathrm{SCI}}_{p=0,q=t}(Q /\!\!/\!\!/ G) = \frac{1}{|W_G|}\int_{T^r} Z^{\mathrm{SCI}}_{p=0,q=t}(Q) K_0(z)^{-2}\Big[\prod_\alpha (1-z^\alpha)\Big]\prod_{i=1}^r \frac{dz_i}{2\pi\sqrt{-1}z_i} \tag{7.16}$$

where

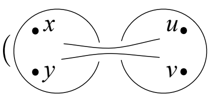
$$K_0(z) = \Big(\prod_{n\geq 1}\frac{1}{1-q^n}\Big)^r \prod_\alpha \prod_{n\geq 1}\frac{1}{1-q^n z^\alpha}. \tag{7.17}$$

Let us define

$\mathcal{W}_{G,3} := \mathit{VOA}(S_G($  $))$. (7.18)

This is a vertex operator algebra with an affine subalgebra $\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g}$, each with level $k = -h^\vee(G)$. Furthermore, there is an action of S_3 on $\mathcal{W}_{G,3}$ permuting three affine subalgebras $\mathfrak{g}$.

For a sphere with four punctures, we have

$\mathcal{W}_{G,4} := \mathit{VOA}(S_G($ 

$)) = (\mathcal{W}_{G,3} \times \mathcal{W}_{G,3})/G_{\mathrm{diag}}$. (7.19)

On the left hand side, we are taking the subalgebra

$$(\mathfrak{g}\oplus\mathfrak{g}\oplus\mathfrak{g}) \oplus (\mathfrak{g}\oplus\mathfrak{g}\oplus\mathfrak{g}) \supset (\mathfrak{g}\oplus\mathfrak{g}) \oplus \mathfrak{g}_{\mathrm{diag}} \oplus (\mathfrak{g}\oplus\mathfrak{g}) \tag{7.20}$$

and therefore $\mathfrak{g}_{\mathrm{diag}}$ has level $k = -2h^\vee(G)$, to which we can apply the construction $\mathcal{V}/G$ defined above. Combining with the discussion of $Z^{\mathrm{SCI}}_{p,q,t}$ in the last section, we have the following conjecture:

Conjecture 13. *For a simply-connected simply-laced G, there are a series of vertex operator superalgebras $\mathcal{W}_{G,n}$ with the following properties:*

- *$\mathcal{W}_{G,n}$ has an affine subalgebra $\mathfrak{g}^{\oplus n}$, together with an action of S_n permuting n copies of $\mathfrak{g}$, each with level $k = -h^\vee(G)$.*

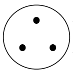
- $(\mathcal{W}_{G,n} \times \mathcal{W}_{G,m})/G_{diag} = \mathcal{W}_{G,n+m-2}$.
- *The character of* $\mathcal{W}_{G,n}$ *is*

$$\mathrm{Str}(\mathcal{W}_{G,n})(z_1, \ldots, z_n) = \sum_\lambda \frac{\prod_i K_0(z_i)\chi_\lambda(z_i)}{(K_{e_{prin}}\chi_\lambda(q^\rho))^{n-2}} \tag{7.21}$$

where z_i *is an element in* G, K_0 *was given in* (7.17)*, and*

$$K_{e_{prin}} = \prod_{i=1}^{r} \prod_{n=0}^{\infty} \frac{1}{1-q^{d_i+n}}. \tag{7.22}$$

Here, again, d_i *is the* i*-th exponent plus one of* G*.*

Since we know $S_{A_1}($$) = \mathrm{Hyp}(V_1 \otimes V_2 \otimes V_3)$ where $V_i \simeq \mathbb{C}^2$, we have an associated conjecture, namely that *the vertex operator algebra* $\mathcal{W}_{A_1,3} := \mathit{SB}(V_1 \otimes V_2 \otimes V_3)$ *satisfies the conditions of the conjecture above, for* $G = A_1$*.*

It is also known that $\mathit{VOA}(S_G(S^2; e_1, e_2, \ldots, e_n))$ is obtained from $\mathcal{W}_{G,n} = \mathit{VOA}(S_G(S^2; 0, 0, \ldots, 0)$ by a quantum Drinfeld-Sokolov reduction. Usually, a quantum Drinfeld-Sokolov reduction is defined as an operation for an affine $\mathfrak{g}$ algebra, given a nilpotent element e in the finite dimensional Lie algebra $\mathfrak{g}$. There is no problem in extending the operation to a vertex algebra $\mathcal{V}$ containing an affine $\mathfrak{g}$ algebra as a vertex operator subalgebra. Let us denote the resulting quantum Drinfeld-Sokolov reduction as $qDS(\mathcal{V}, e)$. Then, the general conjecture is that

$$\mathit{VOA}(S_G(S^2; e_1, e_2, \ldots, e_n)) = qDS(\mathcal{W}_{G,n}, e_1 \oplus \cdots \oplus e_n) \tag{7.23}$$

where we regard $e_1 \oplus \cdots e_n$ to be a nilpotent element of $\mathfrak{g}^{\oplus n}$. This leads to the following conjecture:

Conjecture 14. *In Conjecture 7, we listed a few examples of* (G, e_1, e_2, e_3) *for which we have* $S_G(S^2; e_1, e_2, e_3) = \mathrm{Hyp}(X)$ *for a symplectic vector space* X*. For each of them, we have*

$$qDS(\mathcal{W}_{G,3}, e_1 \oplus e_2 \oplus e_3) = \mathit{SB}(X). \tag{7.24}$$

Finally, we have

Conjecture 15. *Related to the case* $n = 1$ *in Conjecture 2, we have*

$$\mathit{VOA}(S_{A_1}(S^2; [1^2], [1^2], [1^2], [1^2])) = \widehat{\mathfrak{so}(8)}_{-2}, \tag{7.25}$$

$$\mathit{VOA}(S_{A_2}(S^2; [1^3], [1^3], [1^3])) = (\widehat{\mathfrak{e}}_6)_{-3}, \tag{7.26}$$

$$\mathit{VOA}(S_{A_3}(S^2; [2^2], [1^4], [1^4])) = (\widehat{\mathfrak{e}}_7)_{-4}, \tag{7.27}$$

$$\mathit{VOA}(S_{A_5}(S^2; [3^2], [2^3], [1^6])) = (\widehat{\mathfrak{e}}_8)_{-6}. \tag{7.28}$$

Here, $\hat{\mathfrak{g}}_k$ *stands for the vertex operator algebra obtained from the irreducible vacuum representation of the affine* $\mathfrak{g}$ *algebra at level* k*.*

Let us end this section with a few remarks. First, the first one in the list is $\mathcal{W}_{A_1,4}$, which can be constructed from $\mathcal{W}_{A_1,3} = SB(V_1 \otimes V_2 \otimes V_3)$. More explicitly, the conjecture becomes

$$[SB(V_1 \otimes V_2 \otimes V_3) \otimes SB(V_3 \otimes V_4 \otimes V_5)]/\operatorname{SL}(V_3) = \widehat{\mathfrak{so}(8)}_{-2}. \tag{7.29}$$

This is a new free-field construction of this particular affine Lie algebra as a vertex operator algebra.

Second, note that the second one in the list is just $\mathcal{W}_{A_2,3}$. Therefore we can rephrase the conjecture by saying *The vertex operator algebra $\mathcal{W}_{A_2,3} := (\widehat{\mathfrak{e}}_6)_{-3}$ satisfies the conditions listed in Conjecture 13.*

Third, in Conjecture 2, we conjectured that $\mathcal{M}_{\text{Higgs}}$ of these theories for general n gives the n-instanton moduli spaces. We do not, however, expect that the associated vertex operator algebras for general n to be just given by the affine Lie algebra with some level.

8 Hypersurfaces representing minuscule representations

Let us conclude this list of conjectures by discussing a slightly different topic. Given a G-Hitchin system, we often consider its spectral curve, when $G = A_{n-1}$ or $= D_n$. For example, when $G = A_{n-1}$, we take the vector representation as R and consider

$$\det_R(\lambda - \phi) = \lambda^n + u_2(\phi)\lambda^{n-2} + \cdots + u_n(\phi) = 0 \tag{8.1}$$

as an equation giving a curve within T^*C, where λ is the tautological one-form on T^*C.

A method which applies to a general simply-laced simple G is to consider its spectral geometry [26]. Let us illustrate the construction by considering two cases. First consider the case $G = E_6$. The deformation of the simple singularity of type E_6 is given by

$$W_{E_6}(\{x_1, x_2, x_3\}, \{u_d\}) = x_1^4 + x_2^3 + x_3^2 + u_2 x_1^2 x_2 + u_5 x_1 x_2 + u_6 x_1^2 + u_8 x_2 + u_9 x_1 + u_{12} \tag{8.2}$$

where x_1, x_2 and x_3 have degree 3, 4, 6 respectively and u_k are the generators as in (5.15) where the subscripts are renamed to correspond to the degree. The whole expression has the degree $h^\vee(E_6) = 12$.

Then, given ϕ as in (5.8), we consider a three-fold X in the total space of the vector bundle

$$K_C^{\otimes 3} \oplus K_C^{\otimes 4} \oplus K_C^{\otimes 6} \to C \tag{8.3}$$

given by

$$0 = x_1^4 + x_2^3 + x_3^2 + u_2(\phi) x_1^2 x_2 + u_5(\phi) x_1 x_2 + u_6(\phi) x_1^2 + u_8(\phi) x_2 + u_9(\phi) x_1 + u_{12}(\phi) \tag{8.4}$$

where x_1, x_2, x_3 are now sections of $K_C^{\otimes 3}$, $K_C^{\otimes 4}$, $K_C^{\otimes 6}$, respectively. Then the fiber of the Hitchin system is given by the intermediate Jacobian of X.

Next, let us consider the case $G = A_{n-1}$. In this case the type A_{n-1} singularity is given by

$$W_{A_{n-1}}(\{x_1, x_2, x_3\}, \{u_d\}) = x_2 x_3 + x_1^n + u_2(\phi) x_1^{n-2} + \cdots + u_n \tag{8.5}$$

and the spectral geometry is given by

$$0 = x_2 x_3 + x_1^n + u_2(\phi) x_1^{n-2} + \cdots + u_n(\phi) \tag{8.6}$$

where x_1, x_2, x_3 are sections of K_C, $K_C^{\otimes c}$, $K_C^{\otimes(n-c)}$, respectively, with arbitrary c. Note that this is essentially equivalent to the spectral curve (8.1).

Spectral geometries of the Hitchin system, even when its base is zero dimensional, turn out to have an interesting structure. In Sec. 5, we only consider nilpotent residues in the singularities of ϕ in (5.9). A natural way to introduce $m \in \mathfrak{g}(e)$ is to generalize α in (5.9) to be given by

$$\alpha_i \in \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(m_i + d_{LS}^{\mathfrak{l}}(e_i)). \tag{8.7}$$

where $\mathfrak{l}$ is the smallest Levi subalgebra containing e_i and Ind stands for the induction of orbits. In particular, when e_i is principal in $\mathfrak{l}$ and m_i is generic, we just have

$$\alpha_i = m_i. \tag{8.8}$$

Now, by explicitly constructing the spectral geometry of the G-Hitchin systems on $C = S^2$ with three punctures $e_{1,2,3}$ for the cases listed in Conjecture 7 so that $S_G(C, \{e_1, e_2, e_3\}) = \mathrm{Hyp}(V)$, we find the following. We put e_3 at $z = \infty$, where z is a local coordinate of $C = S^2$.

- When $G = A_{n-1}$, $e_1 = e_3 = [1^n] = 0$, $e_2 = [n-1, 1]$, and $X = W \otimes V_1 \otimes V_2^* \oplus W^* \otimes V_1^* \otimes V_2$. Here, $V_i \simeq \mathbb{C}^n$ where we identify $G(e_i) = \mathrm{SL}(V_i) \simeq G$, and $G(e_3) = \mathbb{C}^\times$ with W its natural one-dimensional representation. The spectral geometry, after a change of variables, is given by

$$0 = z \prod_{i=1}^n (x_1 - m_i - \mu) - x_2 x_3 - \prod_{i=1}^n (x_1 - a_i) \tag{8.9}$$

  where $(m_1, \ldots, m_n) \in \mathfrak{g}(e_1) = \mathfrak{sl}(V_1)$, $(a_1, \ldots, a_n) \in \mathfrak{g}(e_1) = \mathfrak{sl}(V_3)$, and $\mu \in \mathbb{C} = \mathfrak{g}(e_2)$. Introduce $X_V(\{x_1, x_2, x_3\}, \{u_d\}, m) = x_1 - m$. Then we can rewrite the equation above as

$$0 = z \prod_{i=1}^n X_V(\{x_1, x_2, x_3\}, \{u_d\}, m) + x_2 x_3 - W_{A_{n-1}}(\{x_1, x_2, x_3\}, \{u_d\}). \tag{8.10}$$

- $G = E_6$, $e_1 = E_6(a_1)$, $e_2 = A_2 + 2A_1$, $e_3 = 0$, and $X = V_{\text{min}} \otimes F \oplus \bar{V}_{\text{min}} \otimes \bar{F}$. Here, $G(e_1) = 1$, $G(e_2) = \mathrm{SL}(2) \times \mathbb{C}^\times$ with F its natural two dimensional representation, and $G(e_3) = E_6$ with V_{min} its minuscule representation of dimension 27. Let $(m_1, m_2) \in \mathfrak{g}(e_2) = \mathfrak{gl}(F)$ and $a \in \mathfrak{e}_6$. By the map $\mathfrak{e}_6/E_6 \simeq \mathbb{C}^6$, we can associate $(u_2, u_5, u_6, u_8, u_9, u_{12})$. The spectral geometry, after a change of variables, is then given by

$$0 = z \prod_{i=1}^2 X_{V_{\text{min}}}(\{x_1, x_2, x_3\}, \{u_d\}, m_i) - W_{E_6}(\{x_1, x_2, x_3\}, \{u_d\}) \tag{8.11}$$

  where W_{E_6} was given in (8.2) and

$$X_{V_{\text{min}}}(\{x_1, x_2, x_3\}, \{u_d\}, m) = -8(x_1^2 - \sqrt{-1}x_3 + \frac{1}{2}u_6) - 4u_2 x_2 + 4mu_5 + m^2(u_2^2 - 12x_2) - 8m^3 x_1 + 2m^4 u_2 + m^6. \tag{8.12}$$

The polynomials X_V for $V \simeq \mathbb{C}^n$ of $G = \mathrm{SL}(n)$ and X_{V_min} for $V_\text{min} \simeq \mathbb{C}^{27}$ of $G = E_6$ have a common feature, which we abstract into a general conjecture:

Conjecture 16. *Given a simply-laced simple G, write the versal deformation of the singularity of type G as*

$$W_G(\{x_1, x_2, x_3\}, \{u_a\}_{a=1}^r) = 0. \tag{8.13}$$

Take an irreducible representation V of G, such that $2k(V) \leq k(\mathfrak{g})$, where k is the eigenvalue of the quadratic Casimir operator.[4] *Then, there is a polynomial*

$$X_V(\{x_1, x_2, x_3\}, \{u_d\}, m) \tag{8.14}$$

such that the hypersurface

$$0 = zX_V(\{x_1, x_2, x_3\}, \{u_d\}, m) - W_G(\{x_1, x_2, x_3\}, \{u_d\}) \tag{8.15}$$

as defining a family $\mathcal{X}$ of three-dimensional hypersurface in $(z, x_1, x_2, x_3) \in \mathbb{C}^4$ parameterized by m and $\{u_d\}$. By the identification $\mathbb{C}[\mathfrak{h}]^W = \mathbb{C}[u_1, \ldots, u_r]$ where $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{g}$ and W the Weyl group, we can pull back the family $\mathcal{X}$ over the space of m and $\{u_d\}$ to a family

$$\mathcal{X} \to \mathbb{C} \oplus \mathfrak{h} \ni m \oplus a. \tag{8.16}$$

Then the fiber develops a singularity of the form $x^2 + y^2 + z^2 + w^2 = 0$ if and only if there is a weight w of V such that $m = w(a)$.

It is very easy to check this statement for $G = \mathrm{SL}(n)$ and $V = \mathbb{C}^n$, using the explicit form of $X_V = x_1 - m$ given above. For $G = E_6$ and $V = V_\text{min}$, it is possible to check the validity of the statement by a heavy use of a computer algebra system. For all possible V with $2k(V) \leq k(\mathfrak{g})$, X_V have been constructed by using various string dualities, and listed in the Appendix of [3]. The conjecture should, however, be solved by a uniform construction of X_V.

References

[1] Y. Tachikawa, "A pseudo-mathematical pseudo-review on 4d $\mathcal{N}$=2 supersymmetric quantum field theories," 2013.
`http://member.ipmu.jp/yuji.tachikawa/not-on-arxiv.html`.

[2] G. W. Moore and Y. Tachikawa, "On 2D TQFTs Whose Values are Holomorphic Symplectic Varieties," `arXiv:1106.5698 [hep-th]`.

[3] Y. Tachikawa and S. Terashima, "Seiberg-Witten Geometries Revisited," *JHEP* **09** (2011) 010, `arXiv:1108.2315 [hep-th]`.

[4] V is then automatically minuscule. But not all minuscule representations satisfy this inequality.

[4] A. Gadde, L. Rastelli, S. S. Razamat, and W. Yan, "Gauge Theories and Macdonald Polynomials," *Commun.Math.Phys.* **319** (2013) 147–193, `arXiv:1110.3740 [hep-th]`.

[5] O. Chacaltana, J. Distler, and Y. Tachikawa, "Nilpotent orbits and codimension-two defects of 6d N=(2,0) theories," *Int.J.Mod.Phys.* **A28** (2013) 1340006, `arXiv:1203.2930 [hep-th]`.

[6] L. F. Alday, D. Gaiotto, and Y. Tachikawa, "Liouville Correlation Functions from Four-Dimensional Gauge Theories," *Lett. Math. Phys.* **91** (2010) 167–197, `arXiv:0906.3219 [hep-th]`.

[7] O. Schiffmann and E. Vasserot, "Cherednik algebras, W algebras and the equivariant cohomology of the moduli space of instantons on $\mathbb{A}^2$," `arXiv:1202.2756 [math.QA]`.

[8] D. Maulik and A. Okounkov, "Quantum Groups and Quantum Cohomology," `arXiv:1211.1287 [math.QA]`.

[9] D. Gaiotto, "$\mathcal{N} = 2$ Dualities," `arXiv:0904.2715 [hep-th]`.

[10] C. Beem, M. Lemos, P. Liendo, W. Peelaers, L. Rastelli, *et al.*, "Infinite Chiral Symmetry in Four Dimensions," `arXiv:1312.5344 [hep-th]`.

[11] R. Bielawski, "Hyperkähler structures and group actions," *J. London Math. Soc.* **55** (1997) 400.

[12] D. Gaiotto and S. S. Razamat, "Exceptional Indices," *JHEP* **1205** (2012) 145, `arXiv:1203.5517 [hep-th]`.

[13] C. A. Keller and J. Song, "Counting Exceptional Instantons," *JHEP* **1207** (2012) 085, `arXiv:1205.4722 [hep-th]`.

[14] A. Hanany, N. Mekareeya, and S. S. Razamat, "Hilbert Series for Moduli Spaces of Two Instantons," `arXiv:1205.4741 [hep-th]`.

[15] D. H. Collingwood and W. M. McGovern, *Nilpotent orbits in semisimple Lie algebras*. Van Nostrand Reinhold Mathematics Series. Van Nostrand Reinhold Co., New York, 1993.

[16] R. Y. Donagi, "Seiberg-Witten Integrable Systems," `arXiv:alg-geom/9705010`.

[17] E. Sommers, "Lusztig's canonical quotient and generalized duality," *J. Algebra* **243** no. 2, (2001) 790–812, `arXiv:math.RT/0104162`.

[18] P. N. Achar and E. N. Sommers, "Local systems on nilpotent orbits and weighted Dynkin diagrams," *Represent. Theory* **6** (2002) 190–201, `arXiv:math/0201248`.

[19] P. N. Achar, "An order-reversing duality map for conjugacy classes in Lusztig's canonical quotient," *Transform. Groups* **8** no. 2, (2003) 107–145, `arXiv:math.RT/0203082`.

[20] A. Gadde, E. Pomoni, L. Rastelli, and S. S. Razamat, "S-Duality and 2D Topological QFT," *JHEP* **03** (2010) 032, `arXiv:0910.2225 [hep-th]`.

[21] A. Gadde, L. Rastelli, S. S. Razamat, and W. Yan, "The Superconformal Index of the E_6 SCFT," `arXiv:1003.4244 [hep-th]`.

[22] A. Gadde, L. Rastelli, S. S. Razamat, and W. Yan, "The 4D Superconformal Index from q-Deformed 2D Yang- Mills," *Phys. Rev. Lett.* **106** (2011) 241602, `arXiv:1104.3850 [hep-th]`.

[23] D. Gaiotto, L. Rastelli, and S. S. Razamat, "Bootstrapping the Superconformal Index with Surface Defects," `arXiv:1207.3577 [hep-th]`.

[24] F. J. van de Bult, "An elliptic hypergeometric integral with $W(F_4)$ symmetry," *Ramanujan J.* **25** no. 1, (2011) 1–20.

[25] S. S. Razamat, "On the $\mathcal{N} = 2$ Superconformal Index and Eigenfunctions of the Elliptic RS Model," `arXiv:1309.0278 [hep-th]`.

[26] D.-E. Diaconescu, R. Donagi, and T. Pantev, "Intermediate Jacobians and ADE Hitchin Systems," `arXiv:hep-th/0607159`.