On some conjectures on VOAs

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In [1], a lot of mathematical conjectures on VOAs were made. Here, we'll provide a more mathematical translation, along the lines of [2]. I'm sure that I used many notations not common in the VOA literature; please point them out. I'm willing to make this as readable as possible for VOA theorists.

1 Category $\mathbb{V}$

In the following, a VOA always stands for vertex operator super-algebra $\mathcal{V} = \oplus_n \mathcal{V}_n$, $\mathcal{V}_n = \mathcal{V}_{n,+} \oplus \mathcal{V}_{n,-}$. The grading n is over integers and half-integers. The part $V_\pm = \oplus_n \mathcal{V}_{n,\pm}$ are called bosonic and fermionic, respectively.

Given two VOAs $\mathcal{V}_1$ and $\mathcal{V}_2$, we denote its product by $\mathcal{V}_1 \times \mathcal{V}_2$. We denote the zero mode of a field $\phi(z)$ by ϕ_0.

Given a VOA $\mathcal{V}$ with a homomorphism $\phi : \hat{\mathfrak{g}} \to \mathcal{V}$, let us denote by $\mathcal{DS}(\mathcal{V}, \mathfrak{g}, e)$ obtained by the quantum Drinfeld-Sokolov reduction of $\mathcal{V}$ with respect to the affine algebra $\hat{\mathfrak{g}}$ and a nilpotent element e of $\mathfrak{g}$.

Again, given a VOA $\mathcal{V} = \oplus_n \mathcal{V}_n$ with a homomorphism $\phi : \hat{\mathfrak{g}} \to \mathcal{V}$, we define its character $\operatorname{ch} \mathcal{V}(z)$ for $z \in G$ where G is a simply-connected group for the Lie algebra $\mathfrak{g}$ by

$$\operatorname{ch} \mathcal{V}(z) = \sum_n q^n \operatorname{Str}_{\mathcal{V}_n} z. \tag{1.1}$$

We consider the following category $\mathbb{V}$:

- Its objects are VOAs of the form

$$\prod_{i=1}^{n} \hat{\mathfrak{g}}_{i,-h^\vee(\mathfrak{g}_i)} \tag{1.2}$$

  where $\mathfrak{g}_i$ is a simple Lie algebra, $\hat{\mathfrak{g}}_k$ is the affine Lie algebra for $\mathfrak{g}$ regarded as a VOA with k its level, and $h^\vee$ is the dual Coxeter number. We allow the case when $n = 0$, which we denote by 1. In the following, we simply denote by $\hat{\mathfrak{g}}$ a VOA of the form (1.2).

- An element in $\mathrm{Hom}(\hat{\mathfrak{g}}, \hat{\mathfrak{g}}')$ is a triple $(\mathcal{V}, \phi, \phi')$ where $\mathcal{V}$ is a VOA and ϕ, ϕ' are homomorphisms
$$\phi : \hat{\mathfrak{g}} \to \mathcal{V}, \quad \phi' : \hat{\mathfrak{g}}' \to \mathcal{V}. \tag{1.3}$$
Writing ϕ, ϕ' explicitly is tedious, and we often just denote $\mathcal{V} \in \mathrm{Hom}(\mathfrak{g}, \mathfrak{g}')$.

- Given $\mathcal{V} \in \mathrm{Hom}(\hat{\mathfrak{g}}, \hat{\mathfrak{g}}')$ and $\mathcal{V}' \in \mathrm{Hom}(\hat{\mathfrak{g}}', \hat{\mathfrak{g}}'')$, its composition $\mathcal{V}' \circ \mathcal{V}$ is defined as a 'VOA version of holomorphic symplectic quotient' as will be defined later.

Before defining the composition, let us discuss easier properties.

- $\mathcal{V}$ is a symmetric monoidal category. Given two objects $\mathfrak{g}$ and $\mathfrak{g}'$, its product is $\mathfrak{g} \times \mathfrak{g}'$. Similarly, given $(\mathcal{V}_1, \phi_1, \phi_1') \in \mathrm{Hom}(\hat{\mathfrak{g}}_1, \hat{\mathfrak{g}}_1')$ and $(\mathcal{V}_2, \phi_2, \phi_2') \in \mathrm{Hom}(\hat{\mathfrak{g}}_2, \hat{\mathfrak{g}}_2')$, its product is $(\mathcal{V}_1 \times \mathcal{V}_2, \phi_1 \times \phi_2, \phi_1' \times \phi_2')$.

- $\mathcal{V}$ 'has duality' in the following technical sense. There is a canonical identification
$$\mathrm{Hom}(\hat{\mathfrak{g}}, \hat{\mathfrak{g}}' \times \hat{\mathfrak{g}}'') \simeq \mathrm{Hom}(\hat{\mathfrak{g}} \times \hat{\mathfrak{g}}'', \hat{\mathfrak{g}}') \tag{1.4}$$
given as follows. Given an element $(\mathcal{V}, \phi, \phi' \times \phi'')$ of the Hom on the left, we assign an element $(\mathcal{V}, \phi \times (\phi'' \circ \pi), \phi')$ on the right, where π is an automorphism of $\hat{\mathfrak{g}}$ obtained by extending the automorphism of $\mathfrak{g} = \prod_i \mathfrak{g}_i$ which sends a finite dimensional representation to its dual.

Now, let us define the composition of morphisms. For this, we define an operation $\mathcal{Q}\mathrm{uot}(\mathcal{V}, \hat{\mathfrak{g}}_{-2h^\vee(\mathfrak{g})}, \phi)$ which gives a new VOA given a VOA $\mathcal{V}$ with a homomorphism ϕ:
$$\phi : \hat{\mathfrak{g}}_{-2h^\vee(\mathfrak{g})} \to \mathcal{V}. \tag{1.5}$$
Note that the level here is $-2h^\vee(\mathfrak{g})$. To define $\mathcal{Q}\mathrm{uot}$, we first introduce a ghost VOA $\mathcal{BC}(\mathfrak{g})$, generated by fermionic fields b^A in $\mathcal{BC}(\mathfrak{g})_{1,-}$ and c_A in $\mathcal{BC}(\mathfrak{g})_{0,-}$ for $A = 1, \dots, \dim \mathfrak{g}$ with the OPE
$$b^A(z) c_B(w) \sim \frac{\delta^A_B}{z - w}. \tag{1.6}$$
This has a subalgebra $\hat{\mathfrak{g}}_{+2h^\vee(\mathfrak{g})}$. Denote by $J^A_{\mathcal{V}}(z)$ and J^A_{ghost} the affine $\mathfrak{g}$ currents of $\mathcal{V}$ and $\mathcal{BC}(\mathcal{V})$ respectively. We define
$$j_{\text{BRST}}(z) = \sum_A (c_A J^A_{\mathcal{V}}(z) + \frac{1}{2} c_A J^A_{\text{ghost}}(z)). \tag{1.7}$$
Then $d = j_{\text{BRST},0}$ is nilpotent, thanks to the condition on the level of $\hat{\mathfrak{g}}$ in $\mathcal{V}$. We take the subspace
$$\mathcal{W} \subset \mathcal{V} \times \mathcal{BC}(\mathfrak{g}) \tag{1.8}$$
defined by
$$\mathcal{W} = \bigcap_A (\mathrm{Ker}\, b^A_0 \cap \mathrm{Ker}(J^A_{\text{total}})) \tag{1.9}$$

where $J^A_{\text{total}} = J^A_{\mathcal{V}} + J^A_{\text{ghost}}$. We can check that the differential d acts within $\mathcal{W}$, and finally we define

$$\mathcal{Q}\text{uot}(\mathcal{V}, \hat{\mathfrak{g}}_{-2h^\vee(\mathfrak{g})}, \phi) = H(\mathcal{W}, d). \tag{1.10}$$

$\mathcal{Q}$uot for products $\prod_i \hat{\mathfrak{g}}_{i,-2h^\vee(\mathfrak{g}_i)}$ is defined in a completely similar way. This is like a VOA version of (holomorphic) symplectic quotient: (1.9) is like setting the moment map to be zero, and (1.10) is like taking the quotient.

Then, given $(\mathcal{V}_1, \phi_1', \phi_1) \in \mathrm{Hom}(\hat{\mathfrak{g}}', \hat{\mathfrak{g}})$ and $(\mathcal{V}_2, \phi_2, \phi_2'') \in \mathrm{Hom}(\hat{\mathfrak{g}}, \hat{\mathfrak{g}}'')$, its composition is given by

$$(\mathcal{V}_2, \phi_2, \phi_2'') \circ (\mathcal{V}_1, \phi_1', \phi_1) = (\mathcal{Q}\text{uot}(\mathcal{V}_1 \times \mathcal{V}_2, \hat{\mathfrak{g}}_{-2h^\vee(\mathfrak{g})}, \phi_1 + (\phi_2 \circ \pi)), \phi_1', \phi_2''). \tag{1.11}$$

Note that this is possible because $\phi_1 + (\phi_2 \circ \pi)$ defines a homomorphism from $\hat{\mathfrak{g}}_{-2h^\vee(\mathfrak{g})}$ to $\mathcal{V}_1 \times \mathcal{V}_2$.

Now, we have cheated at one point, which needs to be stated as a conjecture here:

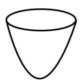
Conjecture 1. *There is a VOA $\mathcal{I}\mathrm{d}_{\hat{\mathfrak{g}}} \in \mathrm{Hom}(\hat{\mathfrak{g}}, \hat{\mathfrak{g}})$ acting as an identity under the composition defined above.*

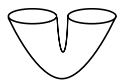
Most probably, in the decomposition $\mathcal{I}\mathrm{d}_{\hat{\mathfrak{g}}} = \oplus_{n=0}^{\infty} \mathcal{V}_0$, $\mathcal{V}_0$ is isomorphic to $\mathbb{C}[G]$ where G is the algebraic group over $\mathbb{C}$ of type $\mathfrak{g}$. In particular, $\mathcal{V}_0$ is infinite dimensnional.

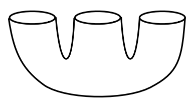
2 Functor $\eta_{\mathfrak{g}}$

We use another category $\mathbb{B}$, i.e. the objects are closed oriented one-dimensional manifolds (i.e. disjoint unions of multiple S^1s) and a morphism from B_1 to B_2 is a two-dimensional oriented manifold C whose boundary is $B_1 \sqcup (-B_2)$. $\mathbb{B}$ is a symmetric monoidal category with duality under the standard operations.

For example, we have

$U =$  $\in \mathrm{Hom}(S^1, \varnothing),$ (2.1)

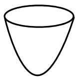
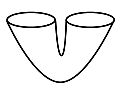
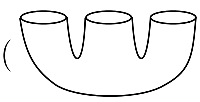
$V =$  $\in \mathrm{Hom}(S^1 \sqcup S^1, \varnothing),$ (2.2)

$W =$  $\in \mathrm{Hom}(S^1 \sqcup S^1 \sqcup S^1, \varnothing).$ (2.3)

Conjecture 2. *For each simple, simply-laced Lie algebra $\mathfrak{g}$. Then there is a functor*

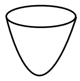
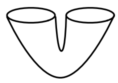
$$\eta_{\mathfrak{g}} : \mathbb{B} \to \mathbb{V} \tag{2.4}$$

with $\eta_{\mathfrak{g}}(S^1) = \hat{\mathfrak{g}}_{-h^\vee(\mathfrak{g})}$, such that the VOAs

$\mathcal{U}_{\mathfrak{g}} = \eta_{\mathfrak{g}}($  $),\qquad \mathcal{V}_{\mathfrak{g}} = \eta_{\mathfrak{g}}($  $),\qquad \mathcal{W}_{\mathfrak{g}} = \eta_{\mathfrak{g}}($  $)$ (2.5)

satisfy the following properties:

1. $\mathcal{V}_{\mathfrak{g}}$ *is the identity* $\mathcal{I}\mathrm{d}_{\mathfrak{g}}$.
2. $\mathcal{U}_{\mathfrak{g}}$ *is obtained from* $\mathcal{V}_{\mathfrak{g}}$ *via the Drinfeld-Sokolov reduction,*

   $\eta_{\mathfrak{g}}($  $) = \mathcal{DS}(\eta_{\mathfrak{g}}($  $), \hat{\mathfrak{g}}, e_{prin})$ (2.6)

   where e_{prin} *is a principal nilpotent element of* $\mathfrak{g}$.
3. $\mathcal{W}_{\mathfrak{g}}$ *has central charge*

   $$c = (3 - 2h^{\vee}(\mathfrak{g}))\dim\mathfrak{g} - \mathrm{rank}\,\mathfrak{g} \tag{2.7}$$

   and the following character:

   $$\mathrm{ch}\,\mathcal{W}_{\mathfrak{g}}(z_1, z_2, z_3) = \frac{K_0(z_1)K_0(z_2)K_0(z_3)}{K_{prin}} \sum_{\lambda} \frac{\chi_\lambda(z_1)\chi_\lambda(z_2)\chi_\lambda(z_3)}{\chi_\lambda(q^\rho)}. \tag{2.8}$$

   Here, $(z_1, z_2, z_3) \in G^3$ *which is the exponential of the action of* $\mathfrak{g}^3$ *on* $\mathcal{W}_n$*, the sum is over all the irreducible representation* λ *of* $\mathfrak{g}$*,* χ_λ *is the character,* K_0 *is the character of* z *in the representation*

   $$\mathcal{K}_0 = \oplus_{m=0}^{\infty} \mathrm{Sym}^m(\oplus_{n=1}^{\infty}(q^n\mathfrak{g})), \tag{2.9}$$

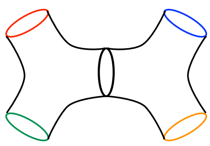
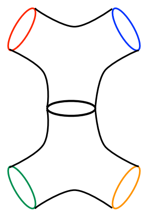
   and

   $$K_{prin} = \prod_{n=1}^{\infty}\prod_{i=1}^{\mathrm{rank}\,\mathfrak{g}} \frac{1}{1 - q^{d_i+n}} \tag{2.10}$$

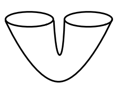
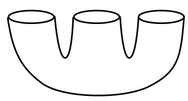
   where d_i *is the* i*-th exponent of* $\mathfrak{g}$ *plus 1, and the element* $q^\rho \in G$ *is defined as usual by the Weyl vector. For example,* $q^\rho = \mathrm{diag}(q^{1/2}, q^{-1/2})$ *when* $\mathfrak{g} = A_1$.

Remark. The generating relations of $\mathbb{B}$ is not very complicated. Therefore, to prove that $\eta_{\mathfrak{g}}$ exists, one just has to do the following.

- Find a VOA $\mathcal{W}_{\mathfrak{g}}$ satisfying (2.7) and (2.8).
- Show

  $\eta_{\mathfrak{g}}($  $) = \eta_{\mathfrak{g}}($  $),$ (2.11)

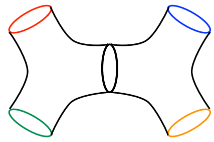
  where the VOA on the both sides are understood to be defined from two copies of $\mathcal{W}_{\mathfrak{g}}$ by (1.11).
- Show

  $\eta_{\mathfrak{g}}($  $) := \mathcal{DS}(\eta_{\mathfrak{g}}($  $), \mathfrak{g}, e_{\mathrm{prin}})$ (2.12)

  is the identity of $\mathrm{Hom}(\hat{\mathfrak{g}}, \hat{\mathfrak{g}})$.

Another remark. From the definition of the operation $\mathcal{Q}\mathrm{uot}$, it is not too difficult to show that given $\phi : \hat{\mathfrak{g}}_{-h^\vee(\mathfrak{g})} \to \mathcal{V}$ and $\phi' : \hat{\mathfrak{g}}_{-h^\vee(\mathfrak{g})} \to \mathcal{V}'$, the character of $\mathcal{W} = \mathcal{Q}\mathrm{uot}(\mathcal{V} \times \mathcal{V}', \hat{\mathfrak{g}}_{-2h^\vee(\mathfrak{g})}, \phi + \phi' \circ \pi)$ is given by

$$\mathrm{ch}\,\mathcal{W} = \int_{G_{\mathrm{cpt}}} \mathrm{ch}\,\mathcal{V}(z)\,\mathrm{ch}\,\mathcal{V}'(z^{-1}) K_0(z)^{-2}[dz] \tag{2.13}$$

where the integral is taken over the compact simply-connected Lie group G_{cpt} of type $\mathfrak{g}$, and $[dz]$ is the Haar measure. From this, it is easy to compute the character of $\eta_{\mathfrak{g}}($  $)$ using orthogonality of the irreducible characters, assuming (2.8). More generally, denote by C_n a sphere with n S^1 boundaries. Then we easily see that

$$\mathrm{ch}\,\eta_{\mathfrak{g}}(C_n)(z_1, \ldots, z_n) = \sum_\lambda \frac{\prod_{i=1}^n K_0(z_i)\chi_\lambda(z_i)}{(K_{\mathrm{prin}}\chi_\lambda(q^\rho))^{n-2}}. \tag{2.14}$$

Note also that, from (2.6), we have

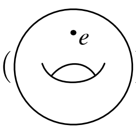
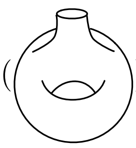
$$\eta_{\mathfrak{g}}(C_{n-1}) = \mathcal{DS}(\eta_{\mathfrak{g}}(C_n), \mathfrak{g}, e_{\mathrm{prin}}). \tag{2.15}$$

This plays nicely with the formula (2.14) due to the following general fact: Given $\phi : \hat{\mathfrak{g}} \to \mathcal{V}$, the character of $\mathcal{W} = \mathcal{DS}(\mathcal{V}, e_{\mathrm{prin}})$ is obtained by

$$\mathrm{ch}\,\mathcal{W} = K_{\mathrm{prin}}(\frac{\mathrm{ch}\,\mathcal{V}(z)}{K_0(z)})\Big|_{z \to q^\rho}. \tag{2.16}$$

Slightly generalized version of $\eta_{\mathfrak{g}}$. We slightly generalize the definition of $\eta_{\mathfrak{g}}$, by allowing surfaces with points labeled by a nilpotent element e of $\mathfrak{g}$ on the 2d surfaces. More precisely, we consider a category $\mathbb{B}_{\mathfrak{g}}$ where the objects are still one-dimensional manifolds, and the morphisms are 2d surfaces with points each labeled by a nilpotent element of $\mathfrak{g}$. We do not allow points on the boundary of the 2d surface.

We then extend the functor $\eta_{\mathfrak{g}} : \mathbb{B} \to \mathbb{V}$ to a functor $\mathbb{B}_{\mathfrak{g}} \to \mathbb{V}$ by defining e.g.

$\eta_{\mathfrak{g}}($  $) = \mathcal{DS}(\eta_{\mathfrak{g}}($  $), \hat{\mathfrak{g}}, e).$ (2.17)

In general, to define $\eta_{\mathfrak{g}}$ for a punctured surface, we replace each marked point by a boundary. We then perform, for each $\hat{\mathfrak{g}}$ associated to the newly introduced boundary, the quantum Drinfeld-Sokolov reduction by the nilpotent e.

We immediately see that a point labeled by $e = 0$ is equivalent to a boundary. Also, due to (2.6), having a point labeled by e_{prin} is equivalent to having no point at all.

3 Examples

Not much is known about $\mathcal{W}_{\mathfrak{g}}$. To state what is known, let us introduce a few VOAs. First, given a symplectic vector space V over $\mathbb{C}$, denote by $\mathcal{SB}(V)$ a VOA such that

$$\mathcal{SB}(V) = \oplus_{n=0}^{\infty} \mathcal{V}_n, \quad \mathcal{V}_0 = \mathbb{C}1, \quad \mathcal{V}_{1/2,+} \simeq V, \tag{3.1}$$

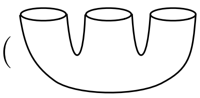
and

$$v(z)w(0) \simeq \frac{\langle v, w\rangle}{z} \tag{3.2}$$

for $v, w \in V \simeq \mathcal{V}_{1/2}$ where $\langle \cdot, \cdot \rangle$ is the symplectic pairing. This has the central charge $c = -(\dim_{\mathbb{C}} V)/2$.

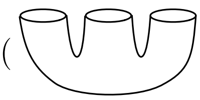
Let us also define $\mathcal{L}(\hat{\mathfrak{g}}, k)$ to be the VOA based on the irreducible, vacuum representation of the affine Lie algebra $\hat{\mathfrak{g}}$ with level k.

Conjecture 3. • *For $\mathfrak{g} = A_1$, we have*

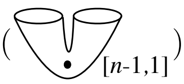
$\eta_{\mathfrak{g}}($  $) = \mathcal{SB}(V_1 \otimes V_2 \otimes V_3),$ (3.3)

where $V_i \simeq \mathbb{C}^2$.

- *For $\mathfrak{g} = A_2$, we have*

$\eta_{\mathfrak{g}}($  $) = \mathcal{L}(\widehat{\mathfrak{e}_6}, -3).$ (3.4)

- *For $\mathfrak{g} = A_{n-1}$, we have*

$\eta_{\mathfrak{g}}($  $) = \mathcal{SB}(V \otimes W^* \oplus W \otimes V^*).$ (3.5)

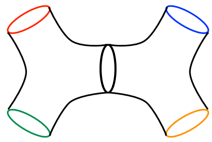
Here, $[n-1, 1]$ is a nilpotent element consisting of two Jordan blocks of size $n-1$, 1 (i.e. an element on the subregular orbit) and $V, W \simeq \mathbb{C}^n$. Combined with (3.4), we have

$$\mathcal{DS}(\mathcal{L}(\widehat{\mathfrak{e}_6}, -3), \widehat{\mathfrak{sl}(3)}_{-3}, e = [2, 1]) = \mathcal{SB}(V \otimes W^* \oplus W \otimes V^*), \tag{3.6}$$

i.e. a variant of the quantum Drinfeld-Sokolov reduction of the affine E_6 algebra gives a free boson.

Note that to complete the proof of the existence of the functor $\eta_{\mathfrak{g}}$ for $\mathfrak{g} = A_1$ and $\mathfrak{g} = A_2$, we just have to show (2.11) and (2.12), starting from $\mathcal{W}_{\mathfrak{g}}$ given by (3.3) and (3.4). In particular, as for the relation (2.11), we have the following conjecture.

Conjecture 4. *For* $\mathfrak{g} = A_1$*,*

$\eta_{\mathfrak{g}}($  $) = \mathcal{L}(\widehat{\mathfrak{so}(8)}, -2).$ (3.7)

Note that $\mathfrak{sl}(2)^4 \subset \mathfrak{so}(8)$*, and the outer automorphism of* $\mathfrak{so}(8)$ *permutes four copies of* $\mathfrak{sl}(2)$*.*

Combined with (3.3)*, we have a free-field realization*

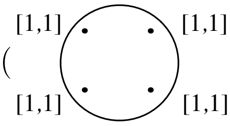
$$\mathcal{L}(\widehat{\mathfrak{so}(8)}, -2) = \mathcal{Q}\mathrm{uot}(\mathcal{SB}(V_1 \otimes V_2 \otimes V_3) \times \mathcal{SB}(V_3 \otimes V_4 \otimes V_5), \widehat{\mathfrak{sl}(2)}_{-4}, \phi) \tag{3.8}$$

where $\phi : \mathfrak{sl}(2) \to \mathcal{SB}(V_1 \otimes V_2 \otimes V_3) \times \mathcal{SB}(V_3 \otimes V_4 \otimes V_5)$ *comes from the action of* $\mathrm{SL}(2)$ *on* V_3*.*

Finally, we have the following, as a further generalization of (3.4) and (3.7) :

Conjecture 5.

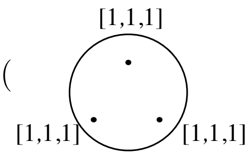
- *For* $\mathfrak{g} = A_1$*,*

  $\eta_{A_1}($ 

$) = \mathcal{L}(\widehat{\mathfrak{so}(8)}, -2).$ (3.9)

  Note that this is equivalent to (3.7)*.*

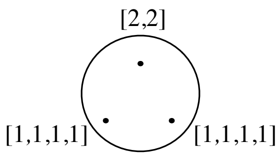
- *For* $\mathfrak{g} = A_2$*,*

  $\eta_{A_2}($ 

$) = \mathcal{L}(\widehat{\mathfrak{e}}_6, -3).$ (3.10)

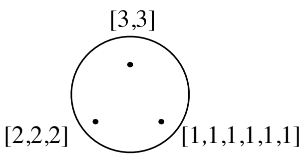
  Note that this is equivalent to (3.4)*.*

- *For* $\mathfrak{g} = A_3$*,*

  $\eta_{A_3}($ 

$) = \mathcal{L}(\widehat{\mathfrak{e}}_7, -4).$ (3.11)

- *For* $\mathfrak{g} = A_5$*,*

  $\eta_{A_5}($ 

$) = \mathcal{L}(\widehat{\mathfrak{e}}_8, -6).$ (3.12)

Note that the number of points on S^2, and the Jordan block structures of the nilpotent elements labeling the points, follow the structure of the extended Dynkin diagrams of D_4, $E_{6,7,8}$, respectively.

References

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