Notes for Skoltech summer school on mathematical physics (week 2)

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Abstract

These are my live-texed notes for (most of) week 2 of the Skoltech International Summer School on Mathematical Physics. (Two of the TA talks do not have notes.) There are definitely many errors and typos, most likely all due to me. Please let me know when you find them.

Contents

1 Andrei Okounkov (Jul 08)

These lectures will be auxiliary to lectures of Nikita. The goal is to introduce important concepts and talk a little bit about them. The important concept for today is Nakajima quiver varieties. Nakajima quiver varieties are algebraic symplectic reductions, of some particular representations of a product of general linear groups $\mathrm{GL}(V)$. These representations are of the form

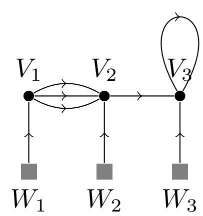
$$T^*\mathsf{Rep}(\text{framed quiver}).$$

A **quiver** is a graph: a bunch of vertices with directed edges, possibly with loops. The idea is that in the end these will act like Dynkin diagrams. A representation of a quiver is an assignment

$$\text{vertices} \mapsto \text{vector spaces } V_i$$
$$\text{arrows} \mapsto \text{operators.}$$

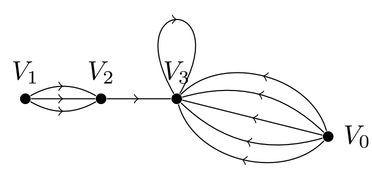
There is a natural action of $\mathrm{GL}(V) \coloneqq \prod \mathrm{GL}(V_i)$, acting on each vertex. The corresponding action on operators is the induced one, i.e. by pre- and post-composition.

A **framing** is a new set of vertices, denoted by squares. Maybe the main technical contribution of Nakajima is the contribution of these framings. Associated to each framing vertex is a new vector space W_i, along with a map $W_i \to V_i$ of corresponding vertices.



While there is an action of $\mathrm{GL}(W) \coloneqq \prod \mathrm{GL}(W_i)$ on these new framing vertices, we will *not* quotient by it later. It will continue to act on the resulting algebraic symplectic reduction.

There is a trick due to Crawley–Boevey that says we can introduce one extra vertex V_0 (of dimension 1) instead of introducing a framing.



The arrows from framing vertices can be replaced by $\dim W_i$ maps from V_0 to the V_i vertex. The group $\mathrm{GL}(V)$ becomes

$$\mathrm{GL}(V) = \mathrm{SL}\left(\bigoplus_{i>0} V_i \oplus V_0\right).$$

A **framed representation** of a framed quiver is then

$$M \coloneqq \bigoplus_{V_i \to V_j} \mathrm{Hom}(V_i, V_j) \oplus \bigoplus_i \mathrm{Hom}(W_i, V_i),$$

as a representation of $\mathrm{GL}(V)$. This is essentially a sum of three kinds of representations: the defining representation V_i, the adjoint representation $\mathrm{Hom}(V_i, V_i)$, and the bifundamental representation $\mathrm{Hom}(V_i, V_j)$. Then we consider T^*M, and use that

$$\mathrm{Hom}(V_1, V_2)^\vee = \mathrm{Hom}(V_2, V_1).$$

This is because given $A \in \mathrm{Hom}(V_1, V_2)$ and $B \in \mathrm{Hom}(V_2, V_1)$, there is a natural pairing

$$\langle A, B \rangle \coloneqq \operatorname{tr} AB = \operatorname{tr} BA.$$

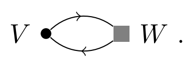
Hence we can write

$$T^*M = \bigoplus_{V_i \to V_j} \mathrm{Hom}(V_i, V_j) \oplus \mathrm{Hom}(V_j, V_i) \oplus \bigoplus_i \mathrm{Hom}(W_i, V_i) \oplus \mathrm{Hom}(V_i, W_i).$$

In particular, after we take T^*, the original orientation of the quiver is not important anymore.

Definition 1.1. A **Nakajima quiver variety** is an algebraic symplectic reduction of T^*framed $\mathrm{rep}(Q)$ of a quiver Q.

To explain algebraic symplectic reduction, we will look at the simplest non-trivial example of a quiver with one vertex and no arrows. A framed representation is therefore given by



If $\dim V = k$ and $\dim W = n$, then we get $T^* \mathrm{Gr}(k, n)$. Recall that the Grassmannian can be written

$$\mathrm{Gr}(k, n) = \{\text{subspaces } L \subset \mathbb{C}^n : \dim L = k\} = \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n)_{\substack{\text{open set}\\ \text{of rank}=k}} / \mathrm{GL}(k).$$

We have to be careful when taking quotients in algebraic geometry. Let's look at the example $k = 1$. Then $\mathrm{Gr}(1, n) = \mathbb{P}(\mathbb{C}^n)$ is the space of lines (through the origin) in $\mathbb{C}^n$. The action of $\mathrm{GL}(1) = \mathbb{C}^\times$ on $\mathbb{C}^n$ has some closed orbits, and some open orbits. Namely, the only closed orbit is $\{0\}$, and all other orbits are open. The closure of all these other orbits intersect at 0. The naive quotient will not be Hausdorff because of this. The first attempt to fix this is to take G-invariant functions and form

$$M/G \coloneqq \operatorname{Spec} \mathbb{C}[M]^G.$$

The points of such a quotient will be closed G-orbits. However in the case of $\mathbb{P}(\mathbb{C}^n)$ this leaves only a single point, which is undesirable. Instead, we want to remember local coordinates x_i/x_j on $\mathbb{P}(\mathbb{C}^n)$, which are all scaled the same way by the action. For Grassmannians, in general, the coordinates are Plücker coordinates, which again are scaled homogeneously. Hence when we construct $\mathbb{P}(\mathbb{C}^n)$ or $\mathrm{Gr}(k, n)$, we should take *ratios* of functions that transform with the same character of G, instead of invariant functions. Equivalently, these are sections of some line bundle L on M. The action of G on M can be lifted to an action of G on this line bundle L. Sections of such an L will be functions that transform by the same character of G. We can use such sections to define coordinates on the quotient M/G. A point $x \in M$ is **semistable** if there exists such a section f such that $f(x) \neq 0$. So such coordinates only exist on semistable points. Orbits which do not hit 0 determine the semistable locus. Note that every orbit has a point which is closest to 0; there is a function $\|gx\|^2$ which has a minimum there. Let $K \subset G$ be the maximal compact subgroup preserving $\|\cdot\|^2$. Then $\|gx\|^2$ is really a function on $K \backslash G$, which is like a Lobachevsky plane. One can check the function is convex. Exercise: the condition that this is a minimum at x is exactly the condition $\mu_\mathbb{R}(x) = 0$, where $\mu_\mathbb{R}$ is the real moment map for the action on the total space of L^{-1}. Hence the condition for the moment map on M is

$$\mu_\mathbb{R}^M(x) = \text{constant}.$$

For the Grassmannian, the conclusion of this whole discussion is that

$$\mathrm{Gr}(k, n) = \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n) / \mathrm{GL}(k) = \mathrm{Iso}(\mathbb{C}^k, \mathbb{C}^n)/U(k).$$

The point is that many important equations in mathematical physics can be interpreted as solving the *real moment map equations* for some infinite-dimensional group. Instead of really trying to solve them, it is much better to think about the corresponding *complex stability condition*.

Suppose we've constructed M/G in the nicest possible situation, e.g. we've already thrown out non-semistable points. How do we construct $T^*(M/G)$ (because in the end we want a symplectic object)? This means we must identify cotangent vectors to spaces of G-orbits in M. Let tangent spaces to orbits be spanned by $\xi \in \mathfrak{g}$. Then cotangent vectors p at $x \in M$ must be perpendicular, i.e.

$$\langle p, \xi x \rangle = 0 \quad \forall \xi \in \mathfrak{g}.$$

So we should first take T^*M, impose this equation, and then quotient by G. This equation is just the Hamiltonian for the flow generated by ξ. This is the *complex* moment map.

The basic feature of Nakajima quiver varieties is that one can just take

$$(T^*M \cap \text{eq.}) /\!\!/_\theta G$$

and this suffices. Here $/\!\!/_\theta$ is a GIT quotient, like we discussed above. The scientific way to phrase this is that a Nakajima quiver variety is the cotangent bundle to the stack, and then we take the semistable locus inside there.

2 Nikita Nekrasov (Jul 08)

This course will be on the BPS/CFT correspondence. We will study certain integrals over moduli spaces of instantons. Today we will discuss in what sense such integrals can be viewed as correlation functions in 4d susy gauge theories (BPS side). Later we will interpret them as (conformal blocks of) correlation functions in 2d conformal field theories (CFT side).

We begin with a brief introduction to supersymmetric field theory as intersection theory. Some people may say it is a very skewed viewpoint on susy gauge theory, but in some sense it is both a restriction and generalization. Intersection theory roughly counts solutions to equations/conditions. It is intersection theory because we can interpret the equations as defining some subvarieties in an ambient variety. If we impose enough equations they will have isolated intersections, and so in the end we are counting intersection points. If the ambient variety is compact and we can represent the varieties as cohomological cycles, we can compute in cohomology and the answer is automatically stable under small deformations of the cycles. However in practice, quite often the ambient variety is *not* compact. So we first compactify by adding some set of "ideal" points, compute the corresponding intersection index, and subtract the undesired intersection points among the "ideal" points.

The physics approach to this problem is straightforward. Instead of counting solutions to the equations $s^a(x) = 0$, we integrate over X a smeared version of a delta function:

$$I_\epsilon \coloneqq \int_X e^{-\frac{1}{\epsilon}\|s\|^2}.$$

When ϵ is small, for any reasonable measure the integral will be supported near the zero locus of the equations. So the integral knows about the intersection points. It must also know a way to compactify the space, dictated by the convergence of the integral.

The supersymmetric version of the theory is one where the integral is set up in such a way that it is *independent* of ϵ, i.e.

$$\frac{d}{d\epsilon} I_\epsilon = 0.$$

Then one can potentially benefit from investigating the *other* limit $\epsilon \to \infty$, where the integral is smeared all over the space X and may be easier to work with.

There is an additional complication to the story where sometimes we may have a group action G. What we are really counting then are not points, but rather orbits. Then these integrals are defined using *equivariant* cohomology. There is the additional benefit that then we can compute many of these integrals using localization on the ambient space X.

To make this discussion more concrete, let's introduce coordinates. The slogan of this part of the story is sometimes called the "holy trinity": fields, equations and symmetries. Fields is what we are given, initially. The equations are conditions on sections s of some vector bundle $E \to X$. Symmetries are a group action of G on both X and E, and we are interested in counting solutions to $s = 0$ modulo the action of G. The output is a moduli space

$$M \coloneqq s^{-1}(0)/G,$$

and some integrals

$$\int_M \Omega.$$

From this definition M need not be compact, and almost never is. Since we want to integrate on it, we compactify it to get $M \hookrightarrow \overline{M}$. Then we go back and ask whether the compactification $\overline{M}$ can be interpreted as solutions to some enhanced equations, or some enhanced space. In other words, whether $\overline{M}$ is itself a moduli space is sometimes an interesting question.

Start with usual de Rham cohomology $\Omega^\bullet(X)$. Here X is the space of fields. For example, on a 4d theory, X is typically the space of functions on a 4-fold, or the space of sections on some bundle over a 4-fold. Introduce local coordinates x^m on X. Forms can be written as

$$\omega_{i_1 \cdots i_k}(x^m) dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

It is better to think of these not as differential forms, but as functions on the super-manifold version of TX. We write this as

$$\Omega^\bullet(X) = \mathsf{Fun}(\Pi TX).$$

So we write forms as

$$\omega_{i_1 \cdots i_k}(x) \psi^{i_1} \cdots \psi^{i_k}$$

and view x as *bosonic* coordinates and ψ as *fermionic* coordinates. The usual de Rham differential becomes a vector field on ΠTX, given by

$$d = \psi^m \frac{\partial}{\partial x^m}.$$

Note that $d^2 = 0$ precisely because of the anticommutation $\psi^m \psi^n = -\psi^n \psi^m$. In conclusion, the space of fields becomes not X, but rather ΠTX.

Now incorporate the action of the symmetry group. What we will do is called the *Cartan model* of equivariant cohomology. For a compact Lie group G, there is a space EG, which is a contractible topological space with free G-action. Its quotient by the G-action is the **classifying space** BG. It classifies principal G-bundles in the sense that given any space Y and principal G-bundle $P \to Y$, there is a map $Y \to BG$ such that P is induced by the universal bundle $EG \to BG$. For example,

$$BU(1) = \mathbb{CP}^\infty, \quad EU(1) = S^\infty \coloneqq \{(z_1, z_2, \ldots, 0, 0, \ldots) : \sum_i |z_i|^2 = 1\}.$$

The action of $U(1)$ on S^∞ is by multiplying the z_i by a phase, and this action clearly has no fixed points. If we imagine S^∞ as an S^2 glued to an S^3, glued to an S^4, etc., then to contract it we can contract the S^2 along the S^3, along the S^4, etc. This is like Hilbert's hotel, where we can always push guests into the next room.

Given any space X with a G-action, the quotient X/G may be bad even set-theoretically. For example, the G-orbits on X may be of different dimension. Instead, we can replace X with $X \times EG$. Then the G-action is free, but topologically we haven't changed anything. Hence we should replace

$$X/G \rightsquigarrow (X \times EG)/G$$

and study the cohomology of the latter space instead. This is the topological definition of **equivariant cohomology** $H_G^*(X)$ of X.

For practical purposes this is not a very good definition, because, as we saw, even for $U(1)$ the space BG is infinite-dimensional. Luckily we don't need differential forms on BG or EG; there is some kind of minimal construction which effectively replaces differential forms on EG by the so-called **Weyl complex**. It is similar to functions on ΠTX, but now X is the Lie algebra $\mathfrak{g}$ of the group G. We can further simplify our lives by observing that if G acts freely on X, then forms $\Omega^\bullet(X/G)$ can be pulled back to $\Omega^\bullet(X)$ and sit inside there as the subspace of **basic** forms:

$$\begin{aligned}\Omega^\bullet_{G\text{-basic}}(X) &\coloneqq \{\omega \in \Omega^\bullet(X) : \iota_V\omega = 0\ \forall V \in \mathsf{Vect}(X) \text{ generates } G\text{-action}\} \\ &= \bigcap_{V\in\mathfrak{g}\subset\mathsf{Vect}(X)} \ker(\iota_V) \cap \ker(\mathrm{Lie}_V).\end{aligned}$$

The operation of contraction is given in coordinates by

$$\iota_V \coloneqq V^m(x)\frac{\partial}{\partial\psi^m},$$

and is an odd operation with $\iota_V^2 = 0$. The Lie derivative is

$$\mathrm{Lie}_V \coloneqq \{d, \iota_V\} = V^m\frac{\partial}{\partial x^m} + \partial_n V^m \psi^n \frac{\partial}{\partial\psi^m}.$$

If $\omega \in \Omega^\bullet(X)$ is such that $\iota_V\omega = \mathrm{Lie}_V\,\omega = 0$, then there exists a form $\widetilde{\omega} \in \Omega^\bullet(X/G)$ such that $\omega = \pi^*\widetilde{\omega}$, where $\pi\colon X \to X/G$ is the projection.

So if we want to integrate over the quotient X/G, it is the same to integrate basic forms over X. However note that basic forms cannot be forms of top degree, because they are annihilated by lots of vector fields. We need to supply additional forms μ:

$$\int_{X/G}\widetilde{\omega} = \int_X \omega\wedge\mu.$$

The form μ should be a "volume form along G-orbits", and is just there to eat up the orbit of the group. We will give a construction of such a form using auxiliary variables, which will enhance our space of fields.

But first let's deal with the problem that the quotient X/G usually does not exist. We typically do not have a space of basic forms on X, but we can discuss basic forms on $X \times EG$. We will not go through the construction because we will not need much of it; it suffices to say that basic forms on $X \times EG$ can be modeled on

$$\left(\Omega^\bullet(X)\otimes\mathsf{Fun}(\mathfrak{g})\right)^G.$$

Equivalently, one can think of this as G-equivariant functions

$$\mathsf{Fun}_G\left(\mathfrak{g}\to\Omega^\bullet(X)\right) \coloneqq \{\alpha : \alpha(\sigma)\in\Omega^\bullet(X),\ \alpha(g^{-1}\sigma g) = g^*\alpha(\sigma)\}.$$

We call this the space of G-equivariant differential forms on X.

A grading on usual differential forms is the same as a $U(1)$-action on ΠTX. This action is generated by another vector field

$$U \coloneqq \psi^m\frac{\partial}{\partial\psi^m}.$$

Now on the space of G-equivariant differential forms on X, the grading is by

$$U \coloneqq \psi\frac{\partial}{\partial\psi} + 2\sigma\frac{\partial}{\partial\sigma}.$$

Hence our space of fields has been enhanced to $\Pi TX \times \mathfrak{g}$. It carries an odd vector field

$$Q \coloneqq d_X + \iota_{V(\sigma)}$$

where $V\colon \mathfrak{g} \to \mathsf{Vect}(X)$ is a homomorphism. In components, this is

$$Q = \psi^m \frac{\partial}{\partial x^m} + V^m(\sigma)\frac{\partial}{\partial \psi^m}.$$

One can check $Q^2 = 0$. This Q is part of the physical supersymmetry, called a **supercharge** or **scalar supersymmetry** or **topological supersymmetry**. For example, in a textbook on 4d $N = 2$ susy gauge theories, there will be many supercharges Q, and one of them will have this form.

What is the form μ? It turns out it is beneficial to multiply $\Pi TX \times \mathfrak{g}$ by $\Pi T\mathfrak{g}$. This extra factor makes it so that we are able to integrate. Typically the notation for coordinates on that space are $\overline{\sigma}$, an even coordinate on $\mathfrak{g}$, and η, an odd coordinate. In some sense we should think

$$\mathsf{Fun}(\mathfrak{g} \times \Pi T\mathfrak{g}) = \Omega^{0,\bullet}(\mathfrak{g}_{\mathbb{C}}).$$

For analysis this is sometimes a good perspective, but for other purposes we should view $\sigma, \overline{\sigma}$ as independent real variables. We extend Q by

$$\begin{aligned} Q &\coloneqq d_X + \iota_{V(\sigma)} + \overline{\partial}_\sigma + \iota_{[\sigma,\overline{\sigma}]\frac{\partial}{\partial\overline{\sigma}}} \\ &= \psi\frac{\partial}{\partial x} + V(\sigma)\frac{\partial}{\partial \psi} + \eta\frac{\partial}{\partial \overline{\sigma}} + [\sigma,\overline{\sigma}]\frac{\partial}{\partial \eta}. \end{aligned}$$

If we explicitly write coordinates $\sigma^A, \overline{\sigma}^A$, and η^A, then the bracket $[\sigma, \overline{\sigma}]$ is

$$\eta^A \frac{\partial}{\partial \overline{\sigma}^A} + f^C_{AB}\sigma^A\overline{\sigma}^B\frac{\partial}{\partial \eta^C}.$$

If the G-action on X is free, there exists a connection form $\Theta \in \Omega^1(X) \otimes \mathfrak{g}$ such that

$$\iota_{V(\sigma)}\Theta = \sigma, \quad \mathrm{Lie}_{V(\sigma)}\Theta = 0.$$

Let $\langle -, - \rangle$ be an invariant bilinear form on $\mathfrak{g}$. Using this form, define

$$\mu \coloneqq \exp\left(i\left(\langle \eta, \Theta\rangle + \langle \overline{\sigma}, \sigma + F\rangle\right)\right)$$

where $F \coloneqq d\Theta + \Theta^2$ is the curvature. One checks that $Q\mu = 0$. The point of this form is as follows. Consider

$$\int_{\Pi T(X\times\mathfrak{g})} \omega \wedge \mu = \int [dx\, d\psi\, d\overline{\sigma}\, d\eta]\omega\mu.$$

Let's assume ω has no η's. The only way to saturate the η's in the integral is to bring down as many Θ's as possible. So the part which takes care of the orbits of G in X comes precisely from a top-degree product of Θ, which thanks to the condition $\mathrm{Lie}_{V(\sigma)}\Theta = 0$ will basically give the volume form of G. Hence

$$\int_{\Pi T(X\times\mathfrak{g})} [dx\, d\psi\, d\overline{\sigma}\, d\eta]\omega\mu \approx \int_{X/G}\widetilde{\omega}. \tag{1}$$

In general, to pass from an integral on ΠTZ to an integral on Z, we have

$$\int_{\Pi TZ} f \,\mathrm{Berezin}\, dz\, d\psi = \int_Z f.$$

On the lhs f is viewed as a function, and on the rhs f is viewed as a differential form.

An interesting feature of this construction is that even if the G-action is *not* free, we can still use it after relaxing the condition $\iota_{V(\sigma)}\Theta = \sigma$. The relaxation is the condition that it is equal to σ *almost* everywhere but not everywhere, i.e. up to a non-degenerate pairing. For example, suppose G has a G-invariant metric g. Then take

$$\Theta \coloneqq g(V(-), -),$$

where we are identifying $\mathfrak{g}$ with $\mathsf{Vect}(X)$.

In general, we need to take the result of (1) and integrate further over σ. This will produce something like

$$\int \frac{d\sigma}{\operatorname{vol} G} \int_{X \times \mathfrak{g}} \omega(\sigma) \wedge \mu = \int_{`X/G'} \frac{\omega(\sigma)\, d\sigma}{\sigma + F} = \int_{`X/G'} \widetilde{\omega(-F)}. \tag{2}$$

Here "X/G" is in quotes because it need not exist. If it exists, we can remove the quotation marks and the equality is literally true.

Note that $Q^2 = 0$. Elements of the cohomology of Q are G-invariant polynomials $P \in (S^\bullet \mathfrak{g}^\vee)^G$, i.e. $P(\sigma) = P(g^{-1}\sigma g)$. If we plug in P for ω in (2), the result is

$$\int_{`X/G'} P(-F),$$

which is some characteristic class.

Now we need to discuss equations. Our setting will be gauge theories on 4-manifolds, e.g. $M^4 = \mathbb{R}^4$ and its various compactifications

$$S^4 = \mathbb{R}^4 \cup \{\infty\}, \quad \mathbb{CP}^2 = \mathbb{C}^2 \cup \mathbb{CP}^1.$$

We will also consider quotients $\mathbb{R}^4/\Gamma$ by discrete subgroups $\Gamma \subset \mathrm{SU}(2)$.

1. **A-type**: take $\Gamma = \mathbb{Z}/\ell$ with action
$$(z_1, z_2) \mapsto (e^{\frac{2\pi i}{\ell}} z_1, e^{-\frac{2\pi i}{\ell}} z_2).$$
2. **D-type**: take $\Gamma = \mathbb{Z}/\ell \rtimes \mathbb{Z}/2$ with action of the $\mathbb{Z}/2$ as
$$(z_1, z_2) \mapsto (-z_2, z_1).$$
3. **E-type**: something.

These quotients can be desingularized to give $\widetilde{\mathbb{R}^4/\Gamma}$. Such desingularizations can be described using quivers, cf. Andrei's lectures. The McKay correspondence tells us these quivers will be ADE-type Dynkin diagrams.

We are not interested in these spaces themselves, but rather in solutions of PDEs defined *over* these spaces. Fix a principal G-bundle $P \to M^4$. The space X which we discussed so far will be

$$X \coloneqq \mathcal{A}_P \coloneqq \{G\text{-connections on } P\}.$$

The gauge group $\mathcal{G}$ will be the group of gauge transformations

$$\mathcal{G} \coloneqq \Gamma_{C^\infty}(M^4, P \times_{\mathrm{Ad}} G)\Big|_{\substack{\to 1 \\ \text{at } \infty}}$$

which tend to 1 at ∞. These are called **framed** gauge transformations. Locally, these gauge transformations are

$$A \mapsto g^{-1} A g + g^{-1} dg, \quad g(x) \to 1 \text{ as } x \to \infty.$$

The equations we are interested in are

$$F_A = - \star F_A$$

where $\star\colon \Omega^2(M^4) \to \Omega^2(M^4)$ is the Hodge star. It satisfies $\star^2 = 1$ and therefore has a $+1$ (self-dual) and a -1 (anti-self-dual) eigenspace; we want the -1 eigenspace. The moduli space we want to work with is

$$\mathcal{M}_P \coloneqq \{A : A \in \mathcal{A}_P,\ F_A^+ = 0\}/\mathcal{G}.$$

This is a very abstract space and is hard to work with unless we have a model for construction. Fortunately we do have one: it is given by a quiver recipe when G is unitary. So we will focus on this case.

Our notation will differ slightly from Andrei's. The gauge group will be $G = U(n)$ for $n \geq 2$. Then G-bundles on S^4 are classified topologically by $k \in \mathbb{Z}$ as follows. View S^4 as two disks $D^4_+ \sqcup_{S^3} D^4_-$ glued along the equator, and trivialize

$$P|_{D_+} \cong G \times D_+, \quad P|_{D_-} \cong G \times D_-.$$

Then the remaining data is an isomorphism

$$P|_{D_+} \xrightarrow{\sim} P|_{D_-},$$

given by a clutching function $\rho: S^3 \to G$. Such maps are classified by $[S^3, G] = \pi_3(G) = \mathbb{Z}$.

Note that tr is a negative-definite form for $\mathfrak{g}$. Hence we have

$$-\int_{S^4} \operatorname{tr} F_A \wedge \star F_A \geq 0.$$

By anti-self-duality, the lhs is

$$-\int_{S^4} \operatorname{tr} F_A \wedge \star F_A = \int_{S^4} \operatorname{tr} F_A \wedge F_A = 8\pi^2 k,$$

and therefore we only have anti-self-dual instantons for $k \geq 0$. We can now categorify these two numbers n and k, to get

$$N := \mathbb{C}^n, \quad K := \mathbb{C}^k,$$

using which we will construct a moduli space $\mathcal{M}(k, n)$.

3 Saebyeok Jeong (Jul 08)

We will discuss a connection between 4d $N = 2$ theory and some symplectic geometry on the moduli of flat connections, first conjectured by Nekrasov–Rosly–Shatashvili.

Take a 4d $N = 2$ theory of class S. Such theories arise from 6d $N = (2, 0)$ theories, for which there is an ADE classification. Restrict to A_{n-1}; for simplicity, take A_1. Compactified on a Riemann surface (possibly with some punctures) and twisting, we get a 4d $N = 2$ theory. We can further compactify on S^1 and look at the Coulomb branch. It is an integrable system, and in fact is a Hitchin moduli $\mathcal{M}_H$. Recall that

$$\mathcal{M}_H(\mathrm{SU}(N), \mathcal{C}) := \left\{ \begin{array}{c} F_A + [\varphi, \overline{\varphi}] = 0 \\ \overline{D}_A \varphi = D_A \overline{\varphi} = 0 \end{array} \right\} / \,\mathrm{SU}(N).$$

It is known that this space is hyperkähler. The main example is $\mathcal{C} = \mathbb{P}^1 - \{0, 1, \infty, q\}$. In this case, the resulting theory is SU(2) gauge theory with 4 fundamental hypermultiplets.

The partition function $Z(a, m, \epsilon_1, \epsilon_2, q)$ we want is an equivariant integral over this moduli space. For example, when $\epsilon_1, \epsilon_2 \to 0$, the asymptotic behavior is

$$\exp\left(\frac{F(a, m, q)}{\epsilon_1 \epsilon_2} + \cdots\right)$$

where $F(a, m, q)$ is the Seiberg–Witten prepotental. This is *not* the limit we will consider; instead, we'll do just $\epsilon_2 \to 0$. This leaves us with a 2d $N = (2, 2)$ theory. A *twisted superpotential* $\widetilde{W}$ governs the dynamics of this theory.

There is a Hitchin map $\mathcal{M}_H \to \bigoplus_{k=1}^N H^*(\mathcal{C}, K_{\mathcal{C}}^k)$ given by mapping to the coefficients of the characteristic polynomial.

We will think about complex Lagrangian submanifolds inside $\mathcal{M}$. Let $\mathcal{O}$ be the *space of opers*, defined as a space of differential operators

$$\hat{D} := \partial_z^N + t_1(z) \partial_z^{N-2} + \cdots + t_N(z),$$

where we allow the coefficients $t_i(z)$ to be meromorphic of up to order N at punctures. To describe $\mathcal{O}$, choose Darboux coordinates. This means we can express $\Omega_J \coloneqq \sum d\alpha_i \wedge d\beta_i$. Since $\mathcal{O}$ is complex Lagrangian,

$$\Omega_J|_{\mathcal{O}} = 0.$$

This means there exists a local function $S(\alpha)$ such that

$$\beta_i = \frac{\partial S}{\partial \alpha_i}.$$

Conjecture 3.1 (Nekrasov–Rosly–Shatashvili)**.** *There exists a certain Darboux coordinate system where*

$$\widetilde{W} = S.$$

The lhs is a purely gauge theoretic quantity governing low-energy dynamics, but the rhs is a purely geometric object.

Example 3.2. Take $\mathcal{C} = \mathbb{P}^1 - \{0, 1, \infty\}$. Then

$$\mathcal{M}_{\text{flat}} = \{(g_i) \in \text{SL}(2) : g_1 g_2 g_3 = 1, \, \text{tr}\, g_i = \text{fixed}\} / \text{SL}(2).$$

A dimension count gives $\dim \mathcal{M} = 0$. There is an oper representation of this element, given by

$$\partial^2 + \frac{\delta_\infty}{z} + \frac{\delta_0}{(z-1)^2} + \frac{\delta_\infty - \delta_0 - \delta_1}{z(z-1)},$$

called the **hypergeometric oper**. The partition function of the resulting 2d $N = (2, 2)$ theory satisfies this differential operator.

Example 3.3. Take $\mathcal{C} = \mathbb{P}^1 - \{0, 1, q, \infty\}$. Then

$$\mathcal{M}_{\text{flat}} = \{(g_i) \in \text{SL}() : g_1 g_2 g_3 g_4 = 1, \, \text{tr}\, g_i = \text{fixed}\} / \text{SL}(2).$$

Here the complex dimension is 2. We want a coordinate system on here. Define

$$\begin{aligned} A &= \text{tr}\, g_q g_0 \\ B &= \text{tr}\, g_0 g_\infty \\ C &= \text{tr}\, g_q g_\infty. \end{aligned}$$

There is a relation

$$0 = G(A, B, C) \coloneqq A^2 + B^2 + C^2 - 4 + \cdots.$$

In particular this space is some hypersurface in $\mathbb{C}^3$. The Poisson structure induced by Ω_J is given by

$$\{A, B\} = \frac{\partial G}{\partial C}.$$

The result is very similar to the hypergeometric case

When we write a perturbative expansion of $Z(a, m, \epsilon_1, \epsilon_2, q)$, we have chosen a particular region of convergence. How do we analytically continue to other punctures? It turns out Z can be written as a contour integral with auxiliary variable x. There is one row of poles distributed horizontally, going toward $+\infty$. There are in addition n rows of poles going toward $-\infty$. Our contour separates these kinds of poles. When $|z| > 1$, we can choose the contour on the right encircling *all* the poles. When $|z| < 1$, we can choose the contour on the left encircling all n rows of left-ward poles.

4 Pavel Etingof (Jul 08)

How do we construct quantizations of ordinary finite simple Lie algebras? We have the powerful notions of Drinfeld double and reconstruction, which not only constructs the quantization but also the universal R-matrix.

Let's begin with $\mathfrak{sl}_2$, where we saw before Drinfeld's mysterious formula for the universal R-matrix.

Let $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ be a simple Lie algebra. For $\mathfrak{sl}_2$, these are spanned by e, h, f respectively. Let

$$\mathfrak{b}_\pm \coloneqq \mathfrak{n}_\pm \oplus \mathfrak{h}$$

be the positive/negative Borel. Define

$$U_q(\mathfrak{b}_+) \coloneqq \langle K, e \rangle, \quad KeK^{-1} = q^2 e$$

where K is a group-like element, i.e.

$$\Delta K = K \otimes K,$$

and e is a skew primitive element, i.e.

$$\Delta e = e \otimes K + 1 \otimes e.$$

Such elements are important, because group-like elements correspond to 1-dimensional comodules and skew primitive elements represent Ext^1 between such comodules.

Say $q = \sqrt[\ell]{1}$. In this case we have a Hopf ideal

$$I \coloneqq \langle K^\ell - 1, e^\ell \rangle$$

which means $\Delta I \subset I \otimes H + H \otimes I$. Then we can form the quotient

$$u_q(\mathfrak{b}_+) \coloneqq H/I,$$

which is a *small* quantum group. This appeared in a problem set and is called the **Taft algebra**, with basis $K^i e^j$ for $0 \le i, j \le \ell - 1$. This is the simplest Hopf algebra which is not co-commutative. Its quantum double can be computed as

$$D(u_q(\mathfrak{b}_+)) = u_q(\mathfrak{b}_+) \oplus u_q(\mathfrak{b}_+)^*.$$

The fact is that the Hopf algebra $u_q(\mathfrak{b}_+)$ is *self-dual*, i.e. invariant under interchanging product and co-product. There are very few such Hopf algebras. We will write instead

$$D(u_q(\mathfrak{b}_+)) = u_q(\mathfrak{b}_+) \oplus u_q(\mathfrak{b}_-).$$

The Hopf algebra $u_q(\mathfrak{b}_+)$ is generated by K and f, with

$$KfK^{-1} = q^{-2} f, \quad \Delta K = K \otimes K, \quad \Delta f = f \otimes 1 + K^{-1} \otimes f.$$

Note now that we have two K's, which we will call K_+ and K_-. Then one checks that $c \coloneqq K_+ K_-^{-1}$ is central.

The object $D(u_q(\mathfrak{b}_+))$ is too big: it has dimension ℓ^4, whereas we want something of dimension ℓ^2. So we should kill something. The thing to kill is exactly this element c. The algebra

$$D(u_q(\mathfrak{b}_+))/\langle c = 1 \rangle$$

then has commutator

$$[e, f] = \frac{K - K^{-1}}{q - q^{-1}},$$

and hence we have verified that

$$D(u_q(\mathfrak{b}_+))/\langle c = 1 \rangle = U_q(\mathfrak{g}).$$

So we do not have to guess this relation; it arises for free. It also automatically equips $U_q(\mathfrak{g})$ with an R-matrix. The R-matrix for the Drinfeld double is supposed to be

$$R = \sum_i a_i \otimes a_i^*$$

for bases $\{a_i\}$ of H and $\{a_i^*\}$ of H^{op}. To compute this pairing we need to know the pairing between $u_q(\mathfrak{b}_+)$ and $u_q(\mathfrak{b}_-)$. This is computed as

$$(K^i, K^j) = q^{2ij}, \quad (e, f) = \frac{1}{q - q^{-1}}.$$

This is a Hopf pairing, so all other pairings arise from these two. The result is

$$R = q^{\frac{h \otimes h}{2}} \left(\sum_{n=0}^{\ell-1} \frac{(q - q^{-1})^n}{[n]_q!} q^{\frac{n(n-1)}{2}} (e^n \otimes f^n) \right).$$

This is exactly the Drinfeld formula. So we do not need to manually guess it and check it satisfies Yang–Baxter.

We can do the same for generic q, with completions. In this case we will not have relations $K^\ell = 1$ and $e^\ell = 0$, and the algebra will become infinite-dimensional. The only difference for the R-matrix is that the finite sum becomes an infinite one:

$$\sum_{n=0}^{\ell-1} \quad \rightsquigarrow \quad \sum_{n=0}^{\infty}.$$

Hence the category of *all* reps will not be a braided tensor category, due to convergence issues. However we don't care about arbitrary reps. Usually we want only finite reps, or category $\mathcal{O}$. So as long as V, W are locally nilpotent under e, then $R|_{V \otimes W}$ makes sense and such reps form a braided tensor category.

For general $\mathfrak{g}$, we start with a Cartan matrix $A = (a_{ij})$. There must be numbers d_i such that

$$d_i a_{ij} = d_j a_{ji},$$

and we let $q_i \coloneqq q^{d_i}$. Simple Lie algebras are generated by $\mathfrak{sl}_2$ with relations. Similarly, quantum groups are generated by $U_q(\mathfrak{sl}_2)$ with relations. We first define

$$\widetilde{U_q(\mathfrak{b}_+)} \coloneqq U_q(\mathfrak{sl}_2^+) \times \cdots \times U_q(\mathfrak{sl}_2^+).$$

In other words, it is generated by K_i, e_i, with $\mathfrak{sl}_2$ relations and

$$K_i K_j = K_j K_i, \quad K_i e_j K_i^{-1} = q_i^{a_{ij}} e_j.$$

Then we define the actual algebra

$$U_q(\mathfrak{b}_+) \coloneqq \widetilde{U_q(\mathfrak{b}_+)}/I$$

where I is the largest Hopf ideal not containing e_i for all i. A theorem of Gabber and Kac says that when $q = 1$ this kills exactly the Serre relations.

Theorem 4.1 (Lusztig)**.** *The Hopf ideal I is generated by q-Serre relations*

$$\sum_{k=0}^{1-a_{ij}} \binom{1 - a_{ij}}{k}_{q_i} e_i^k e_j e_i^{1-a_{ij}-k} = 0.$$

Now we take the double

$$D(U_q(\mathfrak{b}_+)) = U_q(\mathfrak{b}_+) \otimes U_q(\mathfrak{b}_-), \quad (e_i, f_j) = \frac{1}{q_i - q_i^{-1}} \delta_{ij}.$$

Using this Drinfeld inner product, we can identify the ± algebras. The central element is still

$$c_i := K_i^+(K_i^-)^{-1},$$

and we should take

$$U_q(\mathfrak{g}) := D(U_q(\mathfrak{b}_+))/\langle c_i = 1\rangle.$$

In this generality we still have an R-matrix $R = \sum a_i \otimes a_i^*$, but now there is no simple basis so a priori this sum is not so easy. There is a PBW basis, however, the formula is nice, due to Koroshkin–Tolstoy.

In physics we often do perturbation theory, and so we do this in quantum groups as well. We take $q = e^{\hbar/2}$ over $\mathbb{C}[[\hbar]]$ and define the algebras in the same way. This results in the quantum universal enveloping algebra (in the sense of Drinfeld), which is a formal deformation of $U\mathfrak{g}$. This just means we get a Hopf algebra A over $\mathbb{C}[[\hbar]]$ in the topological sense, such that $A/\hbar A = U\mathfrak{g}$. The coproduct in $U\mathfrak{g}$ is co-commutative, so

$$\Delta(x) - \Delta^{\text{op}}(x) = O(\hbar),$$

and we can look at

$$\delta(x) := \lim_{\hbar\to 0} \frac{\Delta(x) - \Delta^{\text{op}}(x)}{\hbar}.$$

This is a map

$$\delta: U\mathfrak{g} \to U\mathfrak{g} \otimes U\mathfrak{g}$$

measuring the first-order failure in co-commutativity.

1. $\delta^*: \wedge^2\mathfrak{g}^* \to \mathfrak{g}^*$ is a Lie bracket.
2. δ is a derivation, i.e.

$$\delta([x,y]) = [x\otimes 1 + 1\otimes x, \delta(y)] + [\delta(x), y\otimes 1 + 1\otimes y].$$

Definition 4.2. A pair $(\mathfrak{g}, \delta)$ with these properties is called a **Lie bi-algebra**. This naming is because both $\mathfrak{g}$ and $\mathfrak{g}^*$ have Lie brackets.

This particular Lie bi-algebra we constructed is called the **quasi-classical limit** of the Hopf algebra A. Exercise: for $U_q(\mathfrak{sl}_2)$, compute that

$$\delta(e) = e\wedge h, \quad \delta(f) = f\wedge h, \quad \delta(h) = 0.$$

Also, A is called a **quantization** of $(\mathfrak{g},\delta)$. While quasi-classical limit is an easy procedure akin to taking the derivative, quantization is a much harder procedure akin to integration. In particular, given a first-order prescription, one has to construct all higher-order structure. It is not even obvious that such higher-order structure exists, and ordinary deformation theory does not give an answer.

Theorem 4.3 (Etingof–Kazhdan)**.** *Any Lie bi-algebra has a quantization.*

Let G be a simply-connected algebraic group. Then

$$U(\mathfrak{g})^*_G = \mathcal{O}(G).$$

Then $\delta^*: \mathcal{O}(G)\otimes\mathcal{O}(G) \to \mathcal{O}(G)$ yields a Poisson bracket

$$\delta^*(f,g) = \{f,g\}$$

such that multiplication $m: G\times G \to G$ is a Poisson map. Such a pair $(G, \{-,-\})$ is called a **Poisson algebraic group**. In a similar way one can define a Poisson Lie group and a Poisson formal group.

Corollary 4.4. *Any Poisson group (in any of these three settings) can be quantized.*

Let A be a quantum universal enveloping algebra (Drinfeld) over $\mathbb{C}[[\hbar]]$, and suppose A has an R-matrix. Recall that this means

$$R\Delta = \Delta^{\mathrm{op}} R, \quad (\Delta \otimes 1)(R) = R^{13}R^{23}, \quad (1 \otimes \Delta)(R) = R^{23}R^{12}. \tag{3}$$

Then we can take the first derivative

$$R = 1 + \hbar r + O(\hbar^2).$$

This is a bad way of writing it; it is only good when we trivialize the deformation. A better way is to say

$$r = \lim_{\hbar \to 0} \frac{R-1}{\hbar} \in \mathfrak{g} \otimes \mathfrak{g}.$$

It satisfies some properties, coming from taking linear parts of (3). In particular,

$$\delta(x) = [x \otimes 1 + 1 \otimes x, r].$$

The Yang–Baxter equation is tautological in linear order, but at $\hbar^2$ order we get

$$r_{12}r_{23} + r_{12}r_{13} + r_{13}r_{23} = 0.$$

This is called the **classical Yang–Baxter equation**.

Definition 4.5. A **quasi-triangular structure** on a Lie algebra $\mathfrak{g}$ is an element

$$r \in \mathfrak{g} \otimes \mathfrak{g}$$

such that the **Casimir tensor**

$$r + r^{21} =: t$$

is $\mathfrak{g}$-invariant, and r satisfies the classical Yang–Baxter equation.

Theorem 4.6 (Etingof–Kazhdan)**.** *Any quasi-triangular Lie bialgebra can be quantized and the quantization, as an algebra, is isomorphic to the undeformed algebra tensor* $\mathbb{C}[[\hbar]]$.

In particular, $U_q(\mathfrak{g}) \cong U\mathfrak{g}[[\hbar]]$ as algebras! The difference is in the coproduct. Actually this is not very hard to prove by classical deformation theory, since $H^2(U\mathfrak{g}, U\mathfrak{g}) = 0$ by Weyl–Kac theorem. But to explicitly construct the isomorphism is very hard; only recently was it done for $\mathfrak{sl}_2$.

If r is skew-symmetric, i.e. $t = 0$, then we get a *triangular* structure instead of just a quasi-triangular one. The reason this is nice is that the classical notion of Drinfeld double, i.e. a classical double, is related to Manin triples, and somehow this gives a way to construct many Hopf algebras.

5 Andrei Okounkov (Jul 09)

In geometry we have manifolds X and maps $X \to Y$. In physics we typically have vector spaces, operators, formulas, and variables. If we have a group G acting on a manifold X, then there are continuous parameters for the action, which are variables. Variables can be separated into *equivariant* ones and *topological* ones. When Nikita was talking about instantons, which are very special connections on $\mathbb{R}^4$, we saw the topological charge which is a topological variable, and we also saw an $\mathrm{SO}(4)$ action on $\mathbb{R}^4$ which provides equivariant variables.

What are the spaces? We should work within algebraic geometry whenever possible, because it is very suitable for analysis of *singular* spaces. The main object we want to associate to a space X with a G-action is the equivariant K-group $K_G(X)$. An element in $K_G(X)$ is a *virtual* equivariant vector bundle. Equivariance means we can lift the G-action to the total space of the vector bundle, such that in fibers it acts by linear operators. One can imagine that X is a moduli of vacua, and the vector bundle over it has fibers all vacua for that particular background. Given two vector bundles, there are many operations on them: $\oplus$, $\otimes$, $\wedge^2$, S^2, etc. This makes the K-group a ring with additional structure, called a Λ-ring. Equivariant K-theory should be thought of as a generalization of the representation ring of G. This is because

$$K_G(\mathrm{pt}) = \mathsf{Rep}(G).$$

There are two ways to define $K_G(X)$: topological and algebraic. In one case we consider *complex* vector bundles, and in the other we consider *algebraic* vector bundles. Within algebraic K-theory we can consider either the K-group of vector bundles, or the K-group of coherent sheaves. These are the same if the variety is smooth. In fact, for instanton moduli spaces, all these notions coincide.

What is a coherent sheaf? It is a "vector bundle" whose fibers are allowed to "jump". Take $X = \mathbb{P}^n$. This is the moduli of lines in $\mathbb{C}^{n+1}$. On it is a tautological bundle $\mathcal{O}(-1)$, whose fiber over a point $[\ell] in \mathbb{P}^n$ is the line ℓ itself. It is called $\mathcal{O}(-1)$ because it has no sections. Its dual $\mathcal{O}(1)$ has sections, because it is the line bundle of linear functions. The coordinates

$$[x_0 : \cdots : x_n] \in \mathbb{P}^n$$

form $n+1$ sections of $\mathcal{O}(1)$. This produces a map

$$\mathcal{O}^{\oplus n} \xrightarrow{x_1,\ldots,x_n} \mathcal{O}(1).$$

It is surjective everywhere except at $[1 : 0 : \cdots : 0]$. Hence the cokernel has zero fibers everywhere except at that point, where it has a 1-dimensional fiber. This is "jumping" behavior.

In algebraic K-theory, if we have three vector bundles forming a short exact sequence

$$0 \to V_1 \to V \to V_2 \to 0,$$

we impose the relation

$$[V] = [V_1] \oplus [V_2].$$

So in particular if we have a resolution, e.g. of the coherent sheaf we discussed above, then we get an alternating sum in algebraic K-theory. In fact by Hilbert's syzygy theorem, on a smooth algebraic variety of dimension d, any resolution terminates after at most d terms.

If we work with algebraic K-theory of vector bundles, then we have to be careful about tensor multiplication. On a smooth variety we can resolve coherent sheaves into vector bundles, and therefore the tensor product of a coherent sheaf with a vector bundle is well-defined.

Theorem 5.1 (McGerty–Nevins)**.** *For Nakajima quiver varieties X, the equivariant K-theory $K_G(X)$ is generated (in the sense of Λ-rings) by tautological bundles.*

Now we discuss some properties of algebraic K-theory. Suppose $\widetilde{X} \to X$ is a G-bundle. If $G \times H$ acts on $\widetilde{X}$, then H still acts on X. For example,

$$K_{\mathrm{PGL}(n)}(\mathbb{P}^{n-1}) = K_{\mathrm{GL}(n)}(\mathbb{C}^n - 0).$$

Operations like this are the origin of all integral formulas in this business. This is because we are effectively computing G-invariants, which is done by averaging over the group G. The Weyl integration formula says it is equivalent to average over the maximal torus $T \subset G$ with weights. The choice of contour is related to the choice of stability condition for GIT quotient.

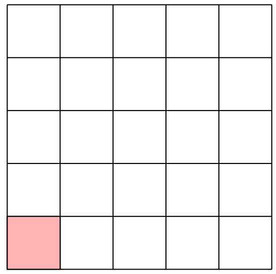
Let $Y \hookrightarrow X$ be a closed embedding. Given a coherent sheaf on Y, we can extend by zero to get a coherent sheaf on X. There is an exact sequence

$$K_G(Y) \to K_G(X) \to K_G(X \setminus Y) \to 0.$$

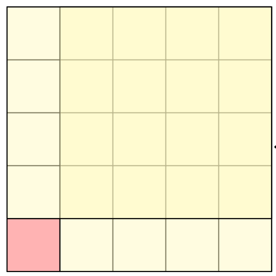
This is the analogue of excision in K-theory. There is some higher K-group K^1 that goes in the kernel, etc. For example,

$$K_{\mathrm{GL}(n)}(\mathbb{C}^n - 0) = K_{\mathrm{GL}(n)}(\mathbb{C}^n)/K_{\mathrm{GL}(n)}(0).$$

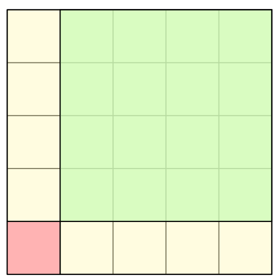
We know $K_{\mathrm{GL}(n)}(\mathbb{C}^n) = \mathsf{Rep}(\mathrm{GL}(n))$, because $\mathbb{C}^n$ is affine. The quotient kills some particular representations. What are they? For $n = 2$, we can phrase this in boxcounting language. The inclusion of 0 is the structure sheaf $\mathcal{O}_0$ of the origin $0 \in \mathbb{C}^2$, drawn as



We need a resolution of $\mathcal{O}_0$ in $\mathbb{C}^2$. Of course $\mathbb{C}[x_1, x_2] = \mathcal{O} \to \mathcal{O}_0$ is a surjection. The kernel is everything not covered by the red square:

.

This is generated by $x_1\mathcal{O}$ and $x_2\mathcal{O}$, as drawn. However now we have counted the shaded region twice.



This corresponds to a kernel $x_1x_2\mathcal{O}$. In conclusion, we have produced a resolution

$$0 \to x_1x_2\mathcal{O} \to x_1\mathcal{O} \oplus x_2\mathcal{O} \to \mathcal{O} \to \mathcal{O}_0 \to 0.$$

We see that for $\mathbb{C}^2$ this resolution stopped after two steps. It is not hard to imagine that in d dimensions such resolutions stop after d steps; this is actually the proof of Hilbert's syzygy theorem.

Now we move this sequence to $\mathbb{P}^{n-1}$. The first term $\mathcal{O}_0$ disappears, and $\mathcal{O}$ becomes $\mathcal{O}_{\mathbb{P}(V)}$. The coordinates x_1, x_2 generate the dual $V^\vee$, so we can rewrite $x_1\mathcal{O} \oplus x_2\mathcal{O}$ as $V^* \otimes \mathcal{O}_{\mathbb{P}(V)}(-1)$. Finally the x_1x_2 generates $\wedge^2 V^\vee$. In conclusion, we get

$$0 \to \wedge^2 V^\vee \otimes \mathcal{O}(-2) \to V^\vee \otimes \mathcal{O}_{\mathbb{P}(V)}(-1) \to \mathcal{O}_{\mathbb{P}(V)} \to 0.$$

If we denote $L \coloneqq \mathcal{O}(1)$, then this exact sequence says L satisfies a relation

$$1 - L^{-1}V^\vee + L^{-2} \wedge^2 V^\vee = 0$$

in K-theory. If we write the weights as

$$\begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix} \in \mathrm{GL}(n)$$

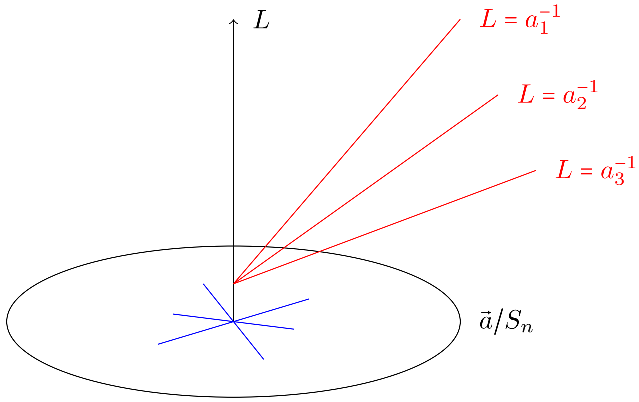
then the character of V is $a_1 + \cdots + a_n$, and $\wedge^2 V = e_2(a)$ where e_2 is the second elementary symmetric polynomial, and so on. Hence the relation can be written

$$\prod_{i=1}^{n} (1 - L^{-1} a_i^{-1}) = 0.$$

In general, there is a map $K_G(\mathrm{pt}) \to K_G(X)$. If we Spec everything, this yields

$$\operatorname{Spec} K_G(X) \to \operatorname{Spec} K_G(\mathrm{pt}) = G/\mathrm{conj}.$$

For $\mathbb{P}^{n-1}$, this means that $K_G(\mathbb{P}^{n-1})$ sits over symmetric functions in $\vec{a}$. The scheme $\operatorname{Spec} K_G(\mathbb{P}^{n-1})$ can be plotted over $\vec{a}/S_n$ as



There is something amazing about this picture. Namely we can ask: what is the fiber over a point? Of course this depends on whether some of the a_i are equal or not, i.e. dependent on some hyperplane arrangement in the base. In fact given $g \in G/\mathrm{conj}$, the fiber is exactly the *non-equivariant* K-theory

$$K(X^g) \to g.$$

For a very generic g, we will just get n copies of $K_G(\mathrm{pt})$. On the hyperplane $a_i = a_j$, we will get the relation

$$(1 - L)^2 = 0,$$

which is a relation in the *non-equivariant* K-theory of $\mathbb{P}^1$, which is the fixed locus.

Take an inclusion $K_T(X^T) \to K_T(X)$. By the excision sequence earlier, we can complete this inclusion as

$$K^1(\cdots) \to K_T(X^T) \to K_T(X) \to K_T(X - X^T) \to 0.$$

The term $K_T(X^T)$ is interpreted as what happens if we take all the branches $L = a_i^{-1}$ and take them apart, i.e. remove all relations. Then these two schemes are isomorphic except on the hyperplane arrangement. A general theorem of Thomason says $K^1(\cdots)$ and $K_T(X - X^T)$ are *torsion*! So we don't care about them so much, because in the end we want to compute a character, which is usually a polynomial or rational function. If we know such functions outside of some positive codimension subvariety, we know it everywhere. Hence if we argue modulo torsion, $K_T(X^T) \to K_T(X)$ becomes an isomorphism.

Let $V \in K_G(X)$ be an equivariant vector bundle. We'd like to find a specific representative on X^T that pushes forward to V. Let $\iota: X^T \to X$ be the inclusion. If X is smooth, there is a pullback map ι^*. We can compute something like

$$\iota_* \iota^* \mathcal{O}_Y = \wedge^\bullet N_{X/Y}^\vee$$

via Koszul resolution. Hence a representative for V, as a pushforward from the fixed locus, is

$$V = \iota_* \left(\frac{\iota^* V}{\wedge^\bullet N_{X/Y}^\vee} \right).$$

6 Nikita Nekrasov (Jul 09)

Today we add equations into the picture. In physics jargon this is called χ-H multiplet, also known as Koszul complex. Then we will discuss global vs local symmetry groups, or how to compute using symmetries. Then we'll return to the discussion of the moduli $\mathcal{M}(k,n)$ of instantons, and their compactifications, symmetries, and fixed points. From this we'll get instanton partition functions, and finally we'll talk about Y-classes, or Y-observables in gauge theory.

Let Q be the supercharge, with $Q^2 = 0$. We had $Qx = \psi$ and $Q\psi = V(\sigma)$. Let G be a group. The objects on which Q acts should be G-invariant. We also introduced objects

$$Q\overline{\sigma} = \eta, \quad Q\eta = [\sigma, \overline{\sigma}].$$

We consider functions $\Gamma(x, \psi, \sigma, \overline{\sigma}, \eta)$ which are G-invariant, so that $Q^2 = 0$ on them.

Now we want to impose constraints $s^a(x) = 0$ on the variables x. To do so, we can either try to solve the equations or we can work on a bigger space and impose these equations *homologically*. In other words, we introduce additional variables in the form of a new multiplet χ_a and H. If the equations are bosonic, χ_a will be fermionic, and vice versa. The transformation by Q extends to these new variables in the following way:

$$Q\chi_a = H_a, \quad QH_a = R(\sigma)_a^\ell \chi_\ell.$$

We want to be able to impose these equations while preserving G-invariance, so we view them as sections of a G-bundle $\mathcal{E}$. This $R(\sigma)$ is the G-action on $\mathcal{E}$. This is of course not the global formula because it assumes $\mathcal{E}$ is trivial. We should take into account the choice of some connection on $\mathcal{E}$, e.g. we should extend to

$$Q\chi_a = H_a - \mathcal{A}_{ma}^b \psi^m \chi_b$$

where $\mathcal{A}$ is some connection 1-form for $\mathcal{E}$.

Last time we discussed integrals of some form. Now we need to add in the integration over χ and H, to get

$$\int_{\mathfrak{g}} \frac{d\sigma}{\operatorname{vol} G} \int_{\Pi T(\Pi\mathcal{E}\times\mathfrak{g})} dx\, d\psi\, d\overline{\sigma}\, d\eta\, d\chi\, dH\ \omega\mu e^{iQ(\chi_a s^a(x))}.$$

Here $e^{iQ(\chi_a s^a(x))}$ is new, and imposes for us the equations, as follows. We can compute

$$\exp i\left(Q(\chi_a s^a(x))\right) = \exp i\left(H_a s^a(x) - \chi_a \psi^m \nabla_m s^a\right).$$

Integrating over H gives a delta function for s, and integrating over χ gives a fermionic delta function for $\psi^m \nabla_m s$, which constrains the tangent directions in the base to be those which are in the kernel of ∇s. In other words, the former constrains bosonic coordinates s, and the latter constrains tangent vectors to those actually tangent to s.

This would not be very useful unless there is some way to deform the integral to make it less sharp as far as the equations are concerned. This is because we don't really want to *solve* the equations; we want to integrate over the zero locus without really solving. We need to somehow *smear* the zero locus. The idea is to use the fact that ω is Q-closed and that $\int Q(\cdots) = 0$. The trick is to add a term:

$$\exp i\left(Q(\chi_a s^a(x) + itG^{ab}\chi_a H_b)\right)$$

where G^{ab} is an invariant metric on the fibers. Applying Q gives

$$\exp\left(-t\|H\|^2 - G\chi\nabla\chi + R\chi\chi\psi\psi\right).$$

Integrating out H now gives

$$\exp\left(-\frac{\|s\|^2}{2t} - G\chi\nabla\chi + R\chi\chi\psi\psi\right)$$

instead of a delta function. If we send $t \to \infty$, we effectively forget about s, and we are left with a bunch of curvature terms forming some sort of Pfaffian. This is like the Chern–Weil representation of Euler classes.

If we spell out what it means to integrate H out in the formulas, we see that it is like finding a critical point. On the critical locus, we have

$$H_a = \frac{1}{t} G_{ab} s^b.$$

Physicists call this "going on-shell". Substituting this into the differential,

$$Q\chi = s(x),$$

which yields exactly the Koszul complex.

Now let's forget about equations, and suppose we are in the simplest case where all equations are already solved. How do we use equivariance to compute things? Suppose X is a G-space, and suppose

$$\omega \in \mathsf{Fun}(\mathfrak{g} \to \Omega^\bullet(X))^G$$

is a G-equivariant form. Moreover suppose $Q\omega = 0$, i.e.

$$(d + \iota_{V(\sigma)})\omega = 0.$$

Then we can compute

$$Z(\sigma) \coloneqq \int_X \omega(\sigma),$$

called the **partition function**. Note that here we are only integrating the top-degree component. The result is a function on the Lie algebra, holomorphic in σ since $\overline{\sigma}$ doesn't enter. We can slightly generalize this and introduce $\overline{\sigma}$ dependence:

$$Z(\sigma, \overline{\sigma}, \eta) = \int_X \omega(\sigma, \overline{\sigma}, \eta)$$

where ω is still Q-closed, with

$$Q = d_X + \iota_{V(\sigma)} + \partial_{\overline{\sigma}} + \iota_{[\sigma,\overline{\sigma}]\frac{\partial}{\partial \overline{\sigma}}}.$$

What can we say about Z? Firstly, it is G-invariant:

$$Z(g^{-1}\sigma g) = Z(\sigma).$$

This is because ω itself is G-invariant. So $Z(\sigma)$ is determined by its restriction to the maximal torus $\mathfrak{t} \subset \mathfrak{g}$, where it is Weyl-invariant. So we should think of Z as some sort of character of the group G.

The great thing about these functions is not only are they W-invariant, they are computable. We can use the same trick of modifying the integrand by Q-exact insertions:

$$Z(\sigma) = \int_X \omega(\sigma) = \int_X \omega(\sigma) e^{Q(R)}$$

for some choice of R. Choose a G-invariant metric $\langle -, - \rangle$ on X and choose an element $\overline{\sigma} \in \mathfrak{g}$ such that

$$\langle V(\sigma), V(\overline{\sigma}) \rangle \geq 0$$

where $V \colon \mathfrak{g} \to \mathsf{Vect}(X)$. Then we should let

$$R = -t g_{mn} \psi^m V^n(\overline{\sigma}) = -t\langle \psi, V(\overline{\sigma}) \rangle$$

Then $Z(\sigma)$ becomes

$$Z(\sigma) = \int_X \omega(\sigma) \exp\left(-t\langle V(\sigma), V(\overline{\sigma})\rangle - t g_{mn} \psi^k \psi^m \nabla_k V^n(\overline{\sigma})\right).$$

The beautiful thing is that $Z(\sigma)$ is independent of t. Now we send $t \to \infty$, so that the resulting integrand is highly localized around $V(\sigma) = 0$.

Assume $V(\sigma)$ has only isolated zeros. Let $p \in X$ be one of these zeros. Then $T_p X$ is a linear representation of the maximal torus $T \subset G$, and it will therefore split as a sum of irreps

$$T_p X = \bigoplus_i \mathbb{R}_i^2.$$

In formulas, since $V^m(\sigma)|_p = 0$, its first derivative is defined and we get a matrix

$$\partial_m V^n(\sigma)|_p$$

representing $\sigma \in \mathrm{End}(T_p X)$. These irreps are invariant subspaces of this matrix, which can be written in block-diagonal form consisting of 2×2 blocks. If X is not even dimensional, by thinking about this block-diagonal form we see that not all zeros can be isolated. The 2×2 blocks are of the form

$$\begin{pmatrix} 0 & w(\sigma) \\ -w(\sigma) & 0 \end{pmatrix}$$

where $w(\sigma)$ are *linear* functions called **weights**. This is the only data we need to know, for localization! The final formula is

$$Z(\sigma) = \sum_p \frac{(\omega(\sigma))^{(0)}|_p}{\prod w_i(\sigma)}.$$

This is the **fixed point formula** for localization.

The famous example of this formula is for a symplectic manifold X, with Hamiltonian action of a torus T. Then each vector field $V^n(\sigma)$ is of the form

$$V^n(\sigma) = \Omega^{nm} \partial_m H(\sigma)$$

where $H: X \to \mathfrak{t}^*$ is a *linear* function called the **moment map**. The differential form we will take is

$$\omega(\sigma) = \exp\left(\Omega - H(\sigma)\right).$$

The corresponding partition function is

$$Z(\sigma) = \int_X \frac{\Omega^n}{n!} \exp\left(-H(\sigma)\right)$$

which is commonly considered in statistical mechanics for X the phase space. The fixed point formula says

$$Z(\sigma) = \sum_{dH(\sigma)=0} \pm \frac{e^{-H(\sigma)}|_p}{\sqrt{\det \partial^2 H}}$$

which is exactly the stationary phase approximation for the phase space. The equality of these two expressions for $Z(\sigma)$ is known as the **Duistermaat–Heckmann formula**.

The slogan is that there are two kinds of groups: those we quotient out, and those that remain and act on the quotient. We used the latter.

Take $X = LY$ to be the loop space of a Riemannian manifold Y. Then tangent vectors in X are vector fields on a loop. Let t be the coordinate on the loop, and define a symplectic form on X by

$$\Omega(\xi, \eta) = \int dt\, g(\xi, \nabla_{\partial_t} \eta).$$

In coordinates we would write

$$\Omega = \frac{1}{2} \int_0^1 dt\, g_{mn} \psi^m (\partial_t \psi^n + \Gamma^n_{\ell k} \dot{x}^\ell \psi^k).$$

There is a $U(1)$ action on X rotating the loop. The moment map for this action is the usual kinetic energy

$$H = \frac{1}{2} \int dt\, g_{mn} \dot{x}^m \dot{x}^n.$$

Now consider the fixed point formula. If we formally compute the lhs, we get the index of the Dirac operator on Y (if it is spin). The rhs will be the Atiyah–Singer index formula. Fixed points are constant loops $Y \subset LY$. Note that Ω is *not* non-degenerate, but this is OK.

Now we apply this whole discussion to $\mathcal{M}(k,n)$, i.e. the study of the anti-self-duality equation $F_A^+ = 0$ on $\mathbb{R}^4$ where A is a $U(n)$-connection. We said last time that

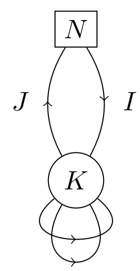
$$\int_{M^4} \operatorname{tr} F_A \wedge \star F_A = 8\pi^2 k$$

where k is the first Pontryagin class of the bundle P extended to the compactification $\overline{M}^4$. For $M^4 = \mathbb{R}^4$, the compactification is $\overline{M}^4 = S^4$. An important observation is that on 4-folds, $\star$ only depends on the *conformal* class of the metric. So it does not change when we switch from $\mathbb{R}^4$ to S^4.

We want to study ASD connections *modulo* gauge equivalence. The gauge equivalences we consider are ones where $g(x) \to 1$ as $x \to \infty$. There is a sort of "Fourier transform" which does not solve the equations explicitly, but maps them to matrix equations. This is the Fourier–Nahm–Mukai–Corrigan–Goddard–ADHM–etc. transform:

$$A \leftrightarrow (B_1, B_2, I, J).$$

Recall that last time we ended with two spaces $K = \mathbb{C}^k$ and $N = \mathbb{C}^n$. These linear maps B_1, B_2, I, J are summarized in the diagram



Suppose $\not{D}_A \psi^\pm = 0$, where $\psi^\pm \in \Gamma(S_\pm \otimes (N \times_G P))$. For irreducible A, the spinors $\psi^\pm$ satisfy

$$\psi^- = \{0\}, \quad \psi^+ = \{K\}.$$

Choose complex coordinates $z_1, z_2 \in \mathbb{C}^2 = \mathbb{R}^4$. The metric is $dz_1^2 + dz_2^2 = ds^2$. The ASD equation says

$$F_{12} = F_{\overline{12}} = 0, \quad F_{1\overline{1}} + F_{2\overline{2}} = 0.$$

The first two equations are equivalent to

$$[D_1, D_2] = 0 = [D_{\overline{1}}, D_{\overline{2}}].$$

Let $\psi_+ = \begin{pmatrix} \psi_1 & \psi_2 \end{pmatrix}$. The Dirac equation $\not{D}_A \psi^+ = 0$ becomes

$$\begin{aligned} D_1 \psi_2 - D_2 \psi_1 &= 0 \\ D_{\overline{1}} \psi_1 + D_{\overline{2}} \psi_2 &= 0. \end{aligned}$$

As $x \to \infty$, we have $A \to 0$. So these equations should become the *ordinary* Dirac equation, without any gauge field. If we write

$$\psi_1 = \partial_1 \phi, \quad \psi_2 = \partial_2 \phi,$$

then this automatically solves the first equation. The constraint for ϕ to solve the second equation is

$$\Delta\phi = 0.$$

This means $\phi \sim 1/r^2$. If we choose another ansatz

$$\psi_1 = \partial_{\overline{2}}\chi, \quad \psi_2 = -\partial_{\overline{1}}\chi,$$

then again this automatically solves the second equation. The constraint for χ to solve the first equation is

$$\Delta\chi = 0.$$

This means $\chi \sim 1/r^2$ as well. Combining them together, we get that

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \sim \frac{1}{r^4} \begin{pmatrix} I^\dagger \overline{z}_1 + J z_2 \\ I^\dagger \overline{z}_2 - J z_1 \end{pmatrix}.$$

Here I and J are recovered from the large-r asymptotics. To recover the matrices B_1 and B_2, we take any such solution and multiply it by the coordinate functions and project back to the space of solutions:

$$B_1 = \int_{\mathbb{R}^4} (\psi^+)^* z_1 \psi^+$$
$$B_2 = \int_{\mathbb{R}^4} (\psi^+)^* z_2 \psi^+$$

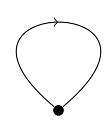
The operation of multiplication by coordinate functions commute, but if instead of looking at all sections we look at the ones annihilated by $\not{D}$, then out of commuting operators we get something which need not commute. In fact,

$$[B_1, B_2] + IJ = 0.$$

This condition arises from the boundary term in the calculation. In addition to this equation, we can also compute

$$[B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = 0.$$

These are the celebrated **ADHM equations**, written in terms of the quiver



with one vertex and one loop.

7 Pavel Etingof (Jul 09)

The goal is to get to Yangians and q-characters. Yesterday we discussed Lie bialgebras and Poisson Lie groups. Recall that G is **Poisson** if it is equipped with a Poisson bracket such that the product $G \times G \to G$ is a Poisson map. The Poisson bracket is determined by a Poisson bivector $\pi \in \Gamma(G, \wedge^2 TG)$:

$$\{f, g\} = \langle df \otimes dg, \pi \rangle.$$

Because we have a Lie group, TG is trivial and there are two canonical ways to trivialize it, by left or right translations. We use right translations. Then we can view π as a function $\Pi\colon G \to \wedge^2 \mathfrak{g}$. We say Π is a **Poisson–Lie structure**. We have

$$\Pi(xy) = \Pi(x) + x\Pi(y)x^{-1} \implies \Pi(1) = 0.$$

The differential of Π is a map

$$d\Pi\colon \mathfrak{g} \to \wedge^2\mathfrak{g},$$

and this is exactly the δ, the co-bracket, which we discussed yesterday. Then δ makes $\mathfrak{g}$ a **Lie bi-algebra**, meaning that

$$\delta^*\colon \wedge^2\mathfrak{g}^* \to \mathfrak{g}^*$$

is a Lie bracket.

The following can be viewed as a Poisson enhancement of Lie's third theorem.

Theorem 7.1 (Drinfeld)**.** *The functor $G \mapsto \mathsf{Lie}(G)$ is an equivalence of categories between simply-connected Poisson Lie groups and finite-dimensional Lie bi-algebras.*

Recall the notion of quasi-triangular Lie bialgebra. This is when the cobracket is given by commutator with some element $r \in \mathfrak{g} \otimes \mathfrak{g}$:

$$\delta(x) = [x \otimes 1 + 1 \otimes x, r].$$

This means the 1-cocycle $\delta \in Z^1(\mathfrak{g}, \wedge^2\mathfrak{g})$ is actually a co-boundary. The element r also solves the **classical Yang–Baxter equation**

$$r_{12}r_{13} + r_{12}r_{23} + r_{13}r_{23} = 0.$$

In general it seems hard to construct cobrackets to produce Lie bialgebras. We can use a tool called **Manin triples**. This is a triple of Lie algebras $(\mathfrak{g}, \mathfrak{g}^+, \mathfrak{g}^-)$ such that

$$\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^-$$

as vector spaces, and there is a non-degenerate invariant inner product on $\mathfrak{g}$ such that $\mathfrak{g}^+, \mathfrak{g}^-$ are isotropic. In particular this means they are Lagrangian.

Given a Manin triple, we can construct a Lie bialgebra in a very simple way. Because of the pairing, $\mathfrak{g}^+ \cong (\mathfrak{g}^-)^*$. This gives a Lie coalgebra structure, because $\mathfrak{g}^-$ is a Lie algebra. Hence $\mathfrak{g}^+$ is a Lie bialgebra, and $\mathfrak{g}^-$ turns out to be the *dual* Lie bialgebra. Recall that Lie bialgebras are *self-dual*: in the dual, bracket and cobracket are exchanged.

Proposition 7.2. *The Lie coalgebra and Lie algebra structures on $\mathfrak{g}^+$ are compatible.*

There is an inverse operation called the **Drinfeld double** of a Lie bialgebra. Namely given a Lie bialgebra $\mathfrak{g}^+$ we can define

$$\mathfrak{g}^- \coloneqq (\mathfrak{g}^+)^*$$

which is another Lie bialgebra, and then we take $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^-$ with standard pairing $\langle -, - \rangle$. Then there exists a unique bracket on $\mathfrak{g}$ which extends the bracket on $\mathfrak{g}^\pm$ such that $\langle -, - \rangle$ is invariant. This is the classical analogue of the quantum double construction. Note that $\mathfrak{g}$ itself is a bialgebra, given by

$$\delta_{\mathfrak{g}} = \delta_{\mathfrak{g}^+} - \delta_{\mathfrak{g}^-}.$$

The minus sign corresponds to taking opposite coproduct in the quantum double construction.

Proposition 7.3. *The Drinfeld double $D(\mathfrak{g}^+)$ is quasitriangular, with*

$$r = \sum a_i \otimes a_i^*$$

where $\{a_i\}$, $\{a_i^*\}$ are bases of $\mathfrak{g}^\pm$.

Manin triples really occur in nature. For example, let $\mathfrak{g}$ be a simple Lie algebra. Then it has subalgebras $\mathfrak{b}_+$ and $\mathfrak{b}_-$ which *almost* form a Manin triple. The issue is that they intersect in the Cartan, and they are not quite isotropic (they are too big dimensional). So we just define

$$\widetilde{\mathfrak{g}} \coloneqq \mathfrak{g} \oplus \mathfrak{h},$$

with Killing form

$$\langle -,-\rangle_{\widetilde{\mathfrak{g}}} \coloneqq \langle -,-\rangle_{\mathfrak{g}} - \langle -,-\rangle_{\mathfrak{h}}.$$

Then we can take the "Borel" subalgebras of $\widetilde{\mathfrak{g}}$

$$\widetilde{\mathfrak{b}}_+ \coloneqq \mathfrak{n}_+ \oplus \langle x, x\rangle$$
$$\widetilde{\mathfrak{b}}_- \coloneqq \mathfrak{n}_- \oplus \langle x, -x\rangle$$

to get a Manin triple. The quasitriangular structure is given by

$$r = \frac{1}{2}\sum x_i \otimes x_i + \sum_{\alpha>0} e_\alpha \otimes f_\alpha$$

where $\{x_i\}$ is an orthonormal basis of $\mathfrak{h}$. That this satisfies the classical Yang–Baxter equation follows from the whole discussion and does not need to be checked. One checks that

$$r + r^{21} = \sum x_i \otimes x_i + \sum e_\alpha \otimes f_\alpha + \sum f_\alpha \otimes e_\alpha$$

is exactly the Casimir of $\mathfrak{g}$.

There is a similar story for infinite-dimensional Lie algebras. There we have to be careful by taking graded duals. The cobracket will be given by formulas like

$$\delta(e_i) = d_i e_i \wedge h_i, \quad \delta(f_i) = d_i f_i \wedge h_i, \quad \delta(h_i) = 0.$$

TOOD: the Yangian Lie bialgebra. In this case, we get

$$r(z) = \frac{\Omega}{z}$$

and the classical Yang–Baxter equation becomes

$$[r^{12}(z_1 - z_2), r^{13}(z_1 - z_3)] + [r^{12}(z_1 - z_2), r^{23}(z_2 - z_3)] + [r^{13}(z_1 - z_3), r^{23}(z_2 - z_3)] = 0$$

in $\mathfrak{g}^{\otimes 3} \otimes \mathbb{C}(z_1, z_2, z_3)$. This is the classical YBE with **spectral parameter**, called a **pseudotriangular structure**. Note that we can expand

$$\frac{1}{t-u} = \frac{1}{t} + \frac{u}{t^2} + \cdots, \quad \frac{1}{u-t} = \frac{1}{u} + \frac{t}{u^2} + \cdots,$$

and hence

$$\frac{1}{t-u} + \frac{1}{u-t} = \sum_{n\in\mathbb{Z}} \frac{t^n}{u^{n+1}} = \delta(u-t).$$

The fact that the middle expression is *not* zero is why the structure is *not* triangular.

Now we discuss representation theory. Start with $U_q(\mathfrak{g})$ where $\mathfrak{g}$ is Kac–Moody (e.g. finite-dimensional or affine) and consider generic q. For usual $\mathfrak{g}$, we have category $\mathcal{O}$. Reps in $\mathcal{O}$ are required to have the following properties:

1. $\mathfrak{h}$ is diagonalizable;

2. e_i are locally nilpotent.

Then there are integrable representations $\mathcal{O}_{\text{int}} \subset \mathcal{O}$, where we additionally require

3. V is locally finite with respect to each $\mathfrak{sl}_2$ triple e_i, f_i, h_i.

Theorem 7.4 (Kac). *This category $\mathcal{O}_{int}$ is semisimple and simple objects are L_λ for $\lambda \in P_+$. The character of L_λ is given by the Weyl–Kac formula.*

Theorem 7.5 (Lusztig)**.** *Kac's theorem also holds for quantum groups.*

Theorem 7.6. $\mathsf{Rep}_{fin}(U_q\mathfrak{g})$ *as a tensor category determines* q *up to* $q \leftrightarrow q^{-1}$*.*

This shows the subtlety of the notion of monoidal category.

It is more interesting to see what happens in category $\mathcal{O}$. There we have Verma modules M_λ, and also simple modules L_λ. There is a surjection

$$M_\lambda \twoheadrightarrow L_\lambda.$$

$\mathcal{O}$ is already not semisimple, and characters of L_λ are already hard to compute. They are computed using Kazhdan–Lusztig polynomials. But it turns out in the quantum case the characters are the same.

Theorem 7.7 (Etingof–Kazhdan)**.** *For generic* q*,*

$$\operatorname{ch} L_\lambda^q = \operatorname{ch} L_\lambda.$$

The idea is to use Lie bialgebras, using the quantization functor.

Finally we should discuss finite-dimensional reps of affine Lie algebras. This will lead to Yangians and quantum affine algebras. In the affine case, besides these categories, we also have a very interesting category of *finite* representations.

Consider $\hat{\mathfrak{g}}$. The beauty of affine Lie algebras is that they have two realizations: one through Kac–Moody generators e_i, f_i, h_i, and the other through the *loop realization* $\mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}c$. In finite-dimensional reps of $\hat{\mathfrak{g}}$, the central element c acts by zero. This is because we compute

$$[ht, ht^{-1}] = 2c.$$

Hence they form a Heisenberg algebra. In finite-dimensional reps of Heisenberg algebras, c must be nilpotent. But also we have $\mathfrak{sl}_2$-subalgebras. Finite-dimensional reps of such algebras have the property that h_i are semisimple. So c is simultaneously semisimple and nilpotent, which means it acts by zero. So this seems uninteresting, because one might think that all the interesting stuff comes from non-zero values of c.

Given $z \in \mathbb{C}^\times$, define the **evaluation homomorphism**

$$phi_z : \hat{\mathfrak{g}} \to \mathfrak{g}, \quad a(t) \mapsto a(z).$$

Given a rep V of $\mathfrak{g}$, there is a pullback $V(z)$ of $\hat{\mathfrak{g}}$ called the **evaluation representation**. These are the building blocks for finite-dimensional reps.

Theorem 7.8.

1. *If* $V_1, \dots, V_n$ *are non-trivial irreps of* $\mathfrak{g}$ *and* $z_1, \dots, z_n \in \mathbb{C}^\times$ *are distinct, then*

$$V_1(z_1) \otimes \cdots \otimes V_n(z_n)$$

is irreducible.

2. *Any irreducible finite-dimensional representation has this form in a unique way (up to permutation).*

Note that when we talk about finite-dimensional reps we do not use the derivations discussed earlier. Otherwise finite-dimensional reps do not exist. If we have $z_i = z_j$, then the following lemma kicks in.

Lemma 7.9. *Let* V, W *be non-trivial irreps of* $\mathfrak{g}$*. Then* $V \otimes W$ *is reducible.*

Proof. There is a one-line proof:

$$\operatorname{Hom}(V \otimes W, V \otimes W) = \operatorname{Hom}(V \otimes V^*, W \otimes W^*),$$

but $V \otimes V^*$ and $W \otimes W^*$ both contain $\mathbb{C}, \mathfrak{g}$. ☐

Hence we see that to get interesting finite-dimensional representations we need to consider *reducible* reps.

8 Andrei Okounkov (Jul 10)

Let G be a compact Lie group. Suppose we have an infinite dimensional representation of G, e.g. $L^2(G)$. It was understood a long time ago that the way to think about the character χ_V of such a representation V is via distributions: If φ is sufficiently nice, like in $C^\infty(G)$, then

$$\langle \chi_V, \varphi \rangle = \operatorname{tr} \int \varphi(g) \pi_V(g) \, dg.$$

Here dg is a Haar measure. The averaging makes the operator $\pi_V(g)$ trace-class, and then we can take its trace. In general, suppose $\pi_V(\varphi)$ is an *integral* operator, i.e.

$$[\pi_V(\varphi) f](g) = \int_G K(g,h) f(h) \, dh$$

for some *kernel* $K(g,h)$. For the regular representation, the kernel is $K(g,h) = \varphi(gh^{-1})$. Then the trace becomes

$$\operatorname{tr} \int_G K(g,g) \, dg,$$

which is nothing more than $\varphi(1)$. This is a sort of localization, because the answer only depends on (functions supported at) $1 \in G$.

We can view a kernel $K(g,h)$ as a *correspondence* on $G \times G$. Similarly one has correspondences in K-theory. Given a K-theory class $\Phi \in K_T(X \times Y)$, we get an operator $K_T(Y) \to K_T(X)$ using the exact same formula:

$$\mathcal{F} \mapsto \pi_{X*}(\Phi \cdot \pi_Y^* \mathcal{F}).$$

Let's do an example of equivariant localization on $\mathbb{P}^1$. It has two fixed points 0 and ∞. Let the action on $\mathbb{P}^1$ be denoted by q. An equivariant vector bundle over $\mathbb{P}^1$ has fibers V_0, V_∞ over $0, \infty$, and being equivariant means V_0, V_∞ carry actions by q as well. Take charts

$$U_0 = \mathbb{P}^1 \setminus \infty, \quad U_\infty = \mathbb{P}^1 \setminus 0,$$

with $U_{0\infty} := U_0 \cap U_\infty$. We can compute cohomology of V using

$$\Gamma(U_0) \oplus \Gamma(U_\infty) \xrightarrow{s_0 - s_\infty} \Gamma(U_{0\infty}),$$

whose kernel is H^0 and cokernel is H^1. So if we want to compute $H^0 - H^1$, we may as well compute using this sequence.

1. Note that $\Gamma(U_{0\infty})$ is a number of copies of the regular representation. If z is a coordinate, then multiplication by z is an *invertible* operator on $\Gamma(U_{0\infty})$. Hence $(1 - q^{-1})\Gamma(U_{0\infty}) = 0$, so $\Gamma(U_{0\infty})$ is torsion and we can forget about it.

2. The equivariant vector bundle is trivial in a neighborhood of 0 and ∞. So for example

   $$\Gamma(U_0) = \Gamma(V_0 \otimes \mathbb{C}[z]).$$

   Hence the character of $\Gamma(U_0)$ is exactly

   $$\frac{\operatorname{tr}_q V_0}{1 - q^{-1}}$$

   because the *function* $z \mapsto z$ has character q^{-1}, not q.

Hence we get

$$\operatorname{tr}_q(H^0 - H^1) = \frac{\operatorname{tr} V_0}{1 - q^{-1}} + \frac{\operatorname{tr} V_\infty}{1 - q}.$$

This is a special case of the general K-theoretic localization formula we stated last time, which said that

$$V = \sum_{\substack{\text{components } F \\ \text{of fixed locus}}} \frac{V|_F}{\wedge^\bullet_{-1}(N_F X)^\vee}.$$

Exercise: take $X = G/B$. If λ is a character of B there is an associated line bundle L_λ. Compute $\chi(G/B, L_\lambda)$ using localization to get the Weyl character formula.

Theorem 8.1 (McGerty–Nevins). *Let X be a Nakajima quiver variety. Then $K_{top}(X) = K_{alg}(X)$ is generated by tautological vector bundles (in the sense of Λ rings). Consequently, classes of the form $c_i(Taut)$ generate integral cohomology.*

Proof. Take $X = \mathrm{Gr}(k,n)$. This is not an actual Nakajima quiver variety. Over X there is the tautological bundle L. Let's look at $X \times X$, with tautological bundles L_1, L_2. Then there is a map

$$\mathrm{Hom}(L_1, \mathbb{C}^n/L_2) \ni L_1 \to \mathbb{C}^n \to \mathbb{C}^n/L_2.$$

This canonical section s vanishes exactly when $L_1 = L_2$, i.e. on the diagonal Δ. Note that $\operatorname{codim}\Delta = \dim X = k(n-k)$, which is exactly the rank of the Hom bundle. Hence the diagonal Δ has a resolution. The corresponding operator in K-theory is the identity, but the resolution allows us to rewrite it as

$$\mathrm{id} = \sum \alpha_i \otimes \beta_j$$

where the α_i are some expressions in L_1, and the β_j are some expressions in L_2. Hence the image of the operator id is spanned by expressions in L_1, which is what we wanted to prove.

For a general Nakajima quiver variety, this proof breaks because they are *not* compact. In that case we can only pushforward if there is some condition on the support. The actual proof requires a compactification. One can think of a Nakajima quiver variety as a moduli of representations of the *path algebra* of the quiver modulo relations coming from the moment map. This is very useful, e.g. in proving there are no strictly semistable points. A priori we have

$$\bigoplus_i \mathfrak{gl}(V_i) \xrightarrow{\text{action}} T^*(\text{framed repr}) \xrightarrow{d\mu} \bigoplus_i \mathfrak{gl}(V_i)^*.$$

But knowing that it is a moduli of representations means T^* is $\mathrm{Ext}^1(R,R)$ over a representation R, and this sequence becomes

$$\mathrm{Hom}(R,R) \to \mathrm{Ext}^1(R,R) \to \mathrm{Ext}^2(R,R).$$

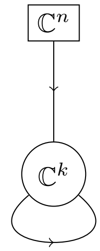
Based on this, we can write a resolution of the diagonal for *any* Nakajima quiver variety. The principle is that if have two stable representations R_1, R_2 of the same dimension, then there is no room for non-trivial Homs unless they are equal. This is the beginning of the general argument; then we need some McGerty–Nevins type classification which we will not discuss. $\square$

Now let's discuss fixed points. Suppose we have a framed quiver. Take the maximal torus

$$\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_{w_i} \end{pmatrix} \subset \mathrm{GL}(W_i).$$

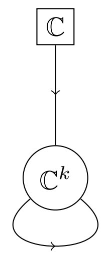
What are the fixed points of this torus acting on the Nakajima quiver variety? This action induces a grading on the $\mathrm{GL}(V_i)$, but does not act on the actual quiver data. Hence the quiver maps must preserve the weight spaces of V_i's. It follows that fixed loci are products of the *same* quiver, but with 1-dimensional framings.

For example, fixed loci in



are products of

$$\mathrm{Hilb}(\mathbb{C}^2, k) =$$



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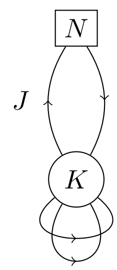
9 Nikita Nekrasov (Jul 10)

Last time we talked about the moduli of framed instantons of charge k on $\mathbb{R}^4 \cong \mathbb{C}^2$, and its relation to the quiver



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For reasons that will be delegated to the exercises, we will work with a slightly modified space $\overline{\mathcal{M}}(k, n)$. The original $\mathcal{M}(k, n)$ arises from imposing the equations

$$\begin{aligned} &\mu_{\mathbb{C}}\colon [B_1, B_2] + IJ = 0 \\ &\mu_{\mathbb{R}}\colon [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = 0. \end{aligned}$$

This produces a non-compact space. Since we want to integrate, we want to get as close to a compact space as possible.

The simplest approach is to solve these equations $\mu_{\mathbb{R}} = \mu_{\mathbb{C}} = 0$ and mod by $U(k)$, to get the *Uhlenbeck* partial compactification. Such a compactification has a scaling symmetry and therefore a conical singularity.

To smooth this singularity, we add a scalar to "smooth out" this singularity. This is the Gieseker–Nakajima compactification, and arises by solving

$$\mu_{\mathbb{C}} = 0, \quad \mu_{\mathbb{R}} = r 1_k \tag{4}$$

for (wlog) $r > 0$. Then the conical singularity is replaced by some non-trivial geometry, with cycles of size r. This partial compactification happens to be the moduli space of something, but we will not discuss this today. Note that solving the equations (4) is equivalent to solving just $\mu_{\mathbb{C}} = 0$ and then imposing stability conditions (and then modding by $\mathrm{GL}(k)$).

We would like to set up the theory of integration over this partial compactification. Instead of solving the equations explictly, we will integrate over the *linear* space of (B_1, B_2, I, J), and then introduce χ-H multiplets to impose equations. This will yield a practical tool for evaluating integrals of closed differential forms over $\overline{\mathcal{M}}(k,n)$ in terms of equivariant cohomology classes of this vector space of matrices. But vector spaces are affine, so their equivariant cohomology is just invariant polynomials on $\mathsf{Lie}U(k)$, which are S_k-invariant polynomials in $x_1, \dots, x_k$, i.e. elements of

$$\mathbb{C}[x_1, \dots, x_k]^{S_k}.$$

Concretely, we have

$$\int_{\overline{\mathcal{M}}(k,n)} \omega = \int_{(B_1,B_2,I,J)} \widetilde{\omega}\mu \cdot (\text{additional multiplets } \sigma, \overline{\sigma}, \eta, \chi, H).$$

In order to be efficient with these calculations, it is useful to have a *global* symmetry which we do not divide by and instead keep in the toolbox. The global symmetry allows us to express this integral as a sum over fixed points.

The global symmetries of these equations include changing basis of $\mathbb{C}^N$. An element $h \in U(N)$ can act by

$$(B_1, B_2, I, J) \mapsto (B_1, B_2, Ih^{-1}, hJ).$$

There is also an SU(2) action on B_1, B_2 rotating them:

$$\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \cdot (B_1, B_2, I, J) = (\alpha B_1 + \beta B_2, -\overline{\beta} B_1 + \overline{\alpha} B_2, I, J).$$

Finally, we can multiply B_1, B_2 by an overall phase $e^{i\phi} \in U(1)$:

$$(B_1, B_2, I, J) \mapsto (e^{i\phi} B_1, e^{i\phi} B_2, e^{2i\phi} I, J).$$

Equivariant parameters are generators of the ring $H^*_G(\text{pt}) = H^*(BG) = \mathbb{C}[\mathfrak{t}]^W$. For the problem at hand, they consist of:

1. eigenvalues $a_1, \dots, a_n$ of a generic element in $\mathfrak{u}(n)$;
2. eigenvalues
$$\frac{\epsilon_1 - \epsilon_2}{2}, \quad \frac{\epsilon_1 + \epsilon_2}{2}$$
for $\mathfrak{su}(2) \times \mathfrak{u}(1)$.

We make this strange choice for $\mathfrak{su}_2$ so that the action generated by such parameters in general is

$$(B_1, B_2, I, J) \mapsto (\epsilon_1 B_1, \epsilon_2 B_2, -I\vec{a}, (\vec{a} + \epsilon_1 + \epsilon_2)J).$$

We want to find fixed points of the global symmetry group on $\overline{\mathcal{M}}(k,n)$. This means that on the cover $\{(B_1, B_2, I, J)\}$, we want to find points where the orbit of the global symmetry group is *contained* in the orbit of the local symmetry group. In physics jargon, we say that the global symmetry transformation can be undone by a local symmetry transformation.

The space K becomes a representation of the torus $T \coloneqq U(1)^N \times U(1)^2 \to \text{GL}(k)$. Every rep of an abelian group splits into characters, so we can write

$$K = \bigoplus_{\gamma} K_\gamma$$

for 1-dimensional irreps K_γ. On these, a generic $\sigma \in T$ acts by multiplication by a phase

$$\exp\left(\langle w, a \rangle + n_1 \epsilon_1 + n_2 \epsilon_2\right).$$

The question is to find this integral data. This is actually pretty easy. The choice of the torus $U(1)^N \subset U(N)$ corresponds to a decomposition

$$N = \bigoplus_{\alpha=1}^{n} N_\alpha$$

into eigenspaces, so that $aN_\alpha = a_\alpha N_\alpha$. Acting by I, we get $I(N_\alpha) \subset K$. This line is an eigenline for σ, so that

$$\sigma I(N_\alpha) = I(aN_\alpha) = a_\alpha I(N_\alpha).$$

Hence $w = (0, 0, \ldots, 1, \ldots, 0)$ with the 1 at position α. The corresponding n_1, n_2 are zero. Acting on this line $I(N_\alpha)$ by $\mathbb{C}[B_1, B_2]$ generates some subspace

$$K_\alpha \coloneqq \mathbb{C}[B_1, B_2] I(N_\alpha),$$

and stability says $\bigoplus_\alpha K_\alpha = K$. It follows from commutation relations that

$$\sigma(B_1^{i-1} B_2^{j-1} I(N_\alpha)) = (\epsilon_1(i-1) + \epsilon_2(j-1) + a_\alpha) I(N_\alpha). \tag{5}$$

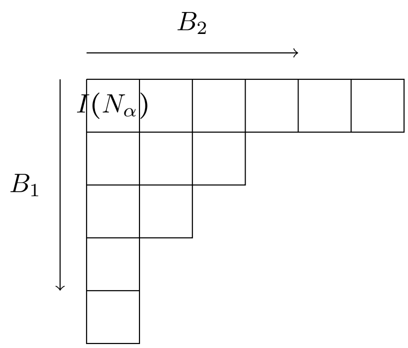
By genericity of the a_α, we get $K_\alpha \cap K_\beta = \varnothing$.

If $N = \mathbb{C}^1$, then it follows from the moment map equations that $J = 0$, and therefore $[B_1, B_2] = 0$. Hence the case of (5) will be the only possibility. First we show $\operatorname{im} I \subset \ker J$, i.e. $JI = 0$. This comes from moment map equations

$$JI = \operatorname{tr}(IJ) = -\operatorname{tr}([B_1, B_2]) = 0.$$

Then we show that J acting on any polynomial in B_1, B_2 applied to I is zero. By stability such polynomials generate the entire space K. Hence $J = 0$.

The space K therefore splits into Young diagrams given by the action of B_1 and B_2 on the image of I. The action of B_1 moves to squares below, until we hit zero; similarly, B_2 moves to the right:



For $\alpha = 1, \ldots, n$, we denote

$$\lambda^{(\alpha)} = \left\{ (i,j) : \begin{array}{l} 1 \le i \le \lambda_j^{(\alpha)t} \\ 1 \le j \le \lambda_i^{(\alpha)} \end{array} \right\}.$$

The action of σ is diagonal, with eigenvalues

$$\operatorname{tr}_k e^\sigma = K = \sum_{\text{rank}=1}^{n} e^{a_k} \sum_{(i,j) \in \lambda^{(\alpha)}} q_1^{i-1} q_2^{j-1}, \quad q_i \coloneqq e^{\epsilon_i}.$$

There is a **tautological complex** given by

$$K \xrightarrow{d_1} K \oplus K \oplus N \xrightarrow{d_2} K \tag{6}$$

with differentials given by

$$d_1 x \coloneqq (B_1 x, B_2 x, J x)$$
$$d_2(x_1, x_2, \xi) \coloneqq -B_2 x_1 + B_1 x_2 + I\xi.$$

The exactness of this complex is equivalent to $\mu_{\mathbb{C}} = 0$. So we can look at the cohomology of this complex.

Suppose $H^2 \neq 0$. Then there exists $\eta \in K$ which is orthogonal to the image of d_2. Then

$$\eta^\dagger(-B_2 x_1 + B_1 x_2 + I\xi) = 0$$

for all x_1, x_2, ξ. For this to hold for all x_1, x_2, ξ, we must have

$$B_1^\dagger \eta = B_2^\dagger \eta = I^\dagger \eta = 0,$$

and therefore by stability $\eta = 0$. So $H^2 = 0$.

For H^0 and H^1, denote

$$S \coloneqq (\mathrm{tr}_{H^1} - \mathrm{tr}_{H^0} - \mathrm{tr}_{H^2})(e^\sigma).$$

By the usual argument, this can be computed on chains. Here we have to be a little careful, because the complex (6) is *not* equivariant. The map d_1 and d_2 carry non-trivial weights. The equivariant version of the complex is

$$q_1 q_2 K \xrightarrow{d_1} q_2 K \oplus q_1 K \oplus N \xrightarrow{d_2} K.$$

It follows that

$$S = N + (q_1 + q_2)K - q_1 q_2 K - K = N - P_{12} K, \quad P_{12} \coloneqq (1 - q_1)(1 - q_2).$$

This is called a *virtual* character, because we are formally "subtracting" characters. It is easier to compute, because when we multiply K by $(1-q_1)(1-q_2)$ there are lots of cancellations. Carrying out these cancellations, we get

$$S = \sum_{\alpha=1}^{n} e^{a_\alpha} \left(\sum_{\square \in \partial_+ \lambda^{(\alpha)}} e^{c_\square} - q_1 q_2 \sum_{\square \in \partial_- \lambda^{(\alpha)}} e^{c_\square} \right)$$

Here $\partial_+ \lambda$ is the **outer boundary**: all squares which can be added to the jagged edge of a Young diagram such that it remains a Young diagram. Similarly, the **inner boundary** is all squares which can be *removed* from the jagged edge. The **content** $c_\square$ is given by

$$c_{\square \text{ at } (i,j)} \coloneqq \epsilon_1(i-1) + \epsilon_2(j-1).$$

Note that there is exactly one more square in the outer boundary than the inner boundary. If I is the ideal corresponding to the partition λ, boxes in the outer boundary $\partial_+\lambda$ correspond to generators of I and boxes in the inner boundary $\partial_-\lambda$ correspond to relations of I. Taking the difference $\partial_+\lambda - \partial_-\lambda$ corresponds in K-theory to the relation given by the SES

$$0 \to I \to \mathcal{O}_X \to \mathcal{O}_Z \to 0$$

where Z is the subscheme cut out by I.

We can introduce an **observable** given by

$$Y(x) \coloneqq \prod_{\alpha=1}^{n} \frac{\prod_{\square \in \partial_+ \lambda^{(\alpha)}} (x - a_\alpha - c_\square)}{\prod_{\square \in \partial_- \lambda^{(\alpha)}} (x - a_\alpha - \epsilon_1 - \epsilon_2 - c_\square)}$$

This is related to the S object by

$$Y(x) = E[e^x S^*]$$

where $E[-]$ is the **plethystic exponential**. If $\chi = \sum \pm e^{w_\pm}$, then the plethystic exponential is

$$E[\chi] \coloneqq \frac{\prod w_-}{\prod w_+}, \quad \chi = \sum \pm e^{w_\pm}.$$

This is an operation which takes sums to products. When we write χ^* we mean

$$\chi^* \coloneqq \sum \pm e^{-w_\pm}.$$

In addition to the tautological complex there is of course also the **tangent complex**. To do things with the fixed point formula we also need to understand the character of that. The tangent space $T_\lambda \overline{\mathcal{M}}(k,n)$ to a fixed point $\lambda = (\lambda^{(1)}, \dots, \lambda^{(n)})$ is realized by a complex as follows. Infinitesimal deformations must satisfy

$$[B_1, \delta B_2] + [\delta B_1, B_2] + \delta I \cdot J + I \cdot \delta J = 0.$$

Because these are *infinitesimal* deformations, we set the following things to be trivial:

$$(\delta B_1, \delta B_2, \delta I, \delta J) = ([\xi, B_1], [\xi, B_2], -\xi I, J\xi) = 0.$$

We can view the first equation as a differential

$$d_2(\delta B_1, \delta B_2, \delta I, \delta J),$$

and the first equation as a differential $d_1(\xi)$. Hence we get the tangent complex

$$C \coloneqq \Big[\mathrm{End}(K) \xrightarrow{d_1} \mathrm{End}(K) \otimes \mathbb{C}^2 \oplus \mathrm{Hom}(N,K) \oplus \mathrm{Hom}(K,N) \to \mathrm{End}(K) \Big].$$

One sees that the only cohomology here is $H^1(C)$, and everything else vanishes. Using this, the character of the tangent space is therefore

$$\mathrm{tr}_{T_\lambda \overline{\mathcal{M}}(k,n)}(e^a, q_1, q_2) = NK^* + q_1 q_2 N^* K - P_{12} K K^*.$$

Even though this looks like a virtual character, it is actually a *pure* character. This reflects the fact that $H^0 = H^2 = 0$.

Now we are in position to define the instanton partition function of *pure* super Yang–Mills theory. Tomorrow we will define more interesting things. This will be

$$Z \coloneqq \sum_{k=0}^{\infty} \mathsf{q}^k \int_{\overline{\mathcal{M}}(k,n)} \exp Q\mathcal{R}$$

where $\mathcal{R}$ is a universally-defined 1-form on $\overline{\mathcal{M}}(k,n)$ given by

$$\mathcal{R} \coloneqq g(\cdot, V(\vec{a}^*, \epsilon_1^*, \epsilon_2^*)).$$

Here Q is the T-equivariant de Rham differential and depends on the parameters $\vec{a}, \epsilon_1, \epsilon_2$. This $\mathcal{R}$ depends on some auxiliary parameters, but as long as they are such that the integral converges the answer is independent of the parameters. So

$$Z = Z(\vec{a}, \epsilon_1, \epsilon_2).$$

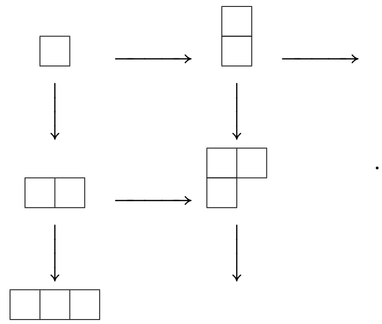
By the arguments from yesterday, we can use the fixed point formula to get the formula

$$Z = \sum_{\lambda = (\lambda^{(1)}, \dots, \lambda^{(n)})} \mathsf{q}^{|\lambda|} \frac{1}{\prod_{w \in T_\lambda \overline{\mathcal{M}}(k,n)} w}.$$

Finally, we can study expectation values of the observable $Y(x)$. In addition to the partition function we can study the correlators

$$\begin{aligned} \langle Y(x) \rangle &\coloneqq \frac{1}{Z} \sum_{k=0}^{\infty} \mathsf{q}^k \int_{\overline{\mathcal{M}}(k,n)} Y(x) \exp Q\mathcal{R} \\ &= \frac{1}{Z} \sum_{\vec{\lambda}} \mathsf{q}^{|\vec{\lambda}|} \frac{1}{\prod_{w \in T_{\vec{\lambda}} \overline{\mathcal{M}}(k,n)} w} \prod_\alpha \frac{\prod_{\square \in \partial_+ \lambda^{(\alpha)}} (x - a_\alpha - c_\square)}{\prod_{\square \in \partial_- \lambda^{(\alpha)}} (x - a_\alpha - \epsilon_1 - \epsilon_2 - c_\square)}. \end{aligned}$$

This function now depends on an auxiliary variable x, and in general will be rational in x. The poles of this function in x are very interesting, because they reveal certain hidden structure of the partition function. Namely, different terms of the sum are *related* to each other in an interesting way, because the terms can be organized along a Young **graph**



The poles of this $\langle Y(x)\rangle$ function correspond to the ways to "go back" along this graph. The residue at the poles of $\langle Y\rangle$ corresponding to *inner* boundary boxes can be related to the residues at the poles of $\langle 1/Y\rangle$ corresponding to *outer* boundary boxes in the *previous* partition. Tomorrow we will see how to take advantage of this to produce combinations that actually have *no* poles, which yields very interesting identities.

10 Noah Arbesfeld (Jul 10)

Start with a single vector space V over a field k. An **R-matrix** is an element

$$R(u) \in \mathrm{GL}(V \otimes V, k(u)).$$

The quantum Yang–Baxter equation is essentially a Reidemeister relation:

$$R^{12}(u_1 - u_2)R^{13}(u_1 - u_3)R^{23}(u_2 - u_3) = R^{23}(u_2 - u_3)R^{13}(u_1 - u_3)R^{12}(u_1 - u_2).$$

These operators should live in

$$\mathrm{End}(V^{\otimes 3}, k(u_1, u_2, u_3)).$$

This is a pretty complicated equation. If $\dim V = n$ then these are n^6 equations and n^4 variables. That there is a solution at all indicates something special is going on. Once we have a solution to this, we get a braiding on some tensor category, from which we can produce a quantum group.

How do we use geometry to find solutions? Maulik–Okounkov have two ways to do this. The first is to let $V = H^*(X_1)$ where X_1 is a Nakajima quiver variety with exactly one 1-dimensional framing node. Then $R(u)$ acts on $V^{\otimes 2}$. As we've seen, X_1 sits as a part of a fixed locus of the *same* quiver but with a 2-*dimensional* framing node. Let X_2 denote this Nakajima variety. A $\mathrm{GL}(2)$ acts on the framing of X_2, so that for a maximal torus $A \subset \mathrm{GL}(2)$,

$$X_2^A = X_1 \times X_1.$$

After suitably localizing, there will be two maps

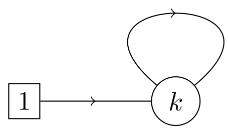
$$H_T^*(X_1)^{\otimes 2} = H_T^*(X_2^T) \xrightarrow[\mathrm{Stab}_-]{\mathrm{Stab}_+} H_T^*(X_2).$$

These maps will depend on some cocharacter of the torus. The R-matrix will then be

$$R := (\mathrm{Stab}_-)^{-1} \circ \mathrm{Stab}_+ .$$

Applying this picture to a bigger picture with *three* Nakajima varieties shows that R satisfies the quantum YBE. The element u, geometrically, is some $u \in H_A^*(\mathrm{pt})$.

If we take $X = \mathrm{Hilb}_k(\mathbb{C}^2)$, which is a Nakajima variety for the quiver



The definition of the Hilbert scheme of points on an arbitrary surface S is that it parameterizes 0-dimensional length-k subschemes of S:

$$\mathrm{Hilb}_k(S) \coloneqq \{Z \subset S : \dim Z = 0,\, \dim_{\mathbb{C}} H^0(\mathcal{O}_Z) = k\}.$$

We think of it as parameterizing "k points on S". For example,

$$\mathrm{Hilb}_1(S) = S.$$

A more interesting example is $\mathrm{Hilb}_2(S)$. There is a locus consisting of two *distinct* points $p, q \in S$. When the two points coincide, something interesting can happen. In this case, we have

$$H^0(\mathcal{O}_Z) = \langle 1, ax + by \rangle$$

for coordinates $x, y \in \mathbb{C}^2$ and scalars $a, b \in \mathbb{C}$ up to rescaling. Hence there is a $\mathbb{P}^1$, which remembers the direction along which the two points collided. In other words,

$$\mathrm{Hilb}_2(S) = \mathrm{Bl}_\Delta(S \times S)/S_2.$$

The second way of producing an R-matrix produces one for $\mathrm{Hilb}(S)$ for *any* surface S, for the vector space $V = H^*(\mathrm{Hilb}(S))$. The construction involves a Virasoro action on $H^*(\mathrm{Hilb}(S))$. Here when we write $\mathrm{Hilb}(S)$ we mean $\bigsqcup_k \mathrm{Hilb}^k(S)$. When $S = \mathbb{C}^2$, Maulik–Okounkov showed that we recover the same R-matrix as in the first construction. For general S we get something new.

Let's discuss the representation theory associated to $H^*(\mathrm{Hilb}(S))$. We put a mild assumption on S: it is a smooth quasiprojective surface over $\mathbb{C}$ which is either proper or equivariantly proper, i.e. there is some action of a torus T whose fixed locus is proper. This is desirable so that we can integrate over S. Then $H_T^*(S)$ has the structure of a Frobenius algebra. The pairing is

$$\langle \gamma_1, \gamma_2 \rangle_S = -\int_S \gamma_1 \cup \gamma_2.$$

As Andrei explained this morning, one way to get interesting actions of algebras on varieties is via correspondences. The Hilbert scheme of points is a particularly rich source of correspondences. Given $S^{[k]}$ and $S^{[k+n]}$, we can define a correspondence

$$Y_\gamma^{[k,k+n]} \subset S^{[k]} \times S^{[k+n]}$$

indexed by some $\gamma \in H_T^*(S)$. This correspondence is

$$Y_\gamma^{[k,k+n]} \coloneqq \{(Z_1, Z_2) : Z_1 \subset Z_2,\, \mathrm{supp}(Z_2/Z_1) = n[p],\, p \in \mathrm{cycle}(\gamma)\}.$$

By push-pull, this yields an operator

$$\alpha_{-n}(\gamma) \colon H_T^*(S^{[k]}) \to H^*(S^{[k+n]}),$$

defined by $p_{2*} \circ p_1^*$ where p_i are projections from $Y_\gamma^{[k,k+n]}$. These are **creation** operators. Their counterparts are given by the same correspondence in reverse:

$$\alpha_n(\gamma) \coloneqq (-1)^n(\text{transposed correspondence}).$$

The sign comes from the minus sign in the pairing. For α_0, which must act by a scalar, we introduce a new variable u and say that α_0 acts by

$$u \int_S \gamma.$$

Theorem 10.1 (Grojnowski, Nakajima)**.** *These operators $\alpha_{\pm n}(\gamma)$ form a Heisenberg algebra labeled by $H_T^*(S)$, i.e.*

$$[\alpha_m(\gamma), \alpha_n(\gamma')] = \delta_{m+n} m \langle \gamma, \gamma' \rangle_S \cdot \mathrm{id}\,.$$

We want to know what this action looks like on the module

$$V_S \coloneqq H_T^*(\bigsqcup_k S^{[k]}).$$

This turns out to be a *lowest weight* representation, also known as a **Fock space**. If we let

$$|\varnothing\rangle \in H^0(S^{[0]}) = H^0(\mathrm{pt}),$$

then $\alpha_m(\gamma)\,|\varnothing\rangle = 0$ for $m > 0$, so that $|\varnothing\rangle$ acts as a vacuum. For $S = \mathbb{C}^2$, a basis over $\mathrm{Frac}(H_T^*(\mathrm{pt}))$ for $H_T^*(\mathrm{Hilb}(S))$ is given by elements

$$\prod_i \alpha_{-\lambda_i}(1)\,|\varnothing\rangle$$

ranging over all partitions λ.

In general, given a Heisenberg module, the Feigin–Fuchs construction makes it a Virasoro module. One can write this purely algebraically. First introduce a formal parameter κ. The Virasoro elements will be functions of κ, and are defined by

$$L_n(\gamma, \kappa) \coloneqq \sum_m {:}\alpha_m(\gamma_i')\alpha_{n-m}(\gamma_i''){:}\; - n\kappa\alpha_n(\gamma) - \delta_n \kappa^2 \int_S \gamma.$$

Here we are using Sweedler notation $\Delta\gamma = \sum_i \gamma_i' \otimes \gamma_i''$. The dots $::$ denote normal ordering, which means we move all annihilation operators to the right. There is also a way to produce these operators geometrically, using c_1 of tautological bundles.

Proposition 10.2. *The L_n form a Virasoro algebra decorated by cohomology classes in $H_T^*(S)$:*

$$[L_m(\gamma, \kappa), L_n(\gamma', \kappa)] = (m-n)L_{m+n}(\gamma\gamma') + \delta_{m+n} \underbrace{\frac{m^3 - m}{12}\left(\int_S \gamma'\gamma''(e(S) - 6\kappa^2)\right)}_{\textit{central charge}} \mathrm{id}\,.$$

Again, V_S is a lowest weight representation for the Virasoro. So

$$L_m(\gamma, \kappa)\,|\varnothing\rangle = 0, \quad \forall m > 0.$$

One can compute, from definition, that

$$L_0(\gamma)\,|\varnothing\rangle = \underbrace{(u^2 - \kappa^2)\int_S \gamma}_{\text{conformal dimension}}\;|\varnothing\rangle\,.$$

Note that both the central charge and conformal dimension are *purely quadratic* in κ. More precisely, they are invariant under $\kappa \leftrightarrow -\kappa$. It is a fact that lowest weight Virasoro representations are classified by their central charge and conformal dimension. So we can make the Virasoro act by either $L_m(\gamma, \kappa)$ or $L_m(\gamma, -\kappa)$, but this classification means we get isomorphic representations. In other words, there exists some isomorphism

$$R^1 \colon \prod_i L_{-\lambda_i}(\gamma, \kappa)\,|\varnothing\rangle \mapsto \prod_i L_{-\lambda_i}(\gamma, -\kappa)\,|\varnothing\rangle\,.$$

This is an isomorphism of vector spaces over $\mathbb{C}(u, \kappa)$.

To produce $R(u) \in \mathrm{End}(V_S^{\otimes 2})$, first write the tensor square in a different way. Replace the generators by

$$\begin{aligned} V_S^{\otimes 2} &= V_S^+ \otimes V_S^- \\ \{\text{generators}\} &\mapsto \{\text{symmetric words}\} \otimes \{\text{antisymmetric words}\}. \end{aligned}$$

The stable envelope construction for $S = \mathbb{C}^2$ guides us to the following construction. We get

$$R(u) = \mathrm{id}_{V_S^+} \otimes R_-^1$$

where R_-^1 acts on the second factor, and is constructed by the same formulas as R^1 but every time we see an α_m we replace it with the antisymmetric version $\alpha_m \otimes 1 - 1 \otimes \alpha_m$, up to some scaling.

11 Pavel Etingof (Jul 10)

Today we will do some computations which will help us get a feel for how things work. We will talk about finite-dimensional representations of $U_q(\hat{\mathfrak{sl}}_2)$. Recall its definition

$$U_q(\hat{\mathfrak{sl}}_2) = \langle e_i, f_i, K_i^{\pm} \rangle / (\cdots)$$

with Cartan matrix

$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

The generators $e_i, f_i, K_i^{\pm}$ satisfy $\mathfrak{sl}_2$-relations for each $i = 0, 1$. Further we have

$$\begin{aligned} K_0 K_1 = K_1 K_0, &\quad [e_0, f_1] = [e_1, f_0] = 0 \\ K_0 e_1 K_0^{-1} = q^{-2} e_1, &\quad K_1 e_0 K_1^{-1} = q^{-2} e_0 \\ K_0 f_1 K_0^{-1} = q^2 f_1, &\quad K_1 f_0 K_1^{-1} = q^2 e_0. \end{aligned}$$

We also have relations

$$e_i^3 e_j - (q^2 + 1 + q^{-2}) e_i^2 e_j e_i + (q^2 + 1 + q^{-2}) e_i e_j e_i^2 - e_j e_i^3 = 0.$$

For $q = 1$, we also have relations coming from $L\mathfrak{sl}_2$

$$e_0 = ft, \quad f_0 = et^{-1}, \quad h_0 = c - h, \quad c = h_0 + h_1.$$

In particular, $K_0 K_1 =: C$ is a *central* element. This will act by 1 in finite-dimensional representations. There is an **evaluation homomorphism**

$$\phi \colon U_q(\hat{\mathfrak{sl}}_2) \to U_q(\mathfrak{sl}_2)$$

given by the formula

$$\phi(e_1) = \phi(f_0) = e, \quad \phi(f_1) = \phi(e_0) = f, \quad \phi(K_1) = \phi(K_0^{-1}) = K.$$

This is a generalization of the evaluation homomorphism $t \mapsto 1$. In fact we have a whole family of evaluation homomorphisms corresponding to a choice of $z \in \mathbb{C}^\times$. There is a $\mathbb{C}^\times$-action τ_z which acts by scaling t, i.e. $e_0 \mapsto z e_0$ and $f_0 \mapsto z^{-1} f_0$, with trivial action on other generators. Then we can look at

$$\phi_z \coloneqq \phi \circ \tau_z.$$

This is good because we know a lot about the representation theory of $U_q(\mathfrak{sl}_2)$, and we can pull them back via ϕ_z. It turns out these pullbacks give all irreducible finite-dimensional representations.

The simplest representation of $U_q(\mathfrak{sl}_2)$ is $V = \mathbb{C}^2$, given by

$$e \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad K \mapsto \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}.$$

For higher-dimensions it may differ, but in two dimensions it is the same. Denote by $V(z)$ the evaluation of V at z, i.e.

$$\Pi_{V(z)}(a) = \Pi_V(\phi_z(a)).$$

Then

$$e_1 \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f_1 \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad e_0 \mapsto \begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix}, \quad f_0 \mapsto \begin{pmatrix} 0 & z^{-1} \\ 0 & 0 \end{pmatrix}, \quad K_0 \mapsto \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix}.$$

It is easy to show that any 2-dimensional irrep will be $V(z)$ for some z. So the *dual* $V(z)^\vee$ of this representation must be $V(w)$ for some w. Let's compute and see what w is. How can we recognize z from $V(z)$? It is clear that

$$z = \operatorname{tr} e_0 e_1.$$

On $V(z)^\vee$, we have

$$\Pi_{V(z)^\vee}(e_0) = \Pi_{V(z)}(Se_0) = \Pi_{V(z)}(-e_0 K_0^{-1})$$
$$\Pi_{V(z)^\vee}(e_1) = \Pi_{V(z)}(Se_1) = \Pi_{V(z)}(-e_1 K_1^{-1}).$$

Hence by algebra relations we get

$$w = \operatorname{tr} e_1 K_1^{-1} e_0 K_0^{-1} = q^2 \operatorname{tr} e_1 e_0 = q^2 z.$$

The result is that

$$V(z)^\vee = V(q^2 z). \tag{7}$$

But then if we take the double dual, we will get

$$V(z)^{\vee\vee} = V(q^4 z).$$

This is an example of a Hopf algebra where taking double dual is *not* a trivial operation, even at the level of simple objects. So not only is $S^2 \neq \mathrm{id}$, it is not an inner automorphism.

Actually, (7) holds for any irrep. For other algebras, the double dual will result in $q^{2h^\vee}$ where $h^\vee$ is the dual Coxeter number.

This implies that $V(z) \otimes V(q^2 z)$ is reducible, because it is the same as $V(z) \otimes V(z)^\vee$, and there is a coevaluation $\mathbb{C} \to V(z) \otimes V(z)^\vee$. The same goes for $V(z) \otimes V(q^{-2} z)$, which maps to $\mathbb{C}$ via the evaluation map. So this category is not semisimple. Exercise: there are SESs

$$0 \to W(q^{-1} z) \to V(z) \otimes V(q^{-2} z) \to \mathbb{C} \to 0$$

where W is the 3-dimensional irrep. Note in particular that this category cannot be braided, because $X \otimes Y$ is not always isomorphic to $Y \otimes X$. However they *do* always have the same composition factors.

Claim: in fact $V(u) \otimes V(z)$ is irreducible except when $u/z = q^2$ or q^{-2}.

For $a \in \mathbb{Z}_{\geq 0}$, let V_a be the irrep of $U_q(\mathfrak{sl}_2)$ with highest weight a. Then we get a representation $V_a(z)$, and we can look at the product

$$V_{a_1}(z_1) \otimes \cdots \otimes V_{a_n}(z_n).$$

When is it irreducible? We saw the answer in the case where there are two factors and both are 2-dimensional. It turns out this has a very nice answer.

Definition 11.1. A **string** in $\mathbb{C}^\times$ is a finite geometric progression with ratio q^2. Two strings S, T are **in special position** if $S \cup T$ is a string containing S and T properly. Otherwise they are in **general** position.

To $V_a(z)$, associate its string $S_a(z)$ given by

$$q^{-(a-1)}z, \quad q^{-(a-3)}z, \quad \dots, \quad q^{a-1}z.$$

This is a string of length a. For example, if V is 2-dimensional, we get the single string $S_1(z)$ which is just z.

Theorem 11.2 (Chari–Pressley)**.**

1. *The tensor product*
$$V_{a_1}(z_1) \otimes \cdots \otimes V_{a_n}(z_n)$$
is irreducible iff the strings $S_{a_i}(z_i)$ are pairwise in general position.
2. *Such irreducible tensor products are isomorphic iff they differ by permutation.*
3. *Any irreducible finite-dimensional representation is isomorphic to such a product.*

It is not clear immediately why if two irreps differ by permutation then they are isomorphic. This is because we have an R-matrix. So no part of this theorem comes for free.

Lemma 11.3. *Any finite set in $\mathbb{C}^\times$ with multiplicities can be uniquely represented as a union of strings pairwise in general position.*

So we view finite-dimensional irreps as parameterized by finite subsets of $\mathbb{C}^\times$ with multiplicities, i.e. polynomials with constant term 1. These are called **Drinfeld polynomials**, which arise in a natural way in the next discussion.

Now let's talk about permuting factors, which comes from R-matrices. Certainly this quantum affine algebra $U_q(\hat{\mathfrak{sl}}_2)$ has a universal R-matrix of the form $R = \sum_i a_i \otimes a_i^*$, which we discussed. It is a huge sum, which even for finite-dimensional V, W does not make sense on $V \otimes W$. Apply the symmetry τ_z to the first component of R. Recall that it has positive weight with respect to this degree. For finite-dimensional V, W,

$$R_{V,W}(z) \coloneqq (\tau_z \otimes 1)(R)|_{V\otimes W} \in \operatorname{End}(V \otimes W)[[z]],$$

which we can think of as $R|_{V(z)\otimes W}$. So while we can't make sense of R on $V \otimes W$, this at least makes sense.

Theorem 11.4 (Drinfeld)**.** *This series $R_{V,W}(z)$ converges for $|z| < r$ for some r. If V, W are irreducible, then*

$$R_{V,W}(z) = \overline{R}_{V,W}(z) f_{V,W}(z)$$

where $\overline{R}$ is a rational function and f is a scalar meromorphic function.

The shift $q \mapsto q^4 z$ gives rise to a non-linear q-difference equation for R. Then one shows using some simple complex analysis that the solution converges on some disk.

The matrix $\overline{R}$ satisfies a unitarity condition, namely

$$\overline{R}(z)\overline{R}^{21}(z) = 1 \otimes 1.$$

This is a *mock* symmetric braiding, because it has poles. For example, if $V = W = \mathbb{C}^2$ with basis v_+, v_-, then

$$\overline{R}(z) = \begin{pmatrix} q & & & \\ & \frac{z-1}{z-q^{-2}} & \frac{q-q^{-1}}{z-q^{-2}} & \\ & \frac{z(q-q^{-1})}{z-q^{-2}} & \frac{z-1}{z-q^{-2}} & \\ & & & q \end{pmatrix}.$$

This matrix has a pole at $z = q^{-2}$, but also we can check that it satisfies the unitarity condition. It is also *not* an isomorphism at $z = q^2$; in fact it has a zero there. These are exactly the points where the tensor product is reducible. This is a general fact that can be formulated for quantum affine algebras in general.

Proposition 11.5. *$V_{a_1}(z_1) \otimes \cdots \otimes V_{a_n}(z_n)$ is irreducible iff all R-matrices*

$$\overline{R}_{ij} \colon \mathrm{End}(V_{a_i}(z_i) \otimes V_{a_j}(z_j))$$

are defined and invertible.

For higher-rank algebras, the situation becomes more complicated. For $U_q(\hat{\mathfrak{sl}}_n)$ we still have an evaluation homomorphism

$$\phi_z \colon U_q(\hat{\mathfrak{sl}}_n) \to U_q(\mathfrak{sl}_n)$$

but it is a little more complicated to define. However it is not true anymore that any irrep is a tensor product of evaluation representations. They will in general be quotients of such products. The ultimate understanding came from Nakajima, who showed that these representations are realized in equivariant K-theory of Nakajima quiver varieties.

For other types, things actually get worse. For example, there are *no* evaluation homomorphisms: there are no arrows making

$$\begin{array}{ccc} U_q(\hat{\mathfrak{g}}) & & U_q(\mathfrak{g}) \\ \uparrow & \overset{\mathrm{id}}{\nearrow} & \\ U_q(\mathfrak{g}) & & \end{array}$$

commute. Drinfeld showed that the adjoint representation for $\mathfrak{g}$ does not lift to $U_q(\hat{\mathfrak{g}})$. To lift it, we have to add a copy of the trivial rep, which then can be twisted by the τ_z. In general, to quantize reps V, one has to add to it a sum:

$$\hat{V}_\lambda = V_\lambda + \sum_{\mu<\lambda} c_{\mu\lambda} V_\mu.$$

This is a complicated phenomenon which was only understood well after Nakajima.

In particular this discussion already shows in the $\mathfrak{gl}_n$ case that there are no "highest weights". We need to pick a different polarization of the Lie algebra. The usual decomposition is

$$\mathfrak{g}[z, z^{-1}] = \hat{\mathfrak{b}}_+ \oplus \mathfrak{h} \oplus \hat{\mathfrak{b}}_-$$

For finite-dimensional reps, we *will* have a nilpotent action if we do a decomposition along the loop direction, i.e.

$$\mathfrak{g}[z, z^{-1}] = \widetilde{\mathfrak{n}}_+ \oplus \widetilde{\mathfrak{h}} \oplus \widetilde{\mathfrak{n}}_-.$$

This is quite simple in the classical case, e.g. we can say what the highest weight for the reps we saw last time are, but what is not completely clear is why it is that after q-deformation we still have a decomposition like this. Elements quantize in a quite complicated way. This was a great discovery of Drinfeld.

12 Andrei Okounkov (Jul 11)

As explained in Alisa's lecture, a correlation function of the form

$$\langle Y(x) + \frac{*}{Y(x + \cdots)} \rangle$$

has no poles. This is a qq character $\mathbb{C}^2(x)$ of $Y(\mathfrak{sl}_2)$. In this case, one can go and check poles cancel by elementary math. In more complicated situations, there is a geometric proof for pole cancellation, involving a localization formula over a compact space. To make this work we would like to make H_T^* or K_T^* of a Nakajima quiver variety into a module for a quantum group.

Let $G \subset \mathrm{GL}(V)$ be a reductive group. Then G acts on all possible tensor products $V^{\otimes d}$. One can consider the category in which objects are such representations and maps are all possible homomorphisms. This gives a symmetric tensor category. It is a simple observation, going back to Weyl, that this category knows everything. In particular it knows what the group G is. For example, if $\dim V = n$, then

$$\mathrm{Hom}(V^{\otimes d}, V^{\otimes d})^{S_d} = \mathrm{Sym}^d(V^* \otimes V) = \text{polynomials on } n \times n \text{ matrices of degree } d.$$

Hence we know all G-invariant polynomials on matrices, yielding coordinates on the quotient

$$\mathrm{GL}(V) \to \mathrm{GL}(V)/G.$$

Since the fiber over the identity $1 \in \mathrm{GL}(V)/G$ is G itself, we have recovered G. The takeaway is that it is typically easier to think about categories than about equations.

Now we want to make the group G quantum. In the classical story, permutation of factors $V_1 \otimes V_2 \to V_2 \otimes V_1$ is a morphism in the category. In the quantum story, there will be a new non-trivial morphism

$$R\colon V_1 \otimes V_2 \to V_2 \otimes V_1$$

called the **R-matrix**. (Note that it need not square to 1.) We require the R-matrix to satisfy the Yang–Baxter equation, coming from two ways to permute three factors in a tensor product. In this setup, the typical reconstruction we just talked about is much *easier*! This is because now, instead of a single symmetric group for all objects, every pair of objects has its *own* R-matrix. Introduce an auxiliary space V_0 and consider

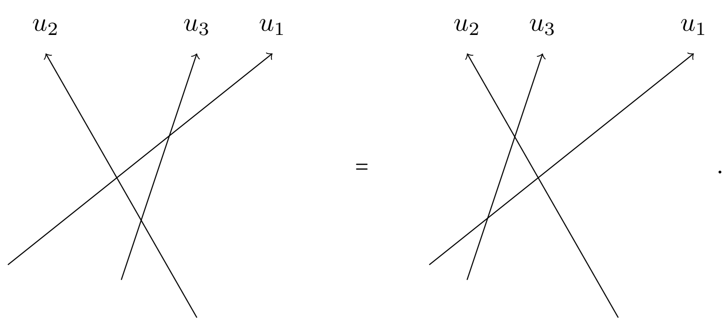
$$V_0 \otimes V_1 \otimes V_2 \otimes V_3 \otimes V_4.$$

Write the R-matrix which braids them by fusion:

$$V_1 \otimes V_2 \otimes V_3 \otimes V_4 \otimes V_0$$

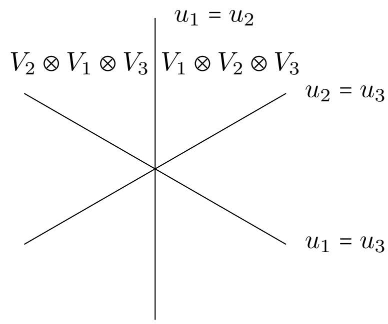
If we take an arbitrary matrix element in the auxiliary space V_0, we get an operator in $\mathrm{End}(V_1 \otimes V_2 \otimes V_3 \otimes V_4)$. These operators are braided by the R-matrix, as claimed. To compose operators we just add tensor factors in the auxiliary space, and the Yang–Baxter equation in the auxiliary space is commutation relations. Any decomposition in the tensor category gives automatically a relation in the auxiliary space. So we get a Hopf algebra.

Let's discuss the Yang–Baxter equation with spectral parameters. The picture to have in mind is



Interpret these as particles with some velocity in spacetime. In principle there is some triple interaction, which factors into double interactions, and it doesn't matter in which order it happens. Now imagine we have a stationary observer. What happens in the momentum plane? At generic times we will see

$V_1(u_1) \otimes V_2(u_2) \otimes V_3(u_3)$, but at special times corresponding to $u_i = u_j$ we will see a bound state.



So we should really think of an R-matrix as factored into two processes: forming a bound state, and unforming. Instead of thinking of the Yang–Baxter equation as a relation on open cells, we should think of there being an operation going between arbitrary strata.

Where have we seen this picture before? Consider the equivariant K-theory

$$K_T\left(\bigsqcup_{k=0}^{n} T^* \operatorname{Gr}(k,n)\right).$$

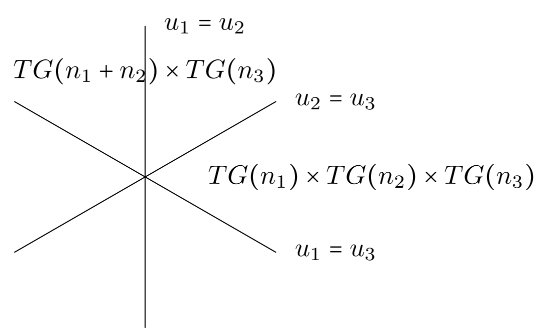
The $T^* \operatorname{Gr}(k,n)$ carries an action of $\mathrm{GL}(n)$. In $\mathrm{GL}(n)$, these variables u_i will be equivariant variables. Take a subtorus of the form

$$\left\{\operatorname{diag}(\overbrace{u_1,\dots,u_1}^{n_1},\overbrace{u_2,\dots,u_2}^{n_2},\overbrace{u_3,\dots,u_3}^{n_3})\right\} \subset \mathrm{GL}(n)$$

with $n_1 + n_2 + n_3 = n$. We discussed that

$$\operatorname{Spec} K_T(\cdots) \to \operatorname{Spec} K_T(\mathrm{pt}) = (u_1, u_2, u_3).$$

Write $TG(n) \coloneqq \bigsqcup_{k=0}^{n} T^* \operatorname{Gr}(k,n)$. Over a generic point in the base, we have $TG(n_1) \times TG(n_2) \times TG(n_3)$. In general, we get a picture



Hence we really need some sort of correspondence

$$TG(n_1) \times TG(n_2) \leftarrow Z \rightarrow TG(n_1 + n_2).$$

All these correspondences are special cases of the following problem. Pick a 1-parameter subgroup $\mathbb{C}^\times \subset A$, and suppose it preserves the symplectic form ω. Then at a component of the fixed locus $X^A \subset X$, there will be as many attracting as repelling directions. The attracting set gives a correspondence, and so does the repelling set. Pick the attracting set to use as a correspondence:

$$\mathrm{Attr} := \{(x, f) : \lim_{a \to 0} ax = f\} \subset X \times X^A.$$

The main issue with Attr is that it is not closed. One would think taking the closure $\overline{\mathrm{Attr}}$ solves this problem, but the closure can be very complicated and will not give something that satisfies the Yang–Baxter equation. The way out, at least in cohomology, is to do a deformation like

$$\mu_{\mathbb{C}} = 0 \quad \rightsquigarrow \quad \mu_{\mathbb{C}} = (\text{scalar}) \cdot 1.$$

After deformation we get something affine, and the resulting cycle is closed. Taking the closure yields a cycle in the central fiber which is the cycle we want.

Exercise: do this computation for $TG(2) = \mathrm{pt} \sqcup T^*\mathbb{P}^1 \sqcup \mathrm{pt}$.

13 Nikita Nekrasov (Jul 11)

In the previous lecture we finally introduced the instanton partition $Z(\mathsf{q}, \vec{a}, \epsilon_1, \epsilon_2)$ function for the simplest gauge theory. We also introduced the $Y(x)$ observable and said we would study correlation functions

$$\langle Y_1(x) \cdots Y_k(x) Y^{-1}(x_{k+1}) \cdots Y^{-1}(x_m) \rangle.$$

Out of all possible insertions Y, there are some that exhibit interesting analytic properties. Today we will see this more explicitly. Namely we'll show that

$$\langle Y(x + \epsilon_1 + \epsilon_2) + \mathsf{q} Y^{-1}(x) \rangle \tag{8}$$

is an entire function in $x \in \mathbb{C}$, and therefore a polynomial in x. This particular combination is the simplest example of a qq-character. Actually we should call this an $\epsilon\epsilon$-character because it is in cohomology; qq-characters live in K-theory. We will generalize Z to quiver gauge theories with matter, and then generalize this qq-character. This particular qq-character corresponds to the fundamental representation of $\mathfrak{sl}_2$. If we look at the formula (8), we see two terms, one of which is the inverse of the other. This should be compared to the trace of elements $g \in \mathrm{SL}_2(\mathbb{C})$. In fact if we remove all deformations, this is literally the trace of a matrix with eigenvalues Y and $1/Y$.

Let's "add adjoint matter". In our paradigm of fields, equations, and symmetries, every term in this triad comes with a supermultiplet. We had (x, ψ) for fields, (χ, H) for equations, and $(\sigma, \overline{\sigma}, \eta)$ for symmetries. The actual form of the equations is not as important as where the equations take values, which determines what variables we introduce.

The complex moment map equation is

$$0 = [B_1, B_2] + IJ.$$

We want to *kill* all degrees of freedom in the system. The space $\overline{\mathcal{M}}(k, n)$ of solutions to these equations has dimension $2kn$. Hence we should introduce a form Ω of degree $4kn$, so that

$$\int_{\overline{\mathcal{M}}(k,n)} \Omega$$

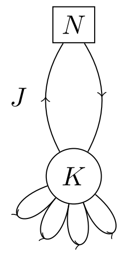
yields a non-zero answer. We could build Ω using polynomials in σ, but there is no canonical choice that way. It would be much better if the space we defined were of dimension 0. Then integration over that space is just a point count. This is what we mean by killing degrees of freedom. We should impose additional equations, at the expense of introducing additional variables, so as to make the dimension 0.

There is a canonical procedure for cooking up the cotangent bundle to $\overline{\mathcal{M}}(k, n)$ and computing its characteristic classes. This is called the **co-field construction**, first introduced by Moore–Cordes–Rangoolam

(1994) and then made explicit by Vafa–Witten (1994) in the context of 4d $N = 4$ super-Yang–Mills. The idea is to introduce new fields, called co-fields, which are dual to both equations and symmetries. In our case we had three Lie algebra-valued equations:

$$
\begin{aligned}
0 &= \mu_{\mathbb{C}} = [B_1, B_2] + IJ \\
0 + \mu_{\mathbb{R}} &= [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J - r \cdot 1.
\end{aligned}
$$

We introduce four copies of $\mathsf{Lie}U(k)$ and call them B_3, B_4. In terms of the quiver we now have



Now we impose as many equations as we have variables, to kill them all. The way to do this in one swoop is as follows. Write

$$(B_3, B_4) = (\vec{\phi} \in \mathsf{Lie}U(k) \otimes \mathbb{R}^3,\ \sigma_r \in \mathsf{Lie}U(k)).$$

Define a function

$$f \coloneqq \operatorname{tr} \vec{\phi} \cdot \vec{\mu} + \operatorname{tr} \phi_1 [\phi_2, \phi_3].$$

The equations we impose are

$$\frac{\partial f}{\partial(\text{variable})} + \delta_{\sigma_r}^{U(k)}(\text{variable}) = 0,$$

for all variables. This is almost like looking for critical points of f, but because we have a symmetry we allow for "generalized" critical points. Note that if we omit the second term, the equations are not independent anymore, by symmetry of multiple derivatives. If we unpack all this, the equations are

$$
\begin{aligned}
0 &= \mu_{\mathbb{C}} + [B_3, B_4]^\dagger \\
0 &= \mu_{\mathbb{R}} + [B_3, B_3^\dagger] + [B_4, B_4^\dagger] \\
0 &= B_3 I + B_4^\dagger J^\dagger \\
0 &= B_4 I - B_3^\dagger J^\dagger \\
0 &= [B_1, B_3] + [B_4, B_2]^\dagger \\
0 &= [B_1, B_4] + [B_2, B_3]^\dagger.
\end{aligned}
$$

If we try to solve these equations, we find that they imply $B_3 = B_4 = 0$, and we are back at the original moduli space $\overline{\mathcal{M}}(k, n)$. But even though we are producing the same space, there is a new symmetry $\mathrm{SU}(2)_{34}$ rotating B_3 and B_4, and a $U(1)_\chi$ extending the one from last time, which scales

$$(B_1, B_2, B_3, B_4, I, J) \mapsto (e^{i\chi} B_1, e^{i\chi} B_2, e^{-i\chi} B_3, e^{-i\chi} B_4, I, e^{2i\chi} J).$$

New symmetries provide new equivariant parameters. In total, we have four parameters

$$\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$$

with the condition that

$$\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 = 0.$$

The geometric origin of this construction in string theory is the worldvolume theory of a stack of D3 branes, in type IIB string theory. These branes span an $\mathbb{R}^4$ sitting inside a ten-dimensional space. There is a transverse six-dimensional space which we write as $\mathbb{R}^4 \times \mathbb{R}^2$. This $\mathbb{R}^2$ represents the σ field which takes care of the gauge symmetry, and the remaining $\mathbb{R}^4$ is represented by B_3, B_4. The coordinates B_1, B_2 are in some sense "non-commutative coordinates" on the worldvolume. The reason we get the original space $\overline{\mathcal{M}}(k,n)$ back out of these equations is that branes don't like to move in transverse directions. The symmetries in this picture, when properly analyzed, are

$$\mathrm{SU}(2)_{12} \times U(1)_\chi \times \mathrm{SU}(2)_{34}.$$

The fixed locus under this larger symmetry group remains the same, because $B_3 = B_4 = 0$. The analysis of characters for (B_1, B_2, I, J) stays the same, but now there are new equivariant parameters. The partition function becomes

$$Z(\mathsf{q}, \vec{a}, \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) = \sum_{\vec{\lambda}=(\lambda^{(1)},\dots,\lambda^{(n)})} \mathsf{q}^{|\vec{\lambda}|} \prod_{w \in T_{\vec{\lambda}}} \frac{w + \epsilon_3}{w}$$

where the character is now

$$T_{\vec{\lambda}} \coloneqq \sum_j e^{w_j} = N K_{\vec{\lambda}}^* + N^* K_{\vec{\lambda}} q_1 q_2 - P_{12} K_{\vec{\lambda}} K_{\vec{\lambda}}^*$$

where

$$K_{\vec{\lambda}} \coloneqq \sum_{\alpha=1}^{k} e^{a_\alpha} \sum_{(i,j)\in\lambda^{(\alpha)}} q_1^{i-1} q_2^{j-1}$$

This space has a symplectic nature, in the sense that

$$T_{\vec{\lambda}}^* q_1 q_2 = T_{\vec{\lambda}}.$$

In other words, the set of weights $\{w_j\}$ is unchanged under the shift

$$\{w_j\} \leftrightarrow \{\epsilon_1 + \epsilon_2 - w_j\}.$$

This parameter ϵ_3 is sometimes called the **mass** of the adjoint hypermultiplet, or adjoint mass for short. We can recover the partition function of the old theory by taking a limit

$$\epsilon_3 \to \infty, \quad \mathsf{q} \to 0, \quad \mathsf{q}_{\text{old}} \coloneqq \mathsf{q}\epsilon_3^n = \text{constant}.$$

So if we can analyze this theory, we can analyze the old theory as well. Note that if we send $\epsilon_3 \to 0$, we get the generating function for partitions, which is some kind of modular form. This is a non-trivial check of S-duality

$$\mathsf{q} = e^{2\pi i \tau}, \quad \tau \mapsto -\frac{1}{\tau}.$$

What we have really computed is a generating function

$$\sum_{k=0}^{\infty} \mathsf{q}^k \int_{\overline{\mathcal{M}}(k,n)} (\text{Chern polynomial of } T^*\overline{\mathcal{M}}(k,n) \text{ evaluated at } \epsilon_3).$$

Explicitly, this is the sum

$$\sum_{\ell=0}^{2nk} \epsilon_3^\ell c_{2nk-\ell}(T^*\overline{\mathcal{M}}(k,n)).$$

Let's do a calculation with Y-observables in this theory. Recall that, in plethystic form, $Y(x)|_{\vec{\lambda}} = E[e^x S_{\vec{\lambda}}^*]$, where at the specific fixed point $\vec{\lambda}$ the character is

$$S_{\vec{\lambda}} = N - P_{12} K_{\vec{\lambda}}.$$

How does the weight of a particular $\vec{\lambda}$ change when we add a box in $\partial_+\lambda$ or remove a box in $\partial_-\lambda$? Let $\vec{\lambda}'$ be obtained from $\vec{\lambda}$ by adding one box $\square \in \partial_+\lambda^{(\alpha)}$ to $\lambda^{(\alpha)}$. We should compare characters of $T_{\vec{\lambda}}$ and $T_{\vec{\lambda}'}$. Note that

$$K_{\vec{\lambda}'} = K_{\vec{\lambda}} + e^{c_\square + a_\alpha}.$$

Plugging this into $T_{\vec{\lambda}'}$ gives

$$\begin{aligned} T_{\vec{\lambda}'} &= T_{\vec{\lambda}} + N^* q_1 q_2 e^{c_\square + a_\alpha} + N e^{-c_\square - a_\alpha} - P_{12}(\text{quadratic term}) \\ &= T_{\vec{\lambda}} + S^*_{\vec{\lambda}} q_1 q_2 e^{c_\square + a_\alpha} + S_{\vec{\lambda}} e^{-c_\square - a_\alpha} - P_{12}. \end{aligned}$$

Let $P_3 \coloneqq 1 - e^{\epsilon_3}$. Taking plethystic exponential,

$$E[T_{\vec{\lambda}'} P_3] = E[T_{\vec{\lambda}} P_3] \frac{Y(a_\alpha + c_\square + \epsilon_1 + \epsilon_2 + \epsilon_3)|_{\vec{\lambda}}}{Y(a_\alpha + c_\square + \epsilon_1 + \epsilon_2)|_{\vec{\lambda}}} \cdot \frac{Y(a_\alpha + c_\square - \epsilon_3)|_{\vec{\lambda}}}{Y(a_\alpha + c_\square)|_{\vec{\lambda}}} \cdot E[-(1-q_1)(1-q_2)(1-q_3)].$$

This last term is problematic, because it contains a fixed weight -1. Fortunately the denominator also contains exactly the same fixed weight. After canceling these fixed weights, we get

$$\operatorname{Res}_{x = a_\alpha + c_\square} \left(Y(x + \epsilon_1 + \epsilon_2)|_{\vec{\lambda}'} + \mathsf{q} \frac{Y(x - \epsilon_3)|_{\vec{\lambda}} Y(x + \epsilon_1 + \epsilon_2 + \epsilon_3)}{Y(x)} \right) = 0.$$

Hence we have shown that

$$Y(x + \epsilon_1 + \epsilon_2) + \mathsf{q} \frac{Y(x - \epsilon_3) Y(x - \epsilon_4)}{Y(x)}$$

has no poles coming from zeros of $Y(x)$. However it has new poles coming from $Y(x - \epsilon_3)$ and $Y(x - \epsilon_4)$ in the numerator. So now we have to add new terms

$$\mathsf{q}^2 \frac{Y(x - \epsilon_3 + \epsilon_4) Y(x) Y(x - \epsilon_4)}{Y(x) Y(x + \epsilon_2)},$$

which takes care of poles coming from $Y(x - \epsilon_3)$. Continuing this procedure, we end up building the Young graph. The q term corresponds to $\square$, the q^2 term we just added corresponds to $\square\square$, etc. The entire series is therefore

$$\sum_\lambda \mathsf{q}^{|\lambda|} \prod_{\square \in \lambda} S(\epsilon_3(\ell_\square + 1) - \epsilon_4 a_\square) \times \frac{\prod_{\square \in \partial_+ \lambda} Y(x + \sigma_\square + \epsilon_1 + \epsilon_2)}{\prod_{\square \in \partial_- \lambda} Y(x + \sigma_\square)} \tag{9}$$

where $a_\square = \lambda_{i-j}$ and $\ell_\square \coloneqq \lambda_j^t - i$ are arm and leg lengths, and

$$\sigma_\square \coloneqq \epsilon_3(i+1) + \epsilon_4(j+1)$$

and

$$S(x) = \frac{(x + \epsilon_1)(x + \epsilon_2)}{x(x + \epsilon_1 + \epsilon_2)}.$$

Note that this formula feels like a localization formula coming from integrating over Hilb of $\mathbb{C}^2$ with ϵ_3, ϵ_4.

What is this formula (9)? It is the qq-character of $\hat{A}_0$, which we denote $\mathcal{X}_{1,0}(x)$.

Theorem 13.1. *The correlator*

$$\langle \mathcal{X}_{1,0}(x) \rangle = \frac{1}{Z} \sum_{k,\ell=0}^{\infty} \mathsf{q}^{k+\ell} \int_{\overline{\mathcal{M}}(k+\ell,n)} \mathcal{X}_{1,0}^{(\ell)}(x) c_{\epsilon_3}(T^* \overline{\mathcal{M}}(k+\ell, n))$$

has no poles in x.

To understand $\mathcal{X}_{1,0}^{(\ell)}(x)$ we need a generalization. Take a finite subgroup $\Gamma \subset \mathrm{SU}(2)$, which corresponds to a graph whose vertices are irreps R_i labeled with their dimensions a_i and

$$R_i \otimes \mathbb{C}^2 = \bigoplus_j \mathbb{C}^{I_{ij}} \otimes R_j.$$

From this it is clear that $2a_i = \sum_j I_{ij} a_j$. Using this data, formulate the following gauge-theoretic problem over $\mathbb{R}^4$. At each vertex, do the ADHM construction. The gauge group will be

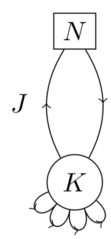
$$G_g \coloneqq \prod_{i \in \text{vertices}} U(N_i)$$

and in addition we introduce bifundamental matter multiplets $\mathrm{Hom}(N_i, N_j)$. This means we will study spin bundles tensored with these bundles. From this we will build a class over $\prod_{i \in \text{vertices}} \overline{\mathcal{M}}(k,n)$. So the resulting partition function involves a whole vector of fugacities $\vec{\mathfrak{q}}$, a whole vector of equivariant parameters $\mathbf{a}$, and so on. The resulting theory is nice when the N_i satisfy the equation

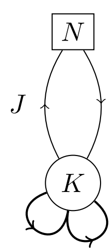
$$2N_i = \sum_j I_{ij} N_j$$

coming from the graph of Γ.

To get this theory from what we already had, we can do the **orbifold construction**. Instead of writing



we should instead combine B_1, B_2 into a single operator $K \to K \otimes \mathbb{C}^2_{12}$ and the same for B_3, B_4. We draw the quiver with fat arrows:



Fix $\Gamma \subset \mathrm{SU}(2)_{34}$, and insist on Γ-invariance in the symmetry group $\mathrm{SU}(2)_{12} \times U(1) \times \mathrm{SU}(2)_{34}$. This means that if g is the 2-dimensional rep of Γ, then

$$g(\gamma)\begin{pmatrix} B_3 \\ B_4 \end{pmatrix} = \begin{pmatrix} H(\gamma)^{-1} B_3 H(\gamma) \\ H(\gamma)^{-1} B_4 H(\gamma) \end{pmatrix}$$

can be undone via some symmetry transformation $H(\gamma)$, and similarly for (B_1, B_2). We should also fix

$$H \colon \Gamma \to U(K), \quad h \colon \Gamma \to U(N),$$

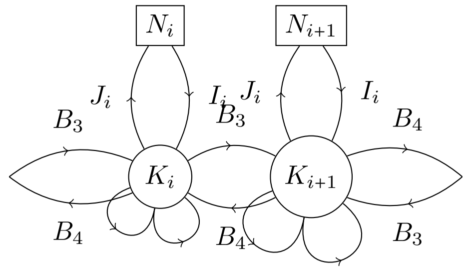
and impose the condition on $I \colon N \to K$ that

$$H(\gamma) I = I h(\gamma)$$

and similarly for J. Producing a representation $\Gamma \to U(K)$ means

$$K = \bigoplus_{i \in \text{vertices}} K_i \otimes R_i$$

and similarly for N. So now our linear algebraic data splits over vertices of the graph Γ, where at each vertex we have the usual ADHM quiver with N_i and K_i. The resulting quiver is



For this theory we can repeat the story for the partition function. We will have observables

$$Y_i(x) := E[-e^x S_i^*], \quad S_i := N_i - P_{12} K_i$$

coming from each vertex. The entire partition function is

$$Z = \sum_{(k_i)} \prod_i \mathfrak{q}_i^{k_i} \sum_{\vec{\lambda}=(\lambda^{(i,\alpha)})} E\Bigg[- \sum_{i \in \text{vertices}} (N_i K_i^* + q_1 q_2 N_i^* K_i - P_{12} K_i K_i^*) + \sum_{e \in \text{edges}} e^{m_e} (N_{t(e)} K_{s(e)}^* + q_1 q_2 N_{s(e)}^* K_{t(e)} - P_{12} K_{t(e)} K_{s(e)}^*) \Bigg].$$

Here $t(e)$ and $s(e)$ are the target and source of the edge e in the McKay graph Γ respectively. The m_e are mass parameters, and correspond to loops in Γ.

The **fundamental qq-character** for Γ is

$$\mathcal{X}_{i,0}(x) := Y_i(x + \epsilon_1 + \epsilon_2) + \mathfrak{q}_i \frac{\prod_{e \to i} Y_{s(e)}(x - m_e) \prod_{e \leftarrow i} Y_{t(e)}(x + \epsilon_1 + \epsilon_2 + m_e)}{Y_i(x)} + \cdots.$$

This $\cdots$ could be fairly complicated, and can be an infinite sum. It is however possible to derive the full expression for ADE quivers, which we'll see tomorrow.

14 Pavel Etingof (Jul 11)

Let's continue from last time, when we wrote a formula

$$R(z) = \begin{pmatrix} 1 & & & \\ & \frac{z-1}{qz-q^{-1}} & \frac{q-q^{-1}}{qz-q^{-1}} & \\ & \frac{z(q-q^{-1})}{qz-q^{-1}} & \frac{z-1}{qz-q^{-1}} & \\ & & & 1 \end{pmatrix}.$$

Call the middle block $B(z)$. For $\mathfrak{sl}_n$, we have an analogous formula, with $V = \mathbb{C}^n = \langle v_1, \ldots, v_n \rangle$. Then R acts by 1 on $v_i \otimes v_i$, and by $B(z)$ on $\langle v_i \otimes v_j, v_j \otimes v_i \rangle$.

The historical approach to this business is the Faddeev–Reshetikhin–Takhtajan formalism (for $\mathfrak{sl}_n$). Given this representation V, define

$$T(z) := (\pi_{V(z)} \otimes \text{id})(\mathcal{R}) \in \mathsf{Mat}_n \otimes U_q(\hat{\mathfrak{sl}}_n)$$

where $\mathcal{R}$ is the universal R-matrix. Then

$$R^{12}(z/w)T^{13}(z)T^{23}(w) = T^{23}(w)T^{13}(z)R^{12}(z/w).$$

What does it mean? If we think about T as a matrix $(t_{ij}(z))$, this is simply the quantum YBE for the universal R-matrix evaluated in $V(z) \otimes V(w)$ in the first two components. The main idea of the FRT approach is to define quantum groups starting from just the R-matrix. It is interesting that in modern appearances of quantum affine algebras in enumerative geometry, R-matrices play the same role.

Given a square matrix A, we can write it as $A = A_- A_0 A_+$ where A_- is strictly lower triangular, A_0 is diagonal, and A_+ is strictly upper triangular. This is true not just for A over a field or commutative ring, but also over a *non-commutative* ring. In particular we can do this for T, over $U_q(\hat{\mathfrak{sl}}_n)$. One has to be careful because some division may be required, but it all works out here because T lies in only one half of the algebra. Hence write

$$T = T_- T_0 T_+.$$

Let the diagonal entries of T_0 be $t^0_{ii}(z)$. They will pairwise commute; this is not immediately obvious. Also,

$$\prod t^0_{ii}(z) = 1,$$

because classically $\det = 1$, and in the quantum setting $\det_q = 1$. Define functions

$$\psi_i(z) \coloneqq \frac{t^0_{ii}(z)}{t^0_{i+1,i+1}(z)}$$

which correspond to roots. Given a finite-dimensional irrep W, there is a highest weight vector $w \in W$ under $U_q(\mathfrak{sl}_n)$ of some weight λ. (A priori there could be several of them, but it turns out they are unique.) If off-diagonal entries of T_+ are t^+_{ij}, then it is clear that $t^+_{ij} w = 0$ for weight reasons. We can choose w to be an eigenvector of $\psi_i(z)$. This gives us a good theory of highest weight, with

$$\psi_i(z) w = d_i(z) w$$

for some power series $d_i(z)$. The elements ψ_i, $t^+_{i,i+1}$, $t^-_{i+1,i}$ are generators.

This is really about a factorization $\mathcal{R} = \mathcal{R}_+ \mathcal{R}_0 \mathcal{R}_-$ of the R-matrix, where $\mathcal{R}_+$ has second component positive, $\mathcal{R}_0$ zero, and $\mathcal{R}_-$ negative $U_q(\mathfrak{sl}_n)$ weights.

Theorem 14.1 (Drinfeld)**.** *There exists a finite-dimensional irrep with highest weight* (d_i) *iff*

$$d_i(z) = \frac{p_i(zq^2)}{p_i(z)}$$

for some polynomials p_i *with constant term* 1*. If this representation exists, it is unique.*

So finite-dimensional irreps are parameterized by $(n-1)$-tuples of subsets of $\mathbb{C}^\times$ with multiplicities, i.e. roots of p_i. These are called **Drinfeld polynomials**. Actually Drinfeld worked them out for Yangians.

The Yangian is a limiting case of this story, where $q = e^{\hbar/2}$, $z = e^{\hbar u}$, and we send $\hbar \to 0$. In this limit,

$$R(z) = \frac{u}{u+1}\left(1 + \frac{P}{u}\right) = \frac{u}{u+1}\begin{pmatrix} 1+\frac{1}{u} & & & \\ & 1 & \frac{1}{u} & \\ & \frac{1}{u} & 1 & \\ & & & 1+\frac{1}{u} \end{pmatrix}$$

where $P(x \otimes y) \coloneqq y \otimes x$ is the permutation. The FRT relation in this limit looks like

$$R^{12}(u-v)T^{13}(u)T^{23}(v) = T^{23}(v)T^{13}(u)R^{12}(u-v)$$

where

$$R(u) \coloneqq 1 + \frac{P}{u}$$

is Yang's R-matrix. This is why the corresponding algebra is called the Yangian. A very nice exercise is to show $R(u)$ satisfies the quantum YBE; it is the simplest non-trivial solution.

Definition 14.2. The **Yangian** $Y(\mathfrak{gl}_n)$ is generated by the entries of the $n{\times}n$ matrices $T^{(k)}$ for $k = 0, 1, 2, \ldots$, with the following relations. Define the generating function

$$T(u) \coloneqq 1 + \sum_{n=0}^{\infty} T^{(n)} u^{-n-1}.$$

Then the relations are

$$R^{12}(u-v)T^{13}(u)T^{23}(v) = T^{23}(v)T^{13}(u)R^{12}(u-v) \in \frac{1}{u-v}\mathbb{C}((u^{-1}, v^{-1})).$$

Note that $T_{ij}^{(0)} = E_{ij}$, so that

$$U(\mathfrak{gl}_n) \subset Y(\mathfrak{gl}_n)$$

and this is true in general as well. One of the advantages of the FRT approach is that the coproduct is very simple. Namely, $\Delta T = T \otimes T$, which means

$$\Delta T_{ij}^{(u)} = \sum T_{ik}^{(u)} \otimes T_{kj}^{(u)},$$

no matter how complicated the algebra.

The Yangian $Y(\mathfrak{gl}_n)$ is almost the same of $Y(\mathfrak{sl}_n)$; the distinction is by a tensor factor of a polynomial ring in infinitely many generators. To get $Y(\mathfrak{sl}_n)$ we need to impose

$$\det_q T = 1.$$

For example, in the 2×2 case the quantum determinant is

$$\det_q \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}(u) = t_1(u)t_{22}(u+1) - t_{12}(u)t_{21}(u+1).$$

Then we can play the same game with factorizing $T = T_+ T_0 T_-$, and then

$$\det_q T(u) = \prod t_{ii}^0(u)$$

which are still pairwise commutative.

For $\mathfrak{sl}_2$, define the roots

$$H_i(z) \coloneqq \frac{t_{ii}^0(z)}{t_{i+1,i+1}^0(z)},$$

and let

$$T^+ = \begin{pmatrix} 1 & X^+ \\ 0 & 1 \end{pmatrix}, \quad T^- = \begin{pmatrix} 1 & 0 \\ X^- & 1 \end{pmatrix}.$$

This yields an infinite collection of series H_n, X_n^+, X_n^-, which we write as

$$H(u) = 1 + \sum_{n \geq 0} H_n u^{-n-1}, \quad X^\pm(u) = \sum_{n \geq 0} X_n^\pm u^{-n-1}.$$

The relations which result from the single RTT relation are

$$[H_k, H_l] = 0, \quad [H_0, X_k^\pm] = \pm 2X_k^\pm, \quad [X_k^+, X_\ell^-] = H_{k+\ell}$$
$$[H_{k+1}, X_\ell^\pm] - [H_k, X_{\ell+1}^\pm] = \pm(H_k X_\ell^\pm + X_\ell^\pm H_k)$$
$$[X_{k+1}^\pm, X_\ell^\pm] - [X_k^\pm, X_{\ell+1}^\pm] = \pm(X_k^\pm X_\ell^\pm + X_\ell^\pm X_k^\pm).$$

In general, if we want $Y(\mathfrak{g})$, we will get such relations for every simple root, and then there will be Serre relations which will look rather horrible. This is another presentation for the Yangian, called the **loop realization**.

There is a third presentation of the Yangian which makes manifest the G-symmetry. The Yangian is a quantization of the enveloping algebra $U(\mathfrak{g}[t])$. Here there is a classical R-matrix

$$r = \frac{\Omega}{z}.$$

So if we take the associated graded with respect to the filtration induced by the k in $T^{(k)}$, we will obtain $U(\mathfrak{g}[t])$. So to produce a presentation of $Y(\mathfrak{g})$ we should start with a presentation of $\mathfrak{g}[t]$ and try to deform it. It has elements $a \in \mathfrak{g}$ and $J(a) \coloneqq at \in \mathfrak{g}t$. If $\mathfrak{g}$ is simple, these suffice as generators. There will be some relations between them, and there is a *unique* way to deform them. This means there is a unique quantization of the Lie bialgebra $U(\mathfrak{g}[t])$, and the result is the Yangian.

Unlike the limit where we degenerate the Yangian to classical algebras, where the representation theory undergoes a drastic change, the degeneration of quantum affine algebras into Yangians does not change the representation theory. Let V_d be a highest weight $d = (d_i)$ representation, i.e.

$$H_i(u)w = d_i(u)w.$$

Proposition 14.3 (Drinfeld)**.** *There exists a finite-dimensional representation (unique) iff*

$$d_i(u) = \frac{p_i(u+1)}{p_i(u)}$$

where p_i is a monic polynomial (Drinfeld polynomial).

So again, representations are parameterized by tuples of such polynomials, or their roots. It is the same story, except we went to logs of all variables.

In the classical theory of compact Lie groups, we have a good notion of character: $\operatorname{tr}_V g$ for reps V. If we restrict to the torus, this is a Weyl-invariant polynomial. So the K-group of the Lie group gets identified with the ring of such polynomials, and knowing the character is equivalent to knowing the representation. However $U(\mathfrak{g}) \subset Y(\mathfrak{g})$, and for every rep of Y we can regard it as just a rep of U and then take its character in the classical Lie theory sense. This will certainly *not* determine the representation, since it will not know about the complex numbers z, i.e. it will not be aware of shifts

$$\tau_a : T(u) \mapsto T(u+a).$$

So we need a notion of character that keeps track of this information. We have some idea where to start because we have these Drinfeld polynomials, which are highest weights.

Let's think about $Y(\mathfrak{sl}_2)$. How do the H_i act on the whole representation? Perhaps to define the character we should look at the eigenvalues of these H_i. They have a generalized eigenbasis, and eigenvalues always have the form

$$\frac{P(u+1)}{P(u)} \frac{Q(u)}{Q(u+1)}$$

for some polynomials P, Q. To record this information, we do the following thing. If

$$\frac{P(u)}{Q(u)} = \frac{\prod_{i=1}^n (u - a_i)}{\prod_{j=1}^m (u - b_j)},$$

then attach to it a polynomial

$$Y_{a_1} \cdots Y_{a_n} Y_{b_1}^{-1} \cdots Y_{b_m}^{-1}$$

for formal variables Y_a indexed by $a \in \mathbb{C}$.

Definition 14.4. The **q-character**

$$\chi_q(V) \in \mathbb{Z}_+[Y_a^{\pm 1} : a \in \mathbb{C}]$$

is the sum of these monomials attached to all eigenvalues.

Theorem 14.5. *The map*

$$\chi_q\colon K_0(\mathsf{Rep}Y) \to \mathbb{Z}[Y_a^{\pm 1} : a \in \mathbb{C}]$$

is an injective ring homomorphism.

This implies all kinds of things, e.g. $K_0(\mathsf{Rep}Y)$ is an integral domain, which is not completely obvious from the start.

Take $V = V_1$ to be a two-dimensional representation of $\mathfrak{sl}_2$ and consider $V_1(a)$ for $a \in \mathbb{C}$. The Drinfeld polynomial in this case is just

$$\frac{P}{Q} = u - a,$$

so the highest weight is Y_a. There is also a second weight, which is the lowest weight. It is a nice computation to show for the lowest weight that

$$\frac{P}{Q} = \frac{1}{u-a-2},$$

Hence the q-character is

$$\chi_q(V_1(a)) = Y_a + Y_{a+2}^{-1}.$$

Now consider the tensor product. By the theorem,

$$\begin{aligned}\chi_q(V_1(a) \otimes V_1(b)) &= (Y_a + Y_{a+2}^{-1})(Y_b + Y_{b+2}^{-1}) \\ &= Y_a Y_b + Y_{a+2}^{-1} Y_b + Y_a Y_{b+2}^{-1} + Y_{a+2} Y_{b+2}^{-1}.\end{aligned}$$

We see immediately that this is irreducible unless $a - b = \pm 2$, because otherwise there are no cancellations and there is only *one* monomial of positive degree, which is the highest weight. To be reducible there must be more than one such monomial. If $a - b = 2$, then $a = b + 2$ and we have

$$Y_a Y_{b+2}^{-1} \rightsquigarrow 1.$$

In this case we split off a trivial rep.

From this we also get the q-character of the three-dimensional rep. Recall that if we take $V_1(0) \otimes V_1(2)$, it fits into a sequence

$$0 \to 1 \to V_1(0) \otimes V_1(2) \to V_2(1) \to 0.$$

Taking q-character gives

$$\chi_q(V_2(1)) = (Y_0 Y_2 + 1 + Y_0 Y_4^{-1} + Y_2^{-1} Y_4^{-1}) - 1.$$

In particular if we want to normalize to 0,

$$\chi_q(V_2(0)) = Y_{-1} Y_1 + Y_{-1} Y_3^{-1} + Y_1^{-1} Y_3^{-1}.$$

Exercise: show that

$$\chi_q(V_{m+1}(0)) = Y_{-1} Y_1 Y_3 \cdots Y_{2m-1} + Y_{-1} Y_1 \cdots Y_{2m-3} Y_{2m+1}^{-1} + \cdots + Y_1^{-1} \cdots Y_{2m+1}^{-1}.$$

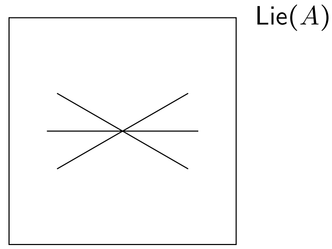
In fact this shows that

$$K_0(\mathsf{Rep}Y) = \mathbb{C}[V_{\omega_i}(a)],$$

because by tensoring we can create an arbitrary highest weight in a unique way.

15 Andrei Okounkov (Jul 12)

Let X be symplectic with an action by $A \subset \mathrm{Aut}(X, \omega)$. Then we had the picture



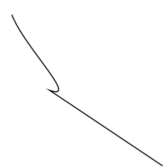
These hyperplanes carry the weights of the A-action on N_{X/X^A}. If $\xi \in \mathsf{Lie}(A)$ is generic, it vanishes only on the fixed locus X^A, but if it sits on one of the hyperplanes then it vanishes on something bigger than X^A. In particular at $0 \in \mathsf{Lie}(A)$ we get X itself. There is a triangle

$$\begin{array}{ccc} X^A & \longrightarrow & X \\ & \searrow \quad \nearrow & \\ & X^{A'} & \end{array}$$

for $A' \subset A$. Each arrow is a correspondence. These correspondences are improved versions of the attracting correspondence

$$\mathrm{Attr} \coloneqq \{(x, f) : \lim_{a \to 0} a \cdot x = f\}$$

where $a = \exp(\xi)$. We saw yesterday that we should fix this by perturbing X and taking the resulting attracting set.



If one does not do this, the resulting correspondences will not make the triangle above commute, and the resulting R-matrix fails Yang–Baxter.

What multiple of the extra cycles do we add? In other words,

$$\overline{\mathrm{Stab}(p_1)} = \mathrm{Attr}(p_1) + c \cdot \overline{\mathrm{Attr}(p_2)} + \cdots$$

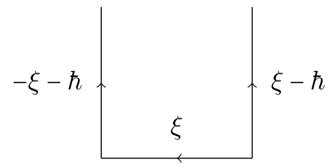
and we want to know what c is. Then we should take the family

$$\begin{array}{ccc} X & \longrightarrow & X_t \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & t \cdot 1. \end{array}$$

Take the *full* torus T, which acts on the base with weight $\hbar^{\pm 1}$. If we turn off the weight $\hbar$, then as we discussed there is an equivariant deformation which makes $\mathrm{Stab}(p_1)|_{p_2}$ miss the point p_2. Hence

$$\mathrm{Stab}(p_1)|_{p_2} = 0 \bmod \hbar.$$

For example, the picture for $T^*\mathbb{P}^1$ (with weights written additively) is



Hence in this case we have

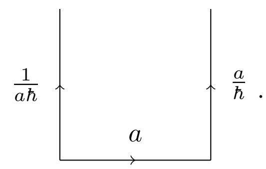
$$\mathrm{Stab}(0)|_\infty = (\xi - \hbar) + c \cdot (-\xi),$$

and if we want this to be 0 mod $\hbar$ we must have $c = 1$.

In K-theory this is not the right thing to do anymore, because there are millions of sheaves we can use to satisfy the analogous condition. We should rephrase the condition in cohomology as follows. Note that in general,

$$\mathrm{Stab}(p_1)|_{p_2} = \text{polynomial in } \xi, \hbar \text{ of degree } \frac{1}{2} \dim X,$$

because it is given by a Lagrangian correspondence. Then the condition that we get $0 \bmod \hbar$ means the degree of this polynomial in ξ is actually *strictly less* than $(1/2) \dim X$. This condition is now sensible in K-theory. For example, $T^*\mathbb{P}^1$ with multiplicative weights is



Let $\mathcal{O}$ be the structure sheaf of the *cohomological* stable envelope. A priori, $\mathcal{O}|_\infty$ is a Laurent polynomial in a and $\hbar$. Actually we have

$$\mathcal{O}|_\infty = 1 - \hbar.$$

What is the "degree" of a Laurent polynomial? It is not a number, and should instead be its Newton polygon, up to translation. So the K-theoretic analogue of the degree condition in cohomology is

$$\text{Newton polygon}_A(\mathrm{Stab}(p_1)|_{p_2}) \subset \text{Newton polygon}_A(\mathrm{Stab}(p_2)|_{p_2}).$$

In general, the K-theoretic stable envelope will be some line bundle supported on the cohomological stable envelope. Because of this line bundle, we can actually allow the lhs Newton polytope to be contained in a *shift* of the rhs polytope. Since these polytopes live in the weight lattice of A, we need to shift by a weight of A. The way to do it is to pick an arbitrary $\mathcal{L} \in \mathrm{Pic}_A(X) \otimes \mathbb{R}$, and write the condition

$$\text{Newton polygon}_A(\mathrm{Stab}(p_1)|_{p_2}) \subset \text{Newton polygon}_A(\mathrm{Stab}(p_2)|_{p_2}) + \text{weight } \mathcal{L}|_{p_2} - \text{weight } \mathcal{L}|_{p_1}.$$

This creates a locally constant dependence on $\mathcal{L}$, and therefore there is some periodic hyperplane arrangement in $\mathrm{Pic}(X) \otimes \mathbb{R}$. We get a *family* of coproducts. Crossing a wall changes the R-matrix by a factor

$$R_{\mathrm{wall}} = \mathrm{Stab}_{\mathcal{L}_-}^{-1} \circ \mathrm{Stab}_{\mathcal{L}_+},$$

which does *not* depend on a spectral parameter. Given an R-matrix $R_\mathcal{L}$, we can cross walls until we get to infinity, where the Drinfeld coproduct lives, and then go back. This gives the Koroshkin–Tolstoy factorization of the R-matrix:

$$R(u) = R_- \underbrace{R^\Delta}_{\text{at } \infty \text{ slope}} R_+.$$

The matrix coefficients of R^Δ generates a commutative subalgebra of which we were taking the trace to get the q-character.

Let's return to cohomology, to talk about R^Δ. If a torus like $\mathrm{diag}(1,1,1,u,u,u)$ acts on the framing for X, we saw that

$$X^u = \bigsqcup X_1 \times X_2$$

for Nakajima varieties. The normal bundle to this product is

$$\mathrm{Ext}^1(F_1, F_2) \oplus \mathrm{Ext}^1(F_2, F_1)$$

in terms of quiver representations. These terms carry weight u and u^{-1} respectively, and are dual to each other with a shift by $\hbar$. Hence R^Δ has the form

$$R^\Delta(u) = \prod \frac{u+w}{u+w+\hbar}$$

where $u+w$ are the Chern roots of $u\,\mathrm{Ext}^1(F_1,F_2)$. When we take matrix elements, it means we fix what happens on F_1 and look at what happens on X_2. In F_2 it is *linear*. Things of the form

$$\mathrm{Ext}^1(\text{fixed}, F)$$

generate *all* characteristic lasses of V_i and W_i.

Now we can discuss qq-characters. First, q-characters are the traces over the K-theory of Nakajima varieties of the operators of classical multiplication. The reason we write them in a funny way is because we use a *particular* generating function. This can be written as

$$(a \otimes -, \mathrm{id}) = \int_{X\times X} i_{\Delta,*}(\alpha)\mathcal{O}_\Delta = \int_X \alpha \cdot \wedge^*_{-1} T^*X.$$

If we have finitely many fixed points, we really get something like a trace, by localization:

$$\sum_{p\in X^T} \alpha|_p.$$

For qq-characters, we just introduce a new variable m:

$$\int_X \alpha \cdot \wedge^*_{-m} T^*X = (\alpha \otimes -, \text{vertex operator}).$$

Recall that $TX = \mathrm{Ext}^1(F,F) = \mathrm{Ext}^1(F_1,F_2)|_{F_1=F_2}$. The vertex operator should be something like

$$\mathrm{Euler}(m \otimes \mathrm{Ext}^1(F_1,F_2)) \tag{10}$$

on $X \times X$. Note that

$$\text{qq-character}(M_1 \otimes M_2) \neq \text{qq-character}(M_1)\text{qq-character}(M_2),$$

precisely because (10) is *bilinear*, and we will get cross-terms corresponding to $\mathrm{Ext}(F_1,F_2)$ and $\mathrm{Ext}(F_2,F_1)$. These cross-terms are dual to each other, and will give contributions like

$$C(m) \coloneqq \prod_w \frac{m+w}{w}\frac{-w-\hbar+m}{-w-\hbar}.$$

What sort of equation does $C(m)$ satisfy? Note that

$$\frac{C(m)}{C(m+\hbar)} = \prod \frac{m+w}{m+w+\hbar}\frac{-w-\hbar+m}{-w+m} = \frac{R^\Delta(m)}{R^\Delta(-m)}.$$

Hence the qq-character is not a homomorphism, but it remembers the Drinfeld coproduct.

16 Nikita Nekrasov (Jul 12)

Let's start by making clearer the class of theories we are studying. Quiver gauge theories is, in a sense, a way of studying

$$\mathsf{Maps}(M^4, \text{quiver varieties}).$$

In Petr's lecture, we saw that in studying maps from two-dimensional manifolds, there is a way to compactify the moduli space called quasimaps. Now we do this for four-dimensional manifolds.

Given a quiver γ, write

$$\mathsf{Edge}_\gamma \overset{t}{\underset{s}{\to}} \mathsf{Vert}_\gamma$$

for source and target of edges. To each vertex $i \in \mathsf{Vert}_\gamma$ of the quiver we associate two complex vector spaces M_i and N_i. The dimensions $m_i \coloneqq \dim M_i$ and $n_i \coloneqq \dim N_i$ are parameters of the theory. We call M_i the space of **flavors** and N_i the space of **colors**. Using these spaces we can study

$$\mathcal{M}_\gamma = \begin{pmatrix} \text{moduli space of collections of torsion-free sheaves} \\ (\mathcal{E}_i)_{i \in \mathsf{Vert}_\gamma} \text{ over } \mathbb{P}^2 \text{ framed by } \mathcal{E}_i|_{\mathbb{P}^1_\infty} \cong N_i \otimes \mathcal{O}_{\mathbb{P}^1_\infty} \end{pmatrix}.$$

The structure of a quiver gives the following structure: take

$$\mathrm{Obs} \coloneqq \bigoplus_{e \in \mathsf{Edge}_\gamma} \mathrm{Hom}(\mathcal{E}_{s(e)}, \mathcal{E}_{t(e)}) \oplus \bigoplus_{i \in \mathsf{Vert}_\gamma} \mathrm{Hom}(\mathcal{E}_i, M_i \otimes \mathcal{O}_{\mathbb{P}^2}).$$

Let $\mathcal{E}_i \to \mathcal{M}_\gamma \times \mathbb{P}^2$ be the *universal* sheaf. Let

$$\pi \colon \mathcal{M}_\gamma \times \mathbb{P}^2 \to \mathcal{M}_\gamma$$

be the projection. Then we can take

$$R\pi_* \mathrm{Obs} \in K(\mathcal{M}_\gamma).$$

In cohomology, we then take its equivariant Euler class and integrate to get the partition function:

$$Z \coloneqq \sum \vec{\mathsf{q}}^{\vec{k}} \int_{\overline{\mathcal{M}}_\gamma^{\vec{k}}} e(\mathrm{Obs}_\gamma).$$

For general quivers, we should view $\mathsf{q}^{\vec{k}}$ as formal parameters; we don't expect any convergence.

The other parameters correspond to $\mathrm{GL}(M_i)$ and $\mathrm{GL}(N_i)$ symmetries. The $\mathrm{GL}(M_i)$ only act on fibers of $M_i \otimes \mathcal{O}_{\mathbb{P}^2}$. Denote the parameters by

$$\underbrace{\vec{m}}_{\text{masses}} \times \underbrace{\vec{a}}_{\text{Coulomb moduli}} \in \mathsf{Lie}\left(\text{maximal torus}(\textstyle\prod \mathrm{GL}(M_i) \times \mathrm{GL}(N_i))\right).$$

Write $\vec{a} = (a_{i,\alpha})$ where $\alpha = 1, \dots, n_i$, and $\vec{m} = (m_{i,f})$ where $f = 1, \dots, m_i$. We can also scale edges, yielding an additional set of parameters

$$(m_e) \in (\mathbb{C}^\times)^{\mathsf{Edge}_\gamma}$$

called *bifundamental* masses. However there is a freedom to use the action on vertices to *undo* this edge action, so the total number of degrees of freedom from edges is actually

$$(\mathbb{C}^\times)^{\mathsf{Edge}_\gamma} / (\mathbb{C}^\times)^{\mathsf{Vert}_\gamma - 1} = (\mathbb{C}^\times)^{b_1(\gamma)}.$$

This relates back to what we said yesterday about these parameters not doing anything for simply-connected quivers.

Finally, there is a $\mathrm{GL}(2)$ acting by rotation on $(\mathbb{P}^2, \mathbb{P}^1_\infty)$. The parameters of the maximal torus of this $\mathrm{GL}(2)$ are called ϵ_1, ϵ_2.

The Y_i observables we introduced arise as the equivariant Euler class

$$\mathrm{Euler}(\text{“}\mathcal{E}_i|_{\mathcal{M}_\gamma \times 0}\text{”}).$$

The formal way to define this is

$$c_x(R\pi_*(\mathcal{E}_i \otimes^L \mathcal{O}_0))$$

where $\mathcal{O}_0$ is the structure sheaf of $0 \in \mathbb{P}^2$ and c_x is the Chern polynomial evaluated at x. We computed yesterday that at a fixed point $\vec{\lambda} \in \mathcal{M}_\gamma$,

$$Y_i(x)|_{\vec{\lambda}} = \prod_{\alpha=1}^{n_i}(x - a_{i,\alpha}) \prod_{\square \in \lambda^{(i,\alpha)}} \frac{(x - a_{i,\alpha} - c_\square - \epsilon_1)(x - a_{i,\alpha} - c_\square - \epsilon_2)}{(x - a_{i,\alpha} - c_\square)(x - a_{i,\alpha} - \epsilon_1 - \epsilon_2)}.$$

Here $c_\square$ for a square at (a,b) is the *content*

$$c_\square \coloneqq \epsilon_1(a-1) + \epsilon_2(b-1).$$

For the top-most square, the first factor in the denominator cancels the prefactor $(x - a_{i,\alpha})$. There are a bunch of cancellations, and the true numerator and denominator only care about $\partial_+\lambda$ and $\partial_-\lambda$ respectively.

As $x \to \infty$, the leading-order term in Y_i is

$$x^{n_i} \exp \sum_{\ell=1}^{\infty} \frac{1}{\ell x^\ell} \sum_\alpha \left(-a_{i,\alpha}^\ell + \sum_{\square \in \lambda^{i,\alpha}} \left((a + c_\square)^\ell + (a + c_\square + \epsilon_1 + \epsilon_2)^\ell - (a + c_\square - \epsilon_1)^\ell - (a + c_\square - \epsilon_2)^\ell \right) \right).$$

When $\ell = 1$, everything cancels, so we get

$$x^{n_i} \exp\left(\frac{1}{x}\left(-\sum_\alpha a_{i,\alpha} \right) + \frac{1}{2x^2}\left(-\sum_\alpha a_{i,\alpha}^2 + 2\epsilon_1\epsilon_2 \sum_\alpha |\lambda^{(i,\alpha)}| \right) + \frac{1}{3x^3}\left(-\sum_\alpha a_{i,\alpha}^3 + \epsilon_1\epsilon_2 \sum (c_\square + \epsilon k_i) \right) + \cdots \right).$$

In the second term, $\sum_\alpha |\lambda^{(i,\alpha)}| = k_i$, and in the third term we did not explicitly identify the numerical factor ϵ.

Fix $w_i \in \mathbb{Z}_{\geq 0}$. Start with a product of the following type:

$$\mathcal{X}_{\vec{w}}(x) \coloneqq \prod_{i \in \mathsf{Vert}_\gamma} \prod_{\beta=1}^{w_i} Y_i(x + w_{i,\beta}) + \begin{pmatrix} \mathfrak{q}\text{-dependent corrections involving } Y, Y^{-1} \\ \text{such that all poles in } x \text{ cancel} \end{pmatrix}$$

where $w_{i,\beta} \in \mathbb{C}$ with $\beta = 1, \ldots, w_i$. Here when we say poles cancel, we mean in the expectation value

$$\langle \mathcal{X}_{\vec{w}}(x) \rangle = \frac{1}{Z} \sum \mathfrak{q}^{\vec{k}} \int_{\mathcal{M}_\gamma^{\vec{k}}} \mathrm{Euler}(\mathcal{M}_\gamma \times \mathcal{X}_{\vec{w}}(x)),$$

i.e. sum over *all* partitions. The quantity $\mathcal{X}_{\vec{w}}(x)$ is the qq**-character** with **highest weight** $\vec{w}$.

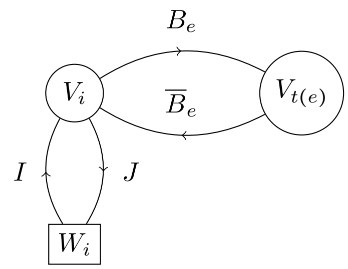
The geometric origin of $\mathcal{X}_{\vec{w}}(x)$ is as follows. Recall that the Nakajima quiver variety associated to data $\vec{v}, \vec{w}$ is

$$\mathcal{M}(\vec{w}, \vec{v}) \coloneqq T^* \left(\bigoplus_{i \in \mathsf{Vert}_\gamma} \mathrm{Hom}(W_i, V_i) \oplus \bigoplus_{e \in \mathsf{Edge}_\gamma} \mathrm{Hom}(V_{s(e)}, V_{t(e)}) \right) /\!\!/ \prod_{i \in \mathsf{Vert}_\gamma} \mathrm{GL}(V_i).$$

Let $G_{\vec{w}} \coloneqq \prod \mathrm{GL}(W_i)$, which acts on $\mathcal{M}(\vec{w}, \vec{v})$. There is another torus $T_\gamma \coloneqq \mathbb{C}^\times \times (\mathbb{C}^\times)^{b_1}$ where the first $\mathbb{C}^\times$ is overall scaling. The terms $(x + w_{i,\beta})$ are equivariant parameters of the framing torus $T_{\vec{w}}$. We identify $-(\epsilon_1 + \epsilon_2)$ with the weight of the overall scaling $\mathbb{C}^\times$, and call the weights of $(\mathbb{C}^\times)^{b_1}$ by $(m_e)_{e \in \mathsf{Edge}_\gamma}$. Now for fixed $\vec{w}$, we have

$$\mathcal{X}_{\vec{w}}(x) = \sum_{\vec{v} \in \mathbb{Z}_{\geq 0}^{\mathsf{Vert}_\gamma}} \vec{\mathfrak{q}}^{\vec{v}} \int_{\mathcal{M}(\vec{w}, \vec{v})} c_{\epsilon_2 \text{ or } \epsilon_1}(T^*\mathcal{M}(\vec{w}, \vec{v})) \cdot \mathcal{C}_x.$$

What is $\mathcal{C}_x$? We need to define the *canonical complex* $\mathcal{C}_i$ over $\mathcal{M}(\vec{w},\vec{v})$. For fixed $i \in \mathsf{Vert}_\gamma$, consider



Form the following map using all arrows coming and going from V_i:

$$V_i \xrightarrow{(J_i, B_e, \overline{B}_e)} W_i \oplus \bigoplus_{s(e)=i} V_{t(e)} \oplus \bigoplus_{t(e)=i} V_{s(e)} \xrightarrow{(I_i, -\overline{B}_e, B_e)} V_i.$$

This is a generalization of the object we called S for the ADHM construction. There are weights q on some arrows, so that in the end the K-class of the complex is

$$\mathcal{C}_i = W_i + \sum_{s(e)=i} e^{m_e} V_{t(e)} + q^{-1} \sum_{t(e)=i} e^{-m_e} V_{s(e)} - V_i - q^{-1} V_i.$$

It turns out, like for S, that $H^2 = 0$ and hence

$$\mathrm{ch}(\mathcal{C}_i) = \mathrm{ch}(H_i^1) - \mathrm{ch}(H_i^0) = \sum_A e^{\xi_{i,A^+}} - \sum_{A'} e^{\xi^-_{i,A'}}.$$

The formula for $\mathcal{C}_x$ in the integrand is therefore

$$\prod_{i \in \mathsf{Vert}_\gamma} \frac{\prod_A Y_i(x + \xi^+_{i,A})}{\prod_{A'} Y_{i'}(x + \xi^-_{i,A'})}.$$

The overall formula looks like a localization formula. It turns out that

$$\langle \mathcal{X}_{\vec{w}}(x) \rangle = \int_{\substack{\text{moduli of quiver} \\ \text{crossed instantons}}} 1.$$

The absence of poles will be a consequence of this new moduli being compact. (Remember that $\mathcal{M}(\vec{w},\vec{v})$ is not compact.)

Let's see $\mathcal{X}_{\vec{w}}(x)$ in some simple examples. The simplest are A-type theories. We will discuss three cases:

1. A_1 case with a general $\vec{w}$;
2. $\hat{A}_0$ case for fundamental $\vec{w}$;
3. finite A_r case for fundamental $\vec{w}$.

In the A_1 case, $M = \mathbb{C}^m$ and $N = \mathbb{C}^n$. The Nakajima variety is

$$\mathcal{M}_{A_1}(w,v) = T^* \operatorname{Gr}(v,w).$$

This is non-empty only when $0 \le v \le w$, so $\mathcal{X}_w(x)$ is a finite sum. The torus is $\mathbb{C}^\times \times (\mathbb{C}^\times)^w$, with $\binom{w}{v}$ fixed points given by coordinate planes. Label them by ways of decomposing $\{1,\dots,w\} = I \sqcup J$. Hence

$$\mathcal{X}_{\vec{w}}(x) = \sum_{I \sqcup J} \mathsf{q}^{|I|} \prod_{\substack{i \in I \\ j \in J}} \frac{(w_i - w_j - \epsilon_1)(w_i - w_j - \epsilon_2)}{(w_i - w_j)(w_i - w_j - \epsilon_1 - \epsilon_2)} \cdot \prod_{i \in I} Y(x + w_i + \epsilon) \prod_{j \in J} \frac{P(x + w_j)}{Y(x + w_j)}.$$

There are 2^w terms. We should think of it as some sort of deformed character of $\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$. The claim is that this expression has no poles in x when we substitute it into the integral. However one can tune the w_i parameters to acquire poles. It is interesting that sometimes one can tune parameters such that the fixed locus is still compact, so the expression is still non-singular, but the fixed points are non-isolated.

17 Nikita Nekrasov (Jul 12)

We already saw the fundamental qq-character for the quiver with one vertex and one edge. This was

$$\mathcal{X}_{1,0}(x) = \sum_\lambda \mathfrak{q}^{|\lambda|} \cdots.$$

The last example is the finite A_r case for the fundamental qq-character. Choose all bifundamental masses $m_e = -\epsilon$. Set all $P_i = 1$ for simplicity. The ℓ-th fundamental qq-character, labeled by the choice of a vertex, is

$$\mathcal{X}_\ell(x) = \frac{1}{z_0 \cdots z_{r+1}} \sum_{\substack{I \subset \{0,1,\ldots,r\} \\ |I|=\ell}} \prod_{i \in I} \Lambda_i(x + \epsilon(h_i + 1 - \ell)).$$

This should be compared to characters $\mathrm{tr}_{\wedge^\ell \mathbb{C}^{r+1}} g$ for $g \in \mathrm{SL}(r+1)$; the Λ_i will be eigenvalues. They are defined as

$$\Lambda_i(x) := z_i \frac{Y_{i+1}(x+\epsilon)}{Y_i(x)}, \quad i = 0, \ldots, r$$

with the convention $Y_0 = Y_{r+1} = 1$. The z_i are redundant variables

$$z_i := z_0 \mathfrak{q}_1 \cdots \mathfrak{q}_i, \quad i = 1, \ldots, r.$$

Finally, h_i are height functions

$$h_i := \#\{i' : i' \in I,\, i' < i\}.$$

The main difference between this and characters in $\mathrm{SL}(r+1)$ over $\wedge^\ell \mathbb{C}^{r+1}$ is the shifts in Λ_i.

The first case in which we cannot escape taking derivatives of Y functions is the D_4 quiver, where there are four fundamental qq-characters $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \mathcal{X}_4$. They each have a simple 8-term structure. D_4 corresponds to the Lie algebra $\mathfrak{spin}(8)$; the outer nodes correspond to vector and spinor representations (which are 8-dimensional) and the middle node correspond to the adjoint rep (which is 28-dimensional). The actual formula contains 28 simple terms, plus a term

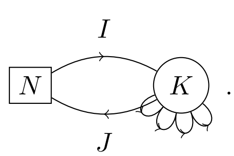
$$\mathfrak{q}_1 \mathfrak{q}_2^2 \mathfrak{q}_3 \mathfrak{q}_4 \frac{Y_2(x)}{Y_2(x-\epsilon)} \left(2\left(\frac{1 - \epsilon_1\epsilon_2}{\epsilon^2} \right) + \frac{\epsilon_1 \epsilon_2}{\epsilon} \partial_x \log\left(\frac{Y_2(x) Y_2(x-\epsilon)}{Y_1(x) Y_3(x) Y_4(x)} \right) \right).$$

This derivative term is not seen in the half-classical limit, where either one of the ϵ_i's go to zero. It comes from the fact that inside the Nakajima variety of D_4 type

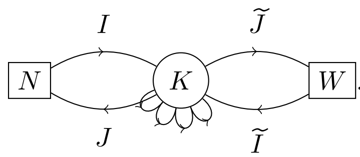
$$\mathcal{M}_{D_4}(\vec{w} = (0,1,0,0), \vec{v} = (1,2,1,1))$$

there is a $\mathbb{P}^1$ fixed by the global symmetry, i.e. non-isolated fixed points.

Let's return now to the ADHM construction, to discuss **crossed instantons**. Recall that we said we'd like to enhance the space of matrices:



This hinted at some symmetry between B_1, B_2, B_3, B_4, but clearly B_1, B_2 were attached to I and J and B_3, B_4 were not. To make it symmetric, let's introduce a second framing space and a new set of $\widetilde{I}, \widetilde{J}$:



This means we should distinguish between the *left* moment map

$$\mu_{\mathbb{C}}^{\ell} = [B_1, B_2] + IJ$$

and the *right* moment map

$$\mu_{\mathbb{C}}^{r} = [B_3, B_4] + \widetilde{I}\widetilde{J}.$$

The moment map equations become

$$\begin{aligned} 0 &= \mu_{\mathbb{C}}^{\ell} + (\mu_{\mathbb{C}}^{r})^{\dagger} \\ 0 &= [B_1, B_3] + [B_4, B_2]^{\dagger} \\ 0 &= [B_1, B_4] + [B_2, B_3]^{\dagger}. \end{aligned}$$

These combine into the equation

$$\sum_{a=1}^{4} [B_a, B_a^{\dagger}] + II^{\dagger} + \widetilde{I}\widetilde{I}^{\dagger} - J^{\dagger}J - J\widetilde{J}^{\dagger}\widetilde{J} = r \cdot 1.$$

Similarly, we can do the same for the real moment maps:

$$\begin{aligned} \mu_{\mathbb{R}}^{\ell} &\coloneqq [B_1, B_1^{\dagger}] + [B_2, B_2^{\dagger}] + II^{\dagger} - J^{\dagger}J \\ \mu_{\mathbb{R}}^{r} &\coloneqq [B_3, B_3^{\dagger}] + [B_4, B_5^{\dagger}] + \widetilde{I}\widetilde{I}^{\dagger} - \widetilde{J}^{\dagger}\widetilde{J}, \end{aligned}$$

and the real moment map equation is

$$\mu_{\mathbb{R}}^{\ell} + \mu_{\mathbb{R}}^{r} = \zeta \cdot 1_K.$$

These are all equations which take values in square matrices, and one can check there are an equal number of variables in $B_1, \ldots, B_4$ and equations. We can now impose equations that govern the interaction between B_i matrices and I, J matrices. These equations "separate" the B_3, B_4 action from the I, J action:

$$\begin{aligned} 0 &= B_3 I + B_4^{\dagger} J^{\dagger} \\ 0 &= B_4 I - B_3^{\dagger} J^{\dagger} \\ 0 &= B_1 \widetilde{I} + B_2^{\dagger} \widetilde{J}^{\dagger} \\ 0 &= B_2 \widetilde{I} - B_1^{\dagger} \widetilde{J}^{\dagger}. \end{aligned}$$

Again, there are an equal number of variables in the I, J variables and equations. But now we will impose more equations! These will forbid relations between I, J and $\widetilde{I}, \widetilde{J}$:

$$0 = \widetilde{J} I - \widetilde{I}^{\dagger} J^{\dagger}.$$

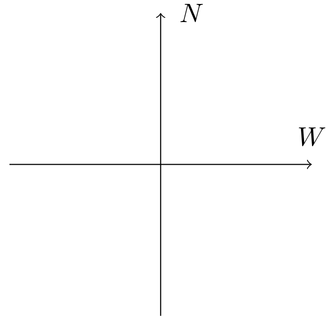
Squaring these equations and expanding, we get relations like

$$\begin{aligned} B_3 I = B_4 I = B_1 \widetilde{I} = B_2 \widetilde{I} = 0 \\ J B_3 = J B_4 = \widetilde{J} B_1 = \widetilde{J} B_2 = 0 \\ \widetilde{J} I = J \widetilde{I} = 0. \end{aligned}$$

These equations also imply that the left and right moment maps vanish *separately*. So we do get ADHM equations for (B_1, B_2, I, J) and $(B_3, B_4, \widetilde{I}, \widetilde{J})$ independently, but without the stability condition and they are not completely independent. The only sort of thing that glues them together is the stability condition, which says

$$K = K_{12} + K_{34}, \quad K_{12} \coloneqq \mathbb{C}[B_1, B_2] I(N), \quad K_{34} \coloneqq \mathbb{C}[B_3, B_4] \widetilde{I}(W).$$

The word *crossed* is in the quaternionic sense: there are two quaternionic planes intersecting in an eight-dimensional space, and we are describing instantons that move along such a space.



There is a sheaf interpretation of this: we get two torsion-free rank n and w sheaves

$$\mathcal{E}_{12} \to \mathbb{P}_{12}, \quad \mathcal{E}_{34} \to \mathbb{P}_{34},$$

with $\mathrm{ch}_2(\mathcal{E}_{12}) = k_{12} \coloneqq \dim K_{12}$ and similarly for $\mathcal{E}_{34}$. They are trivialized at their respective $\mathbb{P}^1_\infty$'s.

But these equations describe more structure, which is the *intersection* of these two spaces $K_{12} \cap K_{34}$, which need not be empty. This is a space which should be annihilated by B_1, B_2 inside K_{12}, and by B_3, B_4 inside K_{34}.

Introduce symmetries $\mathrm{GL}(N) \times \mathrm{GL}(W) \times (\mathbb{C}^\times)^3$, whose maximal torus has weights h and $\widetilde{h}$ and q_i with $\prod_{i=1}^4 q_i = 1$. Then

$$\begin{aligned} B_a &\mapsto q_a B_a \\ J &\mapsto h^{-1} q_1 q_2 J \\ \widetilde{J} &\mapsto \widetilde{h}^{-1} q_3 q_4 \widetilde{J} \\ I &\mapsto Ih \\ \widetilde{I} &\mapsto \widetilde{I}h. \end{aligned}$$

The anti-diagonal torus

$$\mathbb{C}^\times_{\text{anti-diag}} \subset \mathbb{C}^\times_N \times \mathbb{C}^\times_W$$

acts non-trivially on the intersection, and the variable x is exactly the weight of this action. In a single ADHM construction the action is trivial; we don't see it.

Using $\mathrm{GL}(N)$ and $\mathrm{GL}(W)$, we split everything into 1-dimensional eigenspaces. On the fixed locus, $J = \widetilde{J} = 0$, and fixed points should be drawn as *pairs* of Young diagrams λ, μ. In the first Young diagram, we have as usual steps by ϵ_1, ϵ_2 and

$$c_\square = \epsilon_1(i-1) + \epsilon_2(j-1).$$

Every box here is an eigenline in K, and are indexed by $a_\alpha + c_\square$. In the second Young diagram, we have steps by ϵ_3, ϵ_4 and

$$\sigma_\square = \epsilon_3(i-1) + \epsilon_4(j-1).$$

Boxes here are indexed by $w_\beta + \sigma_\square$. Imagine now that the number of boxes keep growing and the two pictures overlap. This is possible when $K_{12} \cap K_{34} \neq \varnothing$. They can only overlap on $\partial_- \lambda$, which are killed by B_1, B_2, and $\partial_- \mu$, which are killed by B_3, B_4. We call this **kissing**.

We look at relations between weights $a_\alpha + c_\square$ and $w_\beta + \sigma_\square$. For certain special points on the torus T, some of these weights may coincide. Then we have a larger fixed locus. Relations will be of the form

$$a_\alpha + n_1\epsilon_1 + n_2\epsilon_2 + n_3\epsilon_3 = w_\beta + m_1\epsilon_1 + m_2\epsilon_2 + m_3\epsilon_3,$$

and therefore will be functions of $x \coloneqq a_\alpha - w_\beta$. This is the *same* x that appears in the definition of Y.

There are special sub-tori for which we do get non-compact fixed loci. But they have nothing to do with the x coordinate; we would avoid them even for the ADHM moduli space. There is a class of sub-tori for which the fixed loci are compact. It applies not only to the moduli we just described, but also to all orbifold versions of it. There was a group $\mathrm{SU}(2)_{12} \times U(1) \times \mathrm{SU}(2)_{34}$. If we take a finite subgroup Γ and fix its homomorphisms $\rho: \Gamma \to U(N) \times U(W)$, then we can look at $(\mathcal{M}_{N,W}^{\text{crossed}})^\Gamma$ and its deformation

$$(\widetilde{\mathcal{M}_{N,W}^{\text{crossed}}})^\Gamma.$$

The deformation means the following. The $U(K)$ symmetry gets broken here into $\prod_i U(K_i)$, where $i \in \mathsf{Irrep}(\Gamma)$. Then $K = \bigoplus_i K_i \otimes R_i$, and

$$(\widetilde{\mathcal{M}_{N,W}^{\text{crossed}}})^\Gamma = \bigsqcup_{\vec{k}} \overline{\mathcal{M}}_{\vec{N},\vec{W}}^{\text{crossed}}(\vec{k}).$$

We can now change the levels, in the real moment map, for each of the irreps. This construction covers quiver theories on ALE spaces, which is where Γ acts on $\mathrm{SU}(2)_{12}$, and in the presence of surface defects, which is where Γ acts on $\mathrm{SU}(2)_{12} \times U(1)$.

Let's distill this theory in the case of A_1 theory with fundamental matter. Recall that this theory has two parameters n and m, and the simplest qq-character has the form

$$\mathcal{X}_{1,0}(x) = Y(x+\epsilon) + \mathfrak{q}\frac{P(x)}{Y(x)}.$$

Now we want to add a surface defect at $z_2 = 0$. This is obtained by saying that we study instantons on the p-fold covering space, with restriction on how they transform as we go around the origin. Let the new coordinates be $(w_1, w_2) \coloneqq (z_1, z_2^p)$. Fix homomorphisms

$$\rho_N, \rho_M: \mathbb{Z}/p \to M, N$$

into the color and flavor spaces; the defect is specified by these homomorphisms. They break the gauge group into the subgroup which commutes with them.

One example which is interesting is called the **regular defect**, where $p = N$ and N is a regular representation of $\mathbb{Z}/N$, i.e. if we decompose N into irreps, each rep occurs once. The good theory is where M is *two* copies of the regular rep. When the number of masses is twice the number of colors, something special happens. In $\mathcal{X}_{1,0}(x)$, the degree of the two terms are n and $m-n$. We would like the second term to be subdominant, so $n \geq m-n$. This happens to coincide with a nice condition in gauge theory called asymptotic freedom and has to do with the ability to do things perturbatively in the coupling constant. The equality case is the critical case.

Now we can repeat the story of universal sheaves and everything. Instantons in the orbifold theory are characterized by n instanton charges, instead of 1; they *fractionalize*. We can interpret this roughly as follows. Given a four-dimensional ambient space $\mathbb{R}^4$ with a surface defect $\mathbb{R}^2$, the surface becomes some space with $U(N)$ symmetry in the orbifold picture, e.g. $U(N)/U(1)^N$ with

$$\pi_2(U(N)/U(1)^N) = \mathbb{Z}^{N-1}.$$

So $k \rightsquigarrow (k_0, k_1, \ldots, k_{N-1})$. Accordingly, the Y observable also gets fractionalized. Instead of a single observable $Y(x)$, we now have N of them:

$$Y(x) \rightsquigarrow (Y_\omega(x))_{\omega=0,\ldots,N-1},$$

These are labeled by irreps of the cyclic group. Since we fixed a homomorphism of the cyclic group into the color space, assume $N = \bigoplus N_\alpha$ commutes with the decomposition into irreps of $\mathbb{Z}/N$; we'll call

$$R_{c(\alpha)} \coloneqq N_\alpha,$$

i.e. for each color there is an irrep. The old formula had the product structure

$$\prod(x-a_\alpha)\frac{\prod(x-a_\alpha-c_\square-\epsilon_1)(x-a_\alpha-c_\square-\epsilon_2)}{\prod(x-a_\alpha-c_\square)(x-a_\alpha-c_\square-\epsilon_1-\epsilon_2)}$$

We will keep only those factors with $\mathbb{Z}/n$-parity equal to ω. For this we need to keep track of what $\mathbb{Z}/n$-reps each variable carries:

$$[x]=0,\quad [\epsilon_1]=0,\quad [\epsilon_2]=1,\quad [c_\square]=j-1.$$

Hence we have a formula

$$Y_\omega(x)\left(\prod_{\alpha,c(\alpha)=\omega}(x-a_\alpha)\right)\prod_\beta\prod_{\substack{\square\in\lambda^{(\beta)}\\ c(\beta)+j-1\equiv\omega(n)}}\frac{x-a_\beta-c_\square-\epsilon_1}{x-a_\beta-c_\square}\times\prod_\beta\prod_{\substack{\square\in\lambda^{(\beta)}\\ c(\beta)+j\equiv\omega \bmod n}}\frac{x-a_\beta-c_\square-\epsilon_4}{x-a_\beta-c_\square-\epsilon_1-\epsilon_2}.$$

The old Y-observable is equal to $\prod_\omega Y_\omega$, and so it really has been fractionalized. Note that the gauge coupling q also gets fractionalized.

It follows from the same compactness theorem that

$$\langle Y_{\omega+1}(x+\epsilon_1+\epsilon_2)+\mathsf{q}_\omega\frac{P_\omega(x)}{Y_\omega(x)}\rangle$$

has no poles in x. It is beneficial to expand it for large x and look at negative-degree terms in x and set them to zero. To get something meaningful in the old setting, we had to expand to something like x^{-n-1}. But in setting it is enough to expand Y_ω only to second-order!

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Now we will expand the qq-character as $x\to\infty$, with the following purpose in mind. Since this expectation value has no poles in x, in its asymptotic expansion

$$x(a+\mathsf{q}_\omega)+\cdots$$

where $\cdots$ means terms of lower degree. But this means it is actually a linear function. Hence we can write it as

$$\langle(1+\mathsf{q}_\omega)x+\hat{u}_\omega+\hat{u}_\omega^{(1)}x^{-1}+\cdots\rangle,$$

and obtain

$$\langle\hat{u}_\omega^{(1)}\rangle=0\quad\forall\omega.$$

Doing $\hat{u}_\omega^{(1)}$ will be enough, even though this is true for all $\hat{u}_\omega^{(k)}$. The large-x expansion of $Y_\omega(x)$ is

$$(x-a)\exp\sum_{\substack{\beta,\square\in\lambda^{(\beta)}\\ c(\beta)+j-1\equiv\omega \bmod n}}\left(-\frac{\epsilon_1}{x-a-c}+\frac{\epsilon_1^2}{2x^2}\right)+\sum_{\substack{\beta,\square\in\lambda^{(\beta)}\\ c(\beta)+j\equiv\omega \bmod n}}\left(\frac{\epsilon_1}{x-a-c_\square}+\frac{\epsilon_1^2}{2x^2}\right).$$

Expanding each term further yields

$$(x-a)\exp\left(-\frac{\epsilon_1k_\omega}{x}-\frac{\epsilon_1\sigma_\omega}{x^2}+\frac{\epsilon_1k_\omega}{2x^2}+\frac{\epsilon_1}{x}k_{\omega-1}+\frac{\epsilon_1(\sigma_{N-1}+\epsilon_2k_{\omega+1})}{x^2}+\frac{\epsilon_1^2k_{\omega-1}}{2x^2}\right).$$

The x^{-1} coefficient in this expression is therefore

$$\frac{\epsilon_1^2}{2}(k_{\omega+1}-k_\omega)^2-a\epsilon_1(k_{\omega-1}-k_\omega)+\epsilon_1(\sigma_{\omega-1}-\sigma_\omega+\epsilon_2k_{\omega-1})+\frac{1}{2}\epsilon_1^2(k_\omega+k_{\omega-1}).$$

Hence we see that

$$\langle \text{some function of } k\text{'s } + \epsilon_1(\sigma_{\omega-1} - \sigma_\omega)\rangle = 0.$$

The function of k's is a second-order differential operator in the $\mathfrak{q}$'s applied to the partition function. So we have obtained some identity like: a differential operator in $\vec{\mathfrak{q}}$ is an expectation value of $\langle \epsilon_1\sigma_{\omega-1} - \sigma_\omega\rangle$. If we now sum over all $\vec{k}$, then these expectation values telescope and we get a non-trivial differential equation satisfied by the partition function.

It is useful to isolate the ω's in the following way. The fugacity for the bulk instanton charge is

$$\mathfrak{q} := \prod_{\omega=0}^{n-1} \mathfrak{q}_\omega.$$

We write these as

$$\mathfrak{q}_\omega = \frac{z_\omega}{z_{\omega-1}}, \quad \omega = 1, \ldots, n-1.$$

This differential operator we obtained is first-order in $\mathfrak{q}$. It is beneficial to multiply the partition function by a prefactor:

$$\Psi := \prod_\omega \mathfrak{q}_\omega^{k_\omega} \prod_\omega z_\omega^{a/\epsilon_1} Z.$$

Then the differential operator is exactly

$$\epsilon_1\epsilon_2\mathfrak{q}\frac{d}{d\mathfrak{q}}\Psi = \mathcal{D}_2(z\frac{\partial}{\partial z})\Psi,$$

which is exactly the KZ equation obeyed by a conformal block of the $\mathfrak{sl}_n$ current algebra at level

$$n + k = \frac{\epsilon_2}{\epsilon_1}.$$

Namely, it is the four-point conformal block

$$\langle V_{\vec{\mu}_0}(0)V_{\mu_q}(q)V_{\mu_1}(1)V_{\vec{\mu}_\infty}(\infty)\rangle$$

on the sphere. Two of these operators (at 0 and ∞) are Verma modules of $\mathfrak{sl}_n$, whose highest weights are determined by the choice of masses $m_1, \ldots, m_n$ and $m_{n+1}, \ldots, m_{2n}$. The other two operators in the middle correspond to reps of $\mathfrak{sl}_n$ in functions of n variables.

It was a priori expected that there is a relation between collections of special observables of 4d susy gauge theory to some conformal blocks of 2d CFT. It was *not* clear how to derive from first principles which 4d theory corresponds to which CFT.

The KZ equation, in 2d terms, is a consequence of the relation between the current algebra and the Virasoro algebra (i.e. the Ward identity). Deeper expansions of the qq-character, i.e. identities from $\hat{u}_\omega^{(k)}$ for $k > 1$, will give objects in the W-algebra. For example, $\hat{u}_\omega^{(2)}$ will give something like

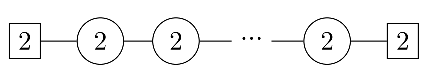
$$\frac{d}{dw_3}\Psi = \mathcal{D}_3(z\frac{\partial}{\partial z})\Psi,$$

where the w-coordinate comes from an additional insertion

$$e^{\sum w_j(\mathrm{ch}_j(\mathcal{E}))}$$

into the partition function.

Take an A_r quiver and take all N spaces to be two-dimensional and M spaces to be two-dimensional attached only to the endpoint nodes.



One can tune mass parameters such that $B_2 I(N_1^{(1)})$ becomes an allowed section. Then we can set it to 0. This means that in the set of Young diagrams, only diagrams which grow in the B_1 direction are allowed. Precisely,

$$m_{1,1} = a_{1,1} + \epsilon_2$$

is what is needed to enforce the equation $B_2 I(N_1^{(1)}) = 0$. Now set

$$m_{1,2} = a_{1,2},$$

which enforces $I(N_1^{(2)}) = 0$. What happens once we make this choice is that, since $\lambda^{(1,1)}$ has this special form and $\lambda_2^{(1,2)} = 0$, we can express

$$\sigma_\square = \sum_{\square\in\lambda}(a+c)$$

as an explicit function of its size

$$ak + \epsilon_1 \frac{k(k-1)}{2}.$$

Now, expanding all qq-characters for the A_r quiver, we get closed-form differential equations on Z. This is because the sub-leading term in the large-x expansion used to require knowledge of $c_\square$, which we now know. Then we get BPZ equations (Liouville CFT)

$$\langle V_{\Delta_0}(z_0) V_{\Delta_1}(z_1) \cdots V_{\Delta_{r+2}}(z_{r+2})\rangle.$$

In general conformal blocks do not obey any equations, but if the dimension of the ? field is a special value corresponding to the appearance of a null vector in the Verma module, then we get a non-trivial relation. For

$$\Delta_0 = -\frac{1}{2} - \frac{3}{4}(b^2 \text{ or } b^{-2})$$

where $c = 1 + 6(b+b^{-1})^2$ is the central charge, then we get a relation (BPZ)

$$\left(\frac{3}{2(2\Delta_0+1)}\frac{\partial^2}{\partial z_0^2} + \frac{\partial^2}{\partial z_0^2} + \sum_{i\neq 0}\left(\frac{\Delta_i}{(z_0-z_i)^2} + \frac{1}{z_0-z_i}\frac{\partial}{\partial z_i}\right)\right)\Psi = 0.$$

In addition to this equation there are also three more relations

$$\begin{aligned}
0 &= \left(\frac{\partial}{\partial z_0} + \sum \frac{\partial}{\partial z_i}\right)\Psi \\
0 &= \left(z_0\frac{\partial}{\partial z_0} + \Delta_0 + \sum (z_i \frac{\partial}{\partial z_i} + \Delta_i)\right)\Psi \\
0 &= \left(z_0^2 + 2\Delta_0 z_0 + \sum (z_i^2 \frac{\partial}{\partial z_i} + 2\Delta_i z_i)\right)\Psi.
\end{aligned}$$

These equations are first-order, so in some sense we can set

$$z_0 = 0, \quad z_{r+2} = \infty, \quad z_{r+1} = 1$$

and then we can identify z_i with q_i. Then we'll get effectively *one* second-order DE, which coincides with the one we got from the qq-characters of the A_r quiver theory.