Some illustrative examples of Argyres-Seiberg-Gaiotto duality

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Abstract. In the past few years we saw significant advancement in understanding the strong-weak duality of $\mathcal{N} = 2$ supersymmetric theories in four dimensions, initiated by the seminal works by Argyres and Seiberg [1], and then greatly generalized by Gaiotto [2]. A most surprising feature of this new duality is that the dual of a normal supersymmetric gauge theory can involve a strange mixture of weakly-coupled gauge multiplets and non-Lagrangian superconformal field theories. Here we illustrate these new dualities by reviewing known examples and then by demonstrating new dual pairs. This work is based on the author's collaboration with Benini and Benvenuti [3].

1. Introduction

The maximally supersymmetric Yang-Mills theory with SU(N) gauge group, which is with $\mathcal{N} = 4$ supersymmetry, is known to be invariant under the S-duality. Let us define the complexified gauge coupling τ to be $\tau = \theta/2\pi + 4\pi i/g^2$ where θ is the theta angle and g the coupling constant. Then the theory is invariant under the transformation $S : \tau \to -1/\tau$, accompanied by the exchange of W-bosons and monopoles. The theory has more mundane invariance $T : \tau \to \tau + 1$ which shifts the theta angle by 2π without affecting anything. As is well-known, S and T generate the group $\mathrm{SL}(2, \mathbb{Z})$ and any point on the upper half plane on which τ lives can be mapped by its action into the fundamental region shown with shaded gray in Fig. 1. The fundamental region does not touch the real axis, which means that you can always go to a duality frame where the theory is not infinitely strongly coupled.

The situation is the same with $\mathcal{N} = 2$ supersymmetric SU(2) gauge theory with four flavors in the fundamental representation. Here it is customary to redefine the coupling constant by a factor of two, and to take $\tau = \theta/\pi + 8\pi i/g^2$. This theory is known to be invariant under the two transformations S and T above.

In general, $\mathcal{N} = 2$ supersymmetric SU(N) gauge theory with $N_f = 2N$ hypermultiplets in the fundamental representation has zero beta function. It is believed to have the strong-weak duality under the transformation S, judging from the behavior of the Seiberg-Witten curve, for example. However, the theory does not have the invariance under T, because it shifts the theta angle by π. The SU(2) gauge theory is special in this sense, because the doublet and the anti-doublet representation are the same in this special case. Thus one could achieve the invariance under T by combining the shift of the theta angle by π and the charge conjugation. But this is not available for $N > 2$, and the duality group is generated by S and T^2. The fundamental

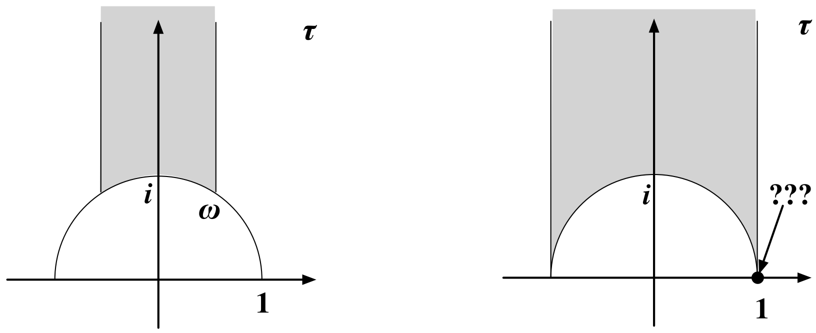

Figure 1. Fundamental regions. Left: $\mathcal{N}=4$ SU(N), and $\mathcal{N}=2$ SU(2) with $N_f=4$. Right: $\mathcal{N}=2$ SU(3) with $N_f=6$.

region then touches the real axis, see Fig. 1. Therefore there is an infinitely strongly coupled point in the space of the coupling constant.

Argyres and Seiberg [1] proposed the dual description of SU(3) with six flavors which solves this conundrum. They argued, using various field theoretical analyses, that the SU(3) theory with coupling τ is equivalent to an SU(2) theory with coupling constant $\tau' = 1/(1-\tau)$ coupled to one flavor of doublet and also to Minahan and Nemeschansky's superconformal theory with flavor symmetry E_6. Then the infinitely-strongly-coupled limit $\tau = 1$ of the original theory becomes a weakly-coupled limit of this dual description. Their proposal was ground-breaking: no one had ever thought of coupling a dynamical gauge field to those mysterious superconformal theories before their work.

At that time, however, their observation seemed rather esoteric and not very relevant to our general understanding of the dynamics of $\mathcal{N}=2$ theories. It was Gaiotto [2] who showed that, on the contrary, this S-duality involving non-Lagrangian superconformal theories coupled to dynamical gauge fields is a common phenomenon which naturally follows from the analysis of M5-branes.

2. Review of the construction via M5-branes

2.1. $\mathcal{N}=4$

Let us first consider the realization of $\mathcal{N}=4$ SU(N) theory in terms of M5-branes. This is surprisingly simple: one only needs to compactify N M5-branes on T^2. First, M5-branes on S^1 is almost by definition give D4-branes, on which lives maximally supersymmetric SU(N) gauge theory in 5 dimensions. Then, a further compactification on S^1 gives maximally supersymmetric SU(N) theory in 4 dimensions when one takes the low energy limit and makes all the Kaluza-Klein modes decouple. One finds that the complexified gauge coupling constant of the resulting four-dimensional theory is given by the modulus of the torus T^2. Now, the way one assigns the modulus to a given torus is not unique: the torus with the modulus τ is equivalent to the torus with the modulus $-1/\tau$. This statement then turns into the equivalence of the $\mathcal{N}=4$ SU(N) theory at coupling τ and at coupling $-1/\tau$, see Fig. 2.

2.2. $\mathcal{N}=2$ SU(2) *with four flavors*

Then let us discuss how the S-duality of $\mathcal{N}=2$ SU(2) theory with four flavors can be geometrically understood using M5-branes. Gaiotto showed in [2] that this theory can be constructed by wrapping two M5-branes on a sphere with four punctures, see Fig 3. The exponential q of the coupling constant is given by the cross ratio of the four points. One can also think of the torus which is a double cover of the same sphere with branch points at these

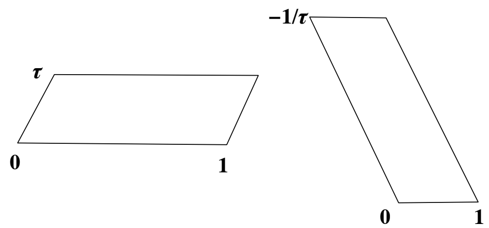

Figure 2. Two ways to label the same torus.

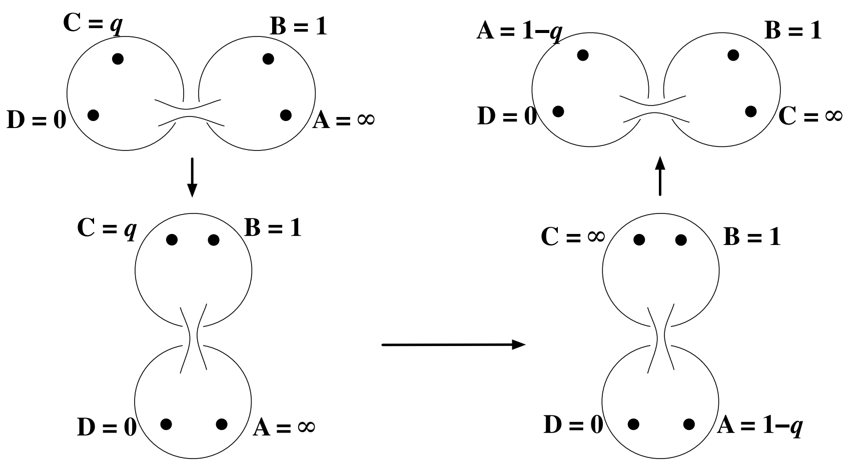

Figure 3. Relabeling of a sphere with four punctures. Step 1: split the sphere vertically. Step 2: perform the coordinate change. Step 3: rotate the sphere by 90 degrees.

four punctures. Then the low-energy coupling constant is given by the modulus of the torus. In either case, there is no unique way to associate a coupling constant to a given four-punctured sphere. This translates to the S-duality of the SU(2) theory with four flavors.

In fact one can read off more about the S-duality from this construction. Four hypermultiplets in the doublet representation transform under the vector representation of SO(8) flavor symmetry. Each of four punctures can be associated to a particular SU(2) subgroup of this flavor symmetry. First let us split four hypermultiplets into two pairs of two hypermultiplets, with flavor symmetry $\mathrm{SO}(4)\times\mathrm{SO}(4)$. Then we decompose each SO(4) into $\mathrm{SU}(2)\times\mathrm{SU}(2)$. Call the punctures at $z=\infty$, 1, q and 0 respectively A, B, C and D, and call the corresponding flavor symmetry $\mathrm{SU}(2)_{A,B,C,D}$. Then the vector representation of SO(8) transforms as

$$2_A \otimes 2_B \oplus 2_C \otimes 2_D. \tag{1}$$

Under the S-duality which sends $q \to 1-q$, the positions of the punctures are reshuffled, see Fig 3. Now the hypermultiplets transform as

$$2_A \otimes 2_D \oplus 2_B \otimes 2_C, \tag{2}$$

which is the decomposition of a chiral spinor representation of SO(8). This clearly exemplifies that the S-duality of this theory involves the triality of the flavor symmetry SO(8). It is surprising that this simple explanation was not known until 2009.

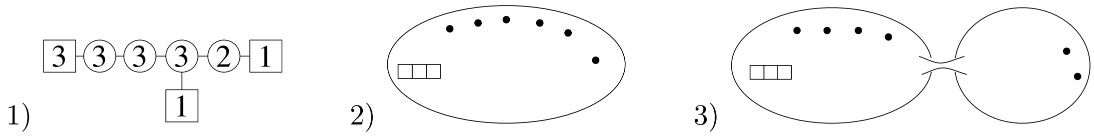

Figure 4. Example of a linear quiver theory and its realization by M5 branes. 1) The gauge theory. 2) M5-branes on a sphere with punctures. 3) The limit where SU(2) becomes weak.

2.3. General procedure

In general, one can consider a superconformal gauge theory with the gauge group

$$\mathrm{SU}(d_1) \times \mathrm{SU}(d_2) \times \cdots \times \mathrm{SU}(d_{n-1}) \times \mathrm{SU}(d_n)\ , \tag{3}$$

with a bifundamental hypermultiplet between each pair of consecutive gauge groups $\mathrm{SU}(d_a) \times \mathrm{SU}(d_{a+1})$. We put $k_a = 2d_a - d_{a-1} - d_{a+1}$ extra fundamental hypermultiplets for $\mathrm{SU}(d_a)$ to make it superconformal. A theory which falls within this construction is called a superconformal linear quiver theory.

Since k_a is non-negative, we have

$$d_1 < d_2 < \cdots < d_l = \cdots = d_r > d_{r+1} > \cdots > d_n\ . \tag{4}$$

We denote $N = d_l = \cdots = d_r$; we refer to the parts to the right of d_r and to the left of d_l as two tails of this superconformal quiver. k_a is non-negative, which means that $d_a - d_{a+1}$ is monotonically non-decreasing for $a > r$. Thus we can associate naturally a Young diagram to the tail by requiring that it has a row of width $d_a - d_{a+1}$ for each $a \geq r$. In the following, a Young diagram with the columns of height 3, 3, 2 will be denoted as $\{3^2, 2\}$, etc.

Gaiotto showed that this gauge theory can be obtained by wrapping N M5-branes on a sphere with $n + 3$ punctures. The punctures are at

$$z = \infty,\ 1,\ q_1,\ q_1q_2,\ \ldots,\ q_1 \cdots q_n,\ 0. \tag{5}$$

The punctures at $z \neq 0, \infty$ are all of the same type, and called the simple punctures. The punctures at $z = \infty, 0$ encode the structure of the left and the right tails, respectively, and thus labeled by the corresponding Young diagrams. q_i is the exponentiated complexified gauge coupling of the i-th SU gauge group. A puncture at $z = z_i$ controls how the worldvolume fields on the coincident M5-branes diverge at $z = z_i$, much as a 't Hoot loop in a gauge theory controls how the gauge field diverges close to the loop. The type of the puncture specifies exactly how they diverge. For more details, see [2].

As an example, consider the quiver in Fig. 4, 1). A circle with a number d stands for $\mathrm{SU}(d)$ gauge group, and a box with a number k stands for k extra fundamental hypermultiplets associated to the gauge group connected by a line to the box in question. The corresponding configuration of punctures on 3 M5-branes is given in Fig. 4, 2). There, the puncture at $z = 0$ for the tail SU(3)–SU(2)–U(1) is denoted by the same symbol as the one for the simple punctures, because the worldvolume fields around a simple puncture and the puncture associated to this tail behave in a same way. This is as it should be, because the S-duality of the rightmostSU(2) can exchange these two punctures.

Now let us make the coupling of the SU(2) gauge group very weak. Equivalently, we take q of this gauge group to be very small. This creates a long neck between the second and the third simple punctures from the right, see Fig. 4, 3). In other words, if we split off two simple punctures on a sphere from the rest of the Riemann surface on which three M5 branes wrap,

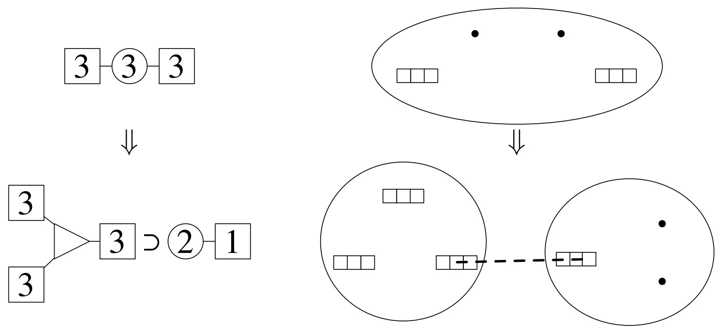

Figure 5. S-duality involving Minahan-Nemeschansky's E_6 SCFT. The configuration of punctures on M5-branes is shown on the right.

there is a weakly coupled SU(2) gauge group with one extra fundamental hypermultiplet coupled to it. In general, if we start by wrapping N M5-branes and split off a sphere with $N-1$ simple punctures from the rest of the Riemann surface, we generate a tail of the form

$$\mathrm{SU}(N-1)\times\mathrm{SU}(N-2)\times\cdots\times\mathrm{SU}(3)\times\mathrm{SU}(2) \tag{6}$$

with one extra fundamental hypermultiplet for the rightmost SU(2), and the coupling constant of the leftmost $\mathrm{SU}(N-1)$ becomes very weak.

3. Application

3.1. E_6

Using the procedure reviewed in the previous section, we can now readily find that the S-dual of a linear quiver often involves non-trivial superconformal field theory. The first example is the original duality of Argyres and Seiberg. SU(3) theory with six flavors can be thought of as a linear quiver of the form shown in the upper row of Fig. 5. The infinite coupling limit involves making two simple punctures collide. In a different coordinate system, this implies that we split off a sphere with two simple punctures. As we saw, this generates a weakly-coupled SU(2) gauge group with one extra hypermultiplet, see the bottom row of Fig. 5.

Let us study what the rest of the Riemann surface stands for. We have three punctures of the same type $\{1^3\}$, which makes $\mathrm{SU}(3)^3$ flavor symmetry manifest. Now, two out of the three punctures came from the punctures which we originally had, and they represented two pairs of three fundamental hypermultiplets. Therefore $\mathrm{SU}(3)^2$ flavor symmetry is a subgroup of SU(6) flavor symmetry of the six fundamental hypermultiplets. From the standpoint of wrapped M5-branes, however, there is no difference between two punctures originally present and the final puncture generated by the decoupling of the tail. Therefore, any two out of three SU(3) flavor symmetry should enhance to SU(6) symmetry. This is possible only when the $\mathrm{SU}(3)^3$ flavor symmetry is in fact inside E_6 flavor symmetry, see Fig. 6. This was discussed in [2].

3.2. E_7

Next, let us consider the quiver shown in the first line of Fig. 7. The gauge group is $\mathrm{SU}(4)\times\mathrm{SU}(2)$ with the bifundamental hypermultiplets charged under the two SU factors, and there are in addition six fundamental hypermultiplets for the node SU(4). In term of four M5-branes wrapping a sphere, we have three simple punctures, one puncture of type $\{1^4\}$ and one of type $\{2^2\}$. We can go to a limit where a sphere with three simple punctures splits. A dual superconformal tail with gauge groups $\mathrm{SU}(3)\times\mathrm{SU}(2)$ appears. After the neck is pinched off, we have a sphere with one puncture of type $\{2^2\}$ and two punctures of type $\{1^4\}$. This

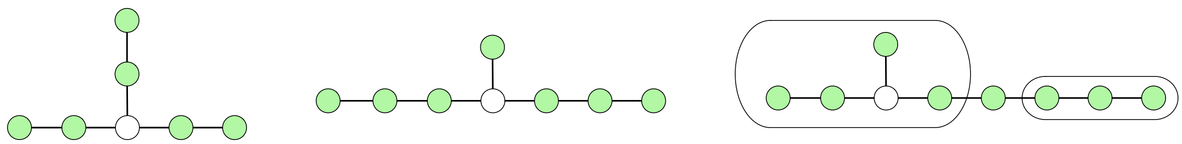

Figure 6. Dynkin diagram of $E_{6,7,8}$. The E_6 diagram for E_6 shows three SU(3) subgroups, any two out of which enhance to SU(6) via the node at the center. The E_7 diagram shows the subgroup $\mathrm{SU}(2) \times \mathrm{SU}(4)^2$. The E_8 diagram shows the subgroup $\mathrm{SO}(10) \times \mathrm{SU}(4)$ and the subgroup $\mathrm{SU}(2) \times \mathrm{SU}(3) \times \mathrm{SU}(6)$.

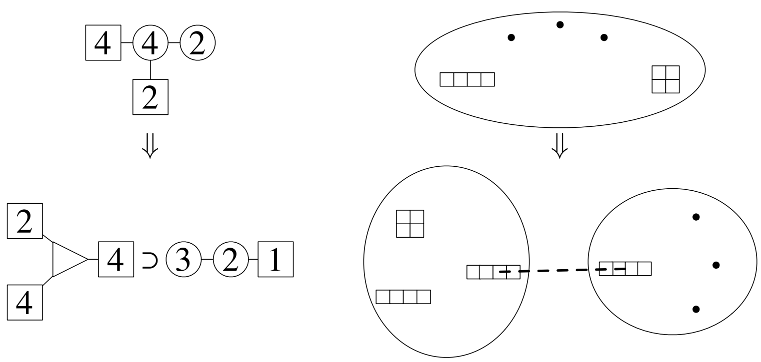

Figure 7. S-duality involving Minahan-Nemeschansky's E_7 SCFT.

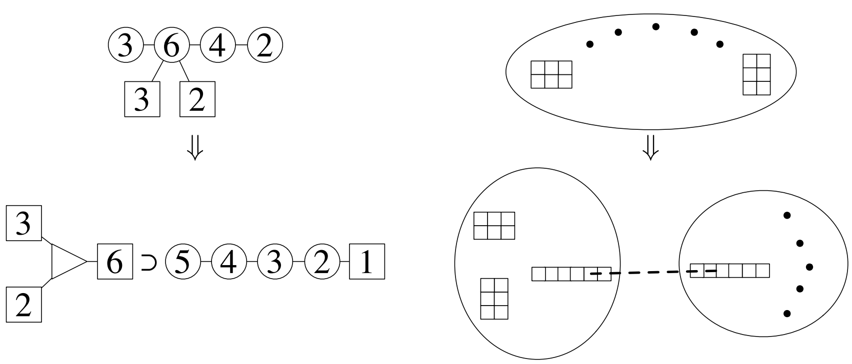

Figure 8. First example of S-duality involving Minahan-Nemeschansky's E_8 SCFT.

description shows the flavor symmetry $\mathrm{SU}(2) \times \mathrm{SU}(4)^2$. In the original description, it is clear that $\mathrm{SU}(2) \times \mathrm{SU}(4)$ enhances to SU(6). From the point of view of the M5-branes, the two SU(4) cannot be distinguished. Therefore, the other combination of $\mathrm{SU}(2) \times \mathrm{SU}(4)$ should also enhance to SU(6). This is only possible when the total flavor symmetry enhances to E_7, see Fig. 6. This was discussed in [3].

3.3. E_8

To discuss S-dualities involving superconformal theory with E_8 flavor symmetry, let us consider the quiver with the gauge group $\mathrm{SU}(3) \times \mathrm{SU}(6) \times \mathrm{SU}(4) \times \mathrm{SU}(2)$ with bifundamental hypermultiplets between the consecutive SU factors; one has in addition five fundamental hypermultiplets for SU(6), see the first line of Fig. 8. We can go to a limit where a

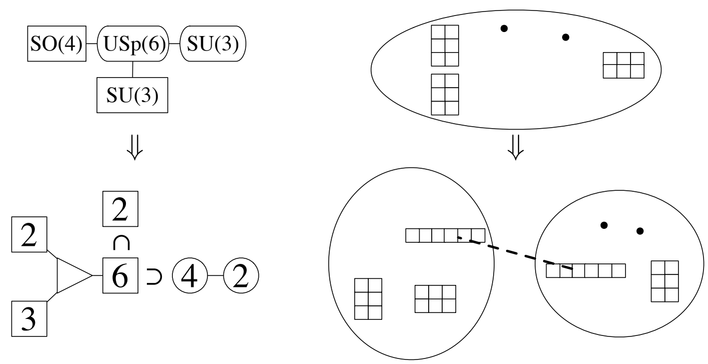

Figure 9. Second example of S-duality involving Minahan-Nemeschansky's E_8 SCFT.

sphere with five simple punctures splits off. A dual superconformal tail with gauge groups $\mathrm{SU}(5)\times\mathrm{SU}(4)\times\mathrm{SU}(3)\times\mathrm{SU}(2)$ appears. We tune the gauge coupling of the SU(5) group to zero, leaving a theory which is given by wrapping six M5-branes on a sphere with one puncture of type $\{1^6\}$, one of type $\{2^3\}$ and another of type $\{3^2\}$. This description shows the flavor symmetry $\mathrm{SU}(2)\times\mathrm{SU}(3)\times\mathrm{SU}(6)$. In the original description, it is clear that $\mathrm{SU}(3)\times\mathrm{SU}(2)$ enhances to SU(5). With a small effort, one can show that the flavor symmetry further enhances to E_8. This was discussed in [3].

Finally, let us consider a new example, whose quiver is shown in the first line of Fig. 9. This theory has the gauge group $\mathrm{USp}(6)\times\mathrm{SU}(3)$, with five fundamental hypermultiplet for the USp(6), and a hypermultiplet in the bifundamental of $\mathrm{USp}(6)\times\mathrm{SU}(3)$. As briefly discussed in [2] and detailed in [4], we get a tail of the form $\mathrm{SU}(2N)$–$\mathrm{USp}(2N)$ with two extra fundamental hypermultiplet for $\mathrm{USp}(2N)$ when we put two punctures of type $\{N^2\}$ on a sphere. From this perspective it is natural to split five hypermultiplets into two and three, and this quiver has the realization as six M5-branes wrapped on a sphere with two simple punctures, two punctures of type $\{3^2\}$, and one of type $\{2^3\}$. By splitting off a sphere with two simple punctures and one of type $\{2^3\}$, a tail with the gauge group $\mathrm{SU}(4)\times\mathrm{SU}(2)$ is generated, with the gauge coupling of SU(4) becoming weak. We are again left with a theory corresponding to a sphere with one puncture of type $\{1^6\}$, one of type $\{2^3\}$ and another of type $\{3^2\}$.

The enhancement of the flavor symmetry to E_8 is more transparent in this description. In the original description we have SO(10) flavor symmetry acting on five fundamental hypermultiplets, of which the subgroup $\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{SU}(3)$ is manifest in the dual description. Here, one SU(2) is directly carried by the puncture $\{3^2\}$, but the other SU(2) toghether with the SU(4) to which the gauge fields couple is a subgroup of SU(6) carried by the puncture $\{1^6\}$. Therefore, the $\mathrm{SU}(2)\times\mathrm{SU}(3)\times\mathrm{SU}(6)$ flavor symmetry of the theory corresponding to this three-punctured sphere should be such that the subgroup $\mathrm{SU}(2)\times\mathrm{SU}(3)\times\mathrm{SU}(2)$ enhances to SO(10). This is only possible when the total flavor symmetry is in fact E_8, see Fig. 6.

Now, let us determine the effective numbers n_v and n_h of the hyper- and vector multiplets of the E_8 theory. In the first realization, the original quiver gauge theory with gauge group $\mathrm{SU}(3)\times\mathrm{SU}(6)\times\mathrm{SU}(4)\times\mathrm{SU}(2)$ had

$$n_v = 8+35+15+3 = 61, \qquad n_h = 18+30+24+8 = 80. \tag{7}$$

The S-dual description has

$$n_v = n_v[E_8]+24+16+8+3, \qquad n_h = n_h[E_8]+20+12+6+2. \tag{8}$$

Equating (7) and (8), we conclude

$$n_v[E_8] = 11, \qquad n_h[E_8] = 40. \tag{9}$$

In the second realization, the original theory with gauge group USp(6) × SU(3) had

$$n_v = 21 + 8 = 29, \qquad n_h = 12 + 18 + 18 = 48. \tag{10}$$

The S-dual description has

$$n_v = n_v[E_8] + 15 + 3, \qquad n_h = n_h[E_8] + 8. \tag{11}$$

Equating (10) and (11), we again find (9), which shows the consistency of our construction.

Another S-duality involving E_8 theory had been discovered in [5]. The original theory was USp(6) gauge theory with 11 half-hypers in the fundamental representation and with a half-hyper in the three-index antisymmetric traceless tensor. The dual is the E_8 theory coupled to an SO(5) gauge multiplet, via the subgroup SO(5) × SO(11) ⊂ SO(16) ⊂ E_8. Here, the original had

$$n_v = 21, \qquad n_h = 33 + 7 = 40. \tag{12}$$

The dual had

$$n_v = n_v[E_8] + 10, \qquad n_h = n_h[E_8]. \tag{13}$$

Again, the consistency of (12) and (13) requires (9). The result (9) is also consistent with [6].

4. Conclusions

In this talk, we saw how the construction of $\mathcal{N} = 2$ theories using wrapped M5-branes helps us understand the S-duality involving strange non-Lagrangian superconformal theories such as the duality of Argyres and Seiberg. As explicit examples, we saw how Minahan and Nemeschansky's superconformal theories with exceptional flavor symmetry arise naturally in S-dual descriptions of standard linear quiver gauge theories.

The general procedure reviewed above makes it clear that M5-branes wrapped on a sphere with three punctures give basic building blocks of $\mathcal{N} = 2$ theories of this class. Such a building block is naturally labeled by three Young diagrams which specify the types of punctures put on the stack of M5-branes. The results presented in Sec. 3 show that some of them are theories which have been long known, but the rest are completely new $\mathcal{N} = 2$ superconformal field theories in four dimensions. They are in some sense as elementary as free hypermultiplets, and warrant further study.

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