Notes for Learning Seminar on Conformal Field Theory (Fall 2019)

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Abstract

These are my live-texed notes for the Fall 2019 student learning seminar on conformal field theory. Let me know when you find errors or typos. I'm sure there are plenty.

Contents

1 Kostya (Sep 24): Intro to CFT

This will be an informal talk from a physics perspective. We'll try to make some mathematical connections, but there will be very few mathematical statements. The main purpose is to give a feeling of how CFT formalism appeared and why.

We'll follow the paper by Belavin, Polyakov and Zamolodchikov from '84. This paper actually arise as an attempt to understand an earlier paper in '81 by Polyakov about string theory. String theory is a theory on Riemann surfaces (Σ, g), with or without boundary, genus, and marked points. It feels as few of the features of the surfaces as possible, because we want it to embed into spacetime $\mathbb{R}^{1,3}$. The theory should be:

1. **covariant**, i.e. independent of all coordinate transformations;

2. **conformal**, i.e. the Lagrangian should only depend on the conformal class of the metric g.

For example, for a quantum particle, there is a worldline and a map from the worldline to $\mathbb{R}^{1,3}$. In principle when we write a Lagrangian for the particle, it depends on the metric of the worldline. But in string theory we don't want this dependence.

The partition function Z of string theory looks like

$$Z = \int_{\text{all } (\Sigma,g)} DX\, Dg \, \exp\left(\frac{1}{\hbar}[S_{\text{matter}}[X] + S_{\text{gravity}}[g]]\right).$$

In general, we think of DX as a measure on something like $\Gamma(E)$ where E is some bundle over the moduli of some (Σ, g). When we consider *conformal* matter, gauge fixing for diffeomorphisms turns this action into

$$Z = \int_{\text{conformal classes}} DX\, Dg \, \exp\left(\frac{1}{\hbar}[S_{\text{matter}}[X] + S_{\text{gravity}}[g] + S_{\text{gauge fixing}}]\right).$$

In a particular case, called **critical strings**, the gravity term $S_{\text{gravity}}[g]$ is trivial and therefore we get a theory that only depends on the topology of Σ. In the critical case, the central charge of matter is $c_{\text{matter}} = 26$; in the supersymmetric version $c_{\text{matter}} = 15$. But in general we can consider arbitrary central charge, giving different gravity theories. One such example is Liouville gravity. Classically it coincides with the Liouville equation, which is also conformal. The most useful result in this direction appeared in the '84 paper, which showed that CFTs could actually be solved. However the technique did not work for Liouville theory, which was worked out about a decade later.

What is a CFT? The first thing to discuss we should discuss, which is very particular to CFTs, is the **state-operator correspondence**. First forget about conformal structure and think only about topological theories. For such 2d theories there are **Atiyah–Segal axioms**, which basically say that such a theory is a functor

$$\mathcal{F} \colon \mathsf{Cob} \to \mathsf{Vect}$$

from the category of cobordisms to the category of vector spaces. Boundaries are always disjoint unions of circles S^1, and

$$\mathcal{F}(S^1) = \mathcal{H} = (\text{state space of the theory}).$$

For a cobordism represented by a 2d manifold Σ with boundary, we get

$$\mathcal{F}(\Sigma) = v \in \mathcal{F}(\partial\Sigma).$$

For example, if Σ is a cylinder, its boundary is two circles with different orientation, i.e.

$$\partial\Sigma = S^1 + (S^1)^{\text{op}}.$$

Part of the axioms say that orientation is encoded by dualizing, so

$$\mathcal{F}(\text{cylinder}) \in \mathcal{H} \otimes \mathcal{H}^\vee$$

which we can view as an operator. This is actually the identity operator of the theory.

A **state** is something living on the boundary. So it is a vector in the Hilbert space $\mathcal{H}$. For example, taking the cylinder, we can plug in two states to get a correlation number of the theory. Another nice way to think of a state is, again, in the path integral formalism. For a non-topological theory, think of $\mathcal{H}$ as $L^2(S^1)$ in some sense. Say Σ is a disk D^2. Then we can take expectation values

$$Z = \int DX \, \exp \frac{1}{\hbar} S \in \mathbb{C}$$

where the action is an integral over a disk

$$S = \int_\Sigma d^2\xi \, \mathcal{L}[X, \partial X].$$

The point is that this integral is not well-defined unless we specify boundary conditions on the disk. A boundary condition is e.g.

$$X(e^{i\varphi}) = X_0(e^{i\varphi}).$$

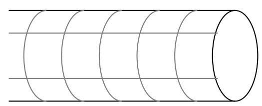
A **(local) operator** are differential polynomials (or slight generalization) of the basic fields of the theory. (Usually operators in the CFT context are always local.) An example of a local operator would be something like $\partial X \cdot X^3$. We can also think about *non-local* operators, e.g. $\langle X(z_0)X(z_1)\rangle$. They depend not on one point, but on multiple.

At the classical level, an example is as follows. Let $X(z) \in \mathbb{C}$ be a section of the trivial line bundle. After quantization, in the Hamiltonian formalism, it will be an operator. Then we can compute correlation functions such as

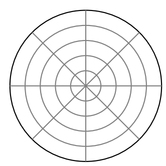
$$\langle X(z_0)\rangle = \int DX \cdot X(z_0) \exp \frac{1}{\hbar} S.$$

This defines an element of $\mathcal{H}^\vee$ associated to the same boundary of the disk, because the action still requires a boundary condition on the disk. Hence local operators define states. In fact *any* state in CFT can be obtained in this way.

In general, think of the operator-state correspondence as follows. Consider a cylinder, infinitely long in the negative direction.



At the negative end, plug in a state $|\phi\rangle$. It produces a state $e^{itH}|\phi\rangle$ on the other S^1. But since the theory is conformal, we can convert this picture into a disk



The boundary on the lhs of the cylinder is mapped to a puncture at the origin, and the boundary on the rhs becomes the boundary S^1. Then the object at the puncture is a local object $\mathcal{A}(0)$, which is a local operator at 0. This is in general called **radial quantization**.

An important consequence is the following. Take a sphere with three punctures. Put two states $|\phi\rangle$ and $|\psi\rangle$ in two of the punctures. Then we get an element $|v\rangle \in \mathcal{H}$ associated to the third puncture. Apply the state-operator correspondence. Pick a basis $\{\phi_\alpha\}_{\alpha\in I}$ for $\mathcal{H}$ and decompose

$$|v\rangle = \sum C^\alpha_{\phi\psi}(z,w)\phi_\alpha(w)$$

for some coefficients C. Hence we get

$$\phi_\alpha(z)\phi_\beta(z) = \sum_\gamma C^\gamma_{\alpha\beta}(z,w)\phi_\gamma(w).$$

This gives an **operator algebra** structure to $\mathcal{H}$. In general, this expression can be put into any correlation function, as

$$\langle \mathcal{A}(z_1,\ldots,z_N)\phi_\alpha(z)\phi_\beta(w)\rangle = \langle \sum_\gamma C^\gamma_{\alpha\beta}\phi_\gamma(w)\mathcal{A}(z_1,\ldots,z_N)\rangle.$$

Associativity of such a product structure imposes constraints on $C^\gamma_{\alpha\beta}$, and allows us to solve for them. This is the **bootstrap** approach to solving CFTs.

A very important object to work with in CFT is the **stress-energy tensor**. It is the generator of time transformations. If we think about a cylinder, pick some slice $t = 0$ in it. The Hamiltonian is

$$H(t) = \int_{t=t_0} dx\, T_{0,0}(\xi,\bar{\xi}).$$

In classical field theory, the Hamiltonian generates time translations, i.e.

$$\{H(t), X\} = \partial_t X.$$

In CFT we can get much more. The theory being conformal is equivalent to the stress-energy tensor T_{ab} being traceless, i.e. $T^a_a = 0$. Noether's theorem gives $\partial_a T^a_b = 0$. In complex coordinates, these two properties become:

$$T_{z\overline{z}} = \operatorname{tr} T = 0, \quad \overline{\partial} T_{zz} = 0, \quad \partial T_{\overline{zz}} = 0.$$

Due to this, we write the holomorphic and anti-holomorphic functions

$$T(z) = T_{zz}(z), \quad \overline{T}(\overline{z}) = T_{\overline{zz}}(\overline{z}).$$

They are rank-2 tensors, so they transform like

$$T(z) \to \left(\frac{\partial \xi}{\partial z}\right)^2 T(\xi).$$

After quantization, $[H(t), X] = \partial_t X$, so

$$[\int_{t=t_0} dx\, T(z), X] = \partial_z X.$$

But more generally we can use *any* complex transformation $\epsilon(z) = \sum_n \epsilon_n z^n$, to get

$$[\int dx\, \epsilon(z) T(z), X] = \delta_\epsilon X.$$

In radial quantization, this commutator becomes just

$$\partial_\epsilon X = \int dt\, \epsilon(z) T(z) X(z_0) |0\rangle$$

where $|0\rangle$ is the state inserted at 0. This is called a **Ward identity**.

A field ϕ is **primary** if it transforms as

$$\phi_\alpha(z) \to \left(\frac{\partial \xi}{\partial z}\right)^{\Delta_\alpha} \left(\frac{\partial \overline{\xi}}{\partial \overline{z}}\right)^{\overline{\Delta}_\alpha} \phi_\alpha(\xi),$$

i.e. it is a "tensor" of rank $(\Delta_\alpha, \overline{\Delta}_\alpha)$, called the **conformal dimension**, which can be non-integers. For such fields, the operator product expansion is

$$T(z)\phi_\alpha(w, \overline{w}) = \frac{\Delta_\alpha}{(z-w)^2}\phi_\alpha + \frac{\partial_w \phi_\alpha}{z-w}.$$

However it turns out that T is *not* a primary field. (In the classical theory it of course is, because it is a rank-2 tensor. But in the quantum theory it is not.) It transforms as

$$T(z) \to \left(\frac{\partial \xi}{\partial z}\right)^2 T(\xi) + \frac{c}{12}\{\xi, z\}$$

where the second (correction) term is the **Schwartz derivative** and c is a constant called the **central charge**. One can try to understand this term physically in many ways, but it is believed to be the most possible such expression. Using this transformation law with the Ward identities, we get

$$T(z)T(w) = \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w}.$$

Out of this, we can get the Virasoro algebra as follows. Formally decompose

$$T(w) = \sum \frac{L_n}{z^{n+2}}.$$

(The +2 here is because of the conformal dimension 2.) Then

$$\oint dz\, z^{n+1} T(z) = L_n.$$

Applying this to the operator product expansion, we get Virasoro relations

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m}.$$

Note that if we define vector fields $\ell_n \coloneqq z^{n+1}\partial_z$, we get just the first term in this commutator:

$$[\ell_n, \ell_m] = (n-m)\ell_{n+m}.$$

This comes from the relation between the classical stress-energy tensor and the Hamiltonian. Note also that this means $\{L_{-1}, L_0, L_1\}$ span an SL_2 subalgebra.

Using the OPE $T(z)\phi_\alpha(0)$, one can show that the state operator correspondence $\phi(z) \to |\phi\rangle = \phi(0)\,|0\rangle$ sends the operator ϕ to a *highest-weight* vector for the Virasoro algebra. In other words,

$$L_n\,|\phi\rangle = 0, \quad n > 0$$

and the highest weight is exactly the conformal weight Δ. Generically,

$$L_{-n_1} \cdots L_{-n_k}\,|\phi\rangle\,, \quad n_1 < \cdots < n_k$$

are independent. By the state-operator correspondence, they can be represented by some contour integrals. These are **descendants** of the primary field $\phi(0)$.

To sum up in representation-theoretic language, the entire Hilbert space $\mathcal{H}$ decomposes into a sum of irreducible highest weight representations of the Virasoro. Primary fields are highest weight vectors, and descendants are generated by primary fields. Correlation functions for all descendants are computed through the ones for primary fields, precisely because we know the OPE of the stress-energy tensor with any field. So to study CFTs, we only care about primary fields.

If in the decomposition $\mathcal{H} = \bigoplus_{\alpha \in \mathcal{A}} V_\alpha$ the indexing set $\mathcal{A}$ is finite, then we have a finite number of primaries and the CFT is called **rational**. Also, when

$$\mathrm{Vir} \subset U(\mathfrak{g})$$

for some $\mathfrak{g}$, e.g. Heisenberg, loop/affine algebra (like WZW theory), then we can ask about reps of this bigger symmetry algebra $U(\mathfrak{g})$. Examples include minimal models, WZW theories, KS models.

Another consequence of the Ward identities, which leads to differential equations for conformal blocks, comes from considering

$$\langle T(z)\phi_{\alpha_1}(z_1)\cdots\phi_{\alpha_n}(z_n)\rangle = \sum_{i=1}^n \left(\frac{\Delta_i}{(z-z_i)^2} + \frac{\partial/\partial z_i}{z - z_i}\right) \times \langle \phi_{\alpha_1}(z_1)\cdots\phi_{\alpha_n}(z_n)\rangle.$$

What is a conformal block? Recall the product $\phi_1(z)\phi_2(w) = \sum C_{12}^\alpha(z)\phi_\alpha(0)$. Decompose ϕ_α into primary/descendants to get

$$\phi_1(z)\phi_2(z) = \sum_\alpha C_{12}^\alpha \sum_{\{k\}} \beta_{12}^{\alpha,\{k\}} z^{\Delta_\alpha - \Delta_1 - \Delta_2 + \sum k_i} L_{-n_1}\cdots L_{-n_k}\phi^\alpha(0).$$

Apart from the structure constants $\beta_{12}^{\alpha,\{k\}}$, everything else is determined by conformal symmetry, e.g.

$$\langle\phi_\alpha(z)\rangle = 0 \quad \text{if } \phi_\alpha \neq \text{id}$$
$$\langle\phi_\alpha(z)\phi_\beta(w)\rangle = \frac{\delta_{\alpha\beta}}{|z-w|^{\Delta_\alpha}}$$
$$\langle\phi_\alpha(z)\phi_\beta(w)\phi_\gamma(\xi)\rangle = \frac{C_{\alpha\beta\gamma}}{|z-w|^{\Delta}|w-\xi||\xi-z|}$$
$$\langle\phi_1(\infty)\phi_2(1)\phi_3(x)\phi_4(0)\rangle = \sum_p C_{12}^p C_{34}^p \sum_{\{k\}} z^{\cdots}\beta_{12}^{p\{k\}}\beta_{34}^{p\{k\}}\langle\phi_p|L_{n_1'}\cdots L_{n_k'}|L_{n_1}\cdots L_{n_k}|\phi_p\rangle.$$

The only unknowns here are the structure constants C_{12}^p and C_{34}^p. In the BPZ paper, the four-point correlators are constructed in terms of objects called **conformal blocks** $\mathcal{F}_{12}^{34}(p|z)$, as

$$\langle\phi_1\phi_2\phi_3\phi_4\rangle = \sum_p C_{12}^p C_{34}^p|\mathcal{F}_{12}^{34}(p|z)|.$$

These are universal quantities which are fully determined by representation-theoretic considerations. In all rational theories, they satisfy differential equations called **BPZ equations** coming from the Ward identities. (In WZW models they are called **KZ equations**, coming from some rep theory of $U(\mathfrak{g})$.) However, in Liouville theory we cannot do this, and there are other things to work with.

2 Davis (Oct 01): The local structure of CFT

(Notes by Davis)

In this talk, I'll try to explain how vertex operator algebras are related to the local structure of conformal field theory. I'll motivate VOAs from this point of view, explain what the Virasoro algebra is from a mathematically motivated perspective, then explain how the Heisenberg Lie algebra is related to the free boson CFT. Time permitting, I hope to say something about the OPE, or about conformal blocks.

In 2D CFT, a general correlator on a manifold Σ,

$$\langle\phi_1(z_1)\dots\phi_n(z_n)\rangle_\Sigma.$$

can be viewed as fields $\phi_1,\dots,\phi_n$ inserted at points $z_1,\dots,z_n$ on the manifold.

Part of the content of Kostya's talk was justifying that we could understand this picture by breaking apart Σ into smaller pieces and cobording them together. Hence, we may break apart Σ into pieces and assume all the fields live together on a genus zero piece. Studying this genus zero piece is what I will call the *local structure* of CFT: understanding higher genus pieces, and how they glue together, will be beyond this talk.

2.1 The standout objects of CFT on the circle

Kostya's talk yesterday introduced a lot of important objects in the study of CFT on a genus zero surface with boundary,

1. The space of states (boundary conditions)
2. The OPE, $C^\alpha(z)C^\beta(w) = \sum C_\gamma^{\alpha\beta}(z,w)C^\gamma(z)$
3. The state operator correspondence
4. The stress energy tensor, $T_{\mu\nu} \to T(z), \overline{T}(\overline{z})$.

Claim. On the *circle*, for a large class of CFTs called 'chiral CFTs', the "holomorphic and antiholomorphic sectors decouple" and we may study the holomorphic part of the CFT independently. I will justify this statement, and explain it in more detail, shortly, but for now take it on faith.

2.2 VOAs

Formalising these ideas, observing that we may derive the OPE from the state-operator correspondence, we arrive at the *long* definition of a VOA.

Definition 2.1. A **vertex operator algebra**

1. A **space of states** V (a vector space)
2. A **vacuum vector** $|0\rangle \in V$
3. A **translation operator** $T : V \to V$,
4. **Vertex operators**, $Y(\bullet, z) : V \to EndV[[z, z^{-1}]]$, sending $A \to Y(A, Z)$, a field.

such that

1. $Y(|0\rangle, z) = id$
2. $Y(A, z)|0\rangle \in V[[z], Y(A, z)|0\rangle_{z=0} = A$;
3. $[T, Y(A, z)] = \partial_z Y(A, z), T|0\rangle = 0$;
4. The $Y(A, z)$ are pairwise local.

where

1. $\sum A_j z^{-j} = A(z) \in EndV[[z, z^{-1}]$ is a **field** if for any $v \in V$, $A_j v = 0$ for j large enough.
2. Two fields $A(z), B(w)$ are **local** pairwise if $\exists N \geq 0$, $(z - w)^N [A(z), B(w)] = 0$.

Where did the stress energy tensor go in this definition? We'll see after we've discussed Virasoro.

I don't want to get into the weeds of formal power series, and BenZvi/Frenkel's book covers this background well. But here's one example:

Example 2.2. The **formal delta function**, $\delta(z - w)$, is the power series

$$\delta(z - w) = \sum z^m w^{-m-1}.$$

It satisfies the property

$$A(z)\delta(z - w) = \sum A_k w^k z^m w^{-m-1} = \sum A_{m+n+1} z^m w^n = A(w)\delta(z - w).$$

In particular, $z\delta(z - w) = w\delta(z - w) \implies (z - w)\delta(z - w) = 0$.

The easiest VOAs are commutative ones.

Example 2.3. Suppose $[Y(A, z), Y(B, w)] = 0$ for all A, B. Then

$$Y(A, z)B = Y(A, z)Y(B, w)|0\rangle_{w=0} = Y(B, w)Y(A, z)|0\rangle_{w=0}.$$

Because this is true for all B and $Y(A, z)|0\rangle \in EndV[[z]$ by the axioms, this means $Y(A, z) \in EndV[[z]$ for all A.

So we can equip the VOA with the structure of a commutative algebra,

$$A \star B := Y(A, 0)B.$$

T is a derivation for this algebra.

In the other direction, with mild assumptions, $Y(A, z) := e^{zT} A$ makes an algebra with a derivation into a VOA.

2.3 Virasoro

Our first examples of interesting vertex operators will come from the Virasoro algebra.

Recall that we have identified states with boundary conditions. So $Diff(S^1)$ acts on states on the disc. We really want the 'complexified' $Diff(S^1)$ action. However, $Diff(S^1)$ admits no complexification as a Lie group. (There *is* a semigroup which serves as a partial answer, as exposited in Andre Henriques cobordism-centred CFT notes).

So, we look at its Lie alg and complexify. Complexified $LieDiff(S^1)$ has a basis

$$\begin{aligned} \ell_n &= -z^{n+1}\partial_z \\ \overline{\ell}_n &= -\overline{z}^{n+1}\overline{\partial_z} \\ [\ell_n, \ell_m] &= (n-m)\ell_{n+m} \\ [\overline{\ell_n}, \overline{\ell_m}] &= (n-m)\overline{\ell}_{n+m} \end{aligned}.$$

So, $(LieDiff(S^1))_{\mathbb{C}} = Witt \oplus Witt$, where $Witt$ has basis $\{\ell_n\}$.

Remark. This is what I mean by decoupling of holomorphic/antiholomorphic sectors. The assumption here is that the $Diff(S^1)$ global action, whatever I really mean by this, upgrades to a local action: this the definition of a chiral CFT.

Primary fields, for instance, are ones where this action so upgrades. For general CFTs, you can only use the VOA to study the **chiral sector**, which basically means fields coming from primary fields.

Goal. We want to study projective representations of the Witt algebra, because scaling by a constant doesn't change the physical state. The standard yoga is that projective representations of Witt are ordinary reps of centrally extended Witt.

Definition 2.4. The **Virasoro algebra** is $[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(m^3 - m)\delta_{n+m,0}$.

It is the universal central extension of the Witt algebra.

Remark. We can now write a generating functional

$$\begin{aligned} T(z) &= \sum L_n z^{-n-2} \\ [T(z), T(w)] &= \frac{c}{12}\partial_w^3 \delta(z-w) + 2T(w)\partial_w \delta(z-w) + \partial_w T(w)\delta(z-w) \end{aligned}.$$

and this relation is equivalent to the $[L_n, L_m]$ relation. So, the stress-energy tensor pops up!

Finally, we can make a VOA

Example 2.5. Let $Vir_c = U(Vir) \otimes_{U(LieDiff(S^1)_{\mathbb{C}} \oplus \mathbb{C}_c} \mathbb{C}_c$, i.e. 'the universal enveloping algebra of Virasoroa with the element c sent to be constant number c.

This algebra has a basis

$$L_{j_1} \dots L_{j_m} |0\rangle$$

where $j_1 \leq \dots \leq j_m \leq -2$

we can declare $T = L_{-1}$, $Y(L_{-2}|0\rangle, z) := T(z)$, and other vertex operators 'generated from this'.

What we mean by 'generated by this' calls for another messy theorem which will let us practically define a lot of VOAs.

Theorem 2.6. *Let V a vector space, $|0\rangle \neq 0 \in V, T \in End(V)$. Let $\{a^\alpha\}_{\alpha \in \mathbb{Z}}$ vectors, and $\{a^\alpha(z)\}$ fields so that*

1. $a^\alpha(z)|0\rangle = a^\alpha + O(z^{\geq 1})$
2. $T|0\rangle = 0, [T, a^\alpha(z)] = \partial_z a^\alpha(z)$

3. $a^\alpha(z)$ are pairwise local;

4. V has a basis of vectors $a^{\alpha_1}_{(j_1)} \dots a^{\alpha_n}_{(j_n)} |0\rangle$, with $j_1 \leq \dots \leq j_m < 0$ and $j_i = j_{i+1} \implies \alpha_i \leq \alpha_{i+1}$

then, we can equip V with a VOA structure by

$$Y(a^{\alpha_1}_{j_1} \dots a^{\alpha_m}_{j_m} |0\rangle, z) = \frac{1}{(-j_1-1)! \dots (-j_m-1)!} : \partial_z^{-j_1-1} a^{\alpha_1} \dots \partial_z^{-j_m-1} a^{\alpha_m}(z) :.$$

*where $: AB :$ denotes **normal ordering**: if $A(z) = \sum A_m z^{-m-1}, B(w) = \sum B_n w^{-n-1}$, we define $: A(z)B(w) :$ to be $\sum_n (\sum_{m<0} A_m B_n z^{-m-1} + \sum_{m\geq 0} B_n A_m z^{-m-1}) w^{-n-1}$. We then extend $: ABC :=: A : BC ::$.*

Normal ordering has a weird definition, basically it means removing the singular point as $z \to w$. In physics speak, we demand that vacuum expectation values of normal ordered correlators vanish.

I hope to explain this theorem's formula via the OPE, time permitting.

Example 2.7. So in the Virasoro case, we may write $Y(L_{j_1} \dots L_{j_m} |0\rangle, z)$ in this way as

$$const \times : \partial_z^{-j_1-2} T(z) \dots \partial_z^{-j_m-2} T(z) :$$

Most VOAs we care about are related to the Virasoro VOAs, in the following sense.

Definition 2.8. A VOA is **conformal with central charge c** if

1. The space of states V is $\mathbb{Z}$-graded;
2. $\exists \omega \in V_2$, a **conformal vector**, so $Y(\omega, z) = \sum L_n^V z^{-n-2}$ and the L_n^V have the Virasoro L_n commutation relations;
3. Further, $T = L_{-1}^V, L_0^V|_{V_n} = n \cdot id$.

Of course, Vir_c is conformal, with central charge c and $\omega = L_{-2} |0\rangle$. (From the physics perspective, all CFTs have stress energy tensors, so the state-operator map should define a conformal vector, so this sort of structure should be universally expected.)

2.4 Free boson

One CFT we care about is the free boson. I will say some physics words to try to make contact with the physics. If these words mean nothing to you, don't worry: it should be brief. It has Lagrangian

$$\frac{g}{2} \int dx (\partial_t \phi)^2 - (\partial_x \phi)^2.$$

On a cylinder, $\phi(x + L, t) = \phi(x, t)$, we may Fourier expand and go to a Hamiltonian, we find

$$H = \frac{1}{2gL} \sum_n \pi_n \pi_{-n} + (2\pi n g)^2 \phi_n \phi_{-n}.$$

Which is an infinite sum of harmonic oscillators, plus one zero mode, π_0^2.

The commutation relations of the π_n, ϕ_n are $[\pi_n, \phi_m] = i\delta_{nm}$. These are like 'energy and momentum pairs'. The algebra generated by this is called the Heisenberg algebra.

Definition 2.9. Heisenberg is the central extension

$$0 \to \mathbb{C}\widetilde{1} \to \mathcal{H} \to \mathbb{C}[t, t^{-1}] \to 0.$$

with basis $\{t^n\}_{n\in\mathbb{Z}}$ and $\widetilde{1}$, and rule $[t^n, t^m] = -m\delta_{n,-m}\widetilde{1}$.

(This is a basis of creation and annihilation operators, related to our ϕ, π by

$$a_n = \frac{1}{\sqrt{4\pi g|n|}}(2\pi g|n|\phi_n + i\pi_{-n})$$
$$n > 0 \implies t^n = -i\sqrt{n}a_n .$$
$$n \leq 0 \implies t^n = i\sqrt{-n}a^\dagger_{-n}$$

The **Weyl algebra** is Heisenberg letting $\widetilde{1} = 1$.

The simplest VOA we can make out of this is the **Fock space** of the CFT, the subalgebra of the Weyl algebra generated by $\{t^i = b_i\}_{i<0}$ acting on $|0\rangle = \widetilde{1}$.

We define

- $T|0\rangle = 0, [T, b_i] = -ib_{i-1}$
- $Y(b_{-1}, z) = b(z) = \sum b_n z^{-n-1}$
- $y(b_{-n}, z) = \frac{1}{(n-1)!}\partial_z^{n-1}b(z)$

Example 2.10. In fact the Fock space VOA admits a one-parameter family of conformal structures.

Let

$$\omega_\lambda = (\frac{1}{2}b^2_{-1} + \lambda b_{-2})|0\rangle.$$

then $Y(\omega_\lambda, z) = \frac{1}{2} : b(z)^2 : +\lambda\partial_z b(z)$ satisfy the conformal relations, and equip the Fock space VOA with the structure of a conformal VOA with central charge $c_\lambda = 1 - 12\lambda^2$.

What's going on here? Because the Hamiltonian is independent of ϕ_0, π_0 'commutes with everything', so we can simultaneously diagonalise eigenstates of H to also be eigenstates of π_0. In the physical picture, we can view the Fock space as being built on a one-parameter family of vacua $|\lambda\rangle$, where λ is related by normalisation to the eigenvalue of π_0 by normalisation. No operators relate the vacua, so the theory decouples into conformal VOAs with different central charges.

By the way, why did we want to consider these vertex operator fields/states rather than our original free boson, ϕ? For one, ϕ itself doesn't factor into holomorphic and antiholomorphic components: it is not primary.

2.5 OPE

OK, the final thing I want to do is give a general derivaton of the OPE from the state-operator correspondence, at the level of physics rigor.

State-operator says that $Y(A, z)Y(B, w)$ is determined by the state $Y(A, z)Y(B, w)|0\rangle$ as $w \to 0$. (This is a theorem in Ben-Zvi/Frenkel, called Goddard's uniqueness theorem). Translate to $Y(A, z-w)Y(B, 0)|0\rangle$ and expand $Y(A, z-w) = \sum A_n(z-w)^{-n-1}$. We find

$$Y(A, z)Y(B, w) = \sum_{n\leq 0} \frac{Y(A_nB, w)}{(z-w)^{n+1}} + : Y(A, z)Y(B, w) :.$$

which is the OPE for vertex operators. (The non-singular term, sort of by definition, is $: Y(A, z)Y(B, w) :$, but I haven't justified this adequately.)

OPE can be used to construct the weird formula we had for the VOA of a product of endomorphisms.

We need one more formula:

Claim. $Y(B,z)|0\rangle = e^{zT}B$

Proof. It suffices to prove $B_{(-n-1)}|0\rangle = \frac{T^n}{n!}B$.
By the axioms, $\partial_z Y(B,z)|0\rangle = [T, Y(B,z)]|0\rangle = TY(B,z)|0\rangle$.
Equate coefficients to find $nB_{-n-1}|0\rangle = TB_{-n}|0\rangle$. Induct. □

Claim. $Y(Ta,z) = \partial_z Y(A,z)$ for all A.

Proof. $Y(Ta,z)|0\rangle = \partial_z Y(A,z)|0\rangle$, use state-operator to go back. □

Which implies $Y(B_{(-n-1)}|0\rangle, z) = \frac{\partial_z^n}{n!}Y(B,z)$.
Contour integrating, the OPE we find

$$Y(A_nB,z) = \frac{1}{(-n-1)!} : \partial_z^{-n-1}Y(A,z)Y(B,z) :.$$

Now using the above claim to expand replace B with B_{-n-1}, if desired, and inducting to include more fields, we get the desired 'reconstruction formula'.

3 Cailan (Oct 08): VOA: Examples and Representations

Definition 3.1. A **VOA** is:

1. a vector space V;
2. a vacuum vector $|0\rangle \in V$;
3. a translation operator $T\colon V \to V$;
4. vertex operators, which are linear maps

$$Y(-,z)\colon V \to \operatorname{End} V[[z,z^{-1}]]$$

such that $Y(A,z) = \sum_{n\in\mathbb{Z}} A_n z^{-n-1}$ is a field, i.e. for all $v \in V$ we have $A_n v = 0$ for $n \gg 0$. We call the A_n the **Fourier coefficients** of A.

This data is subject to the following axioms:

1. $Y(|0\rangle, z) = \mathrm{id}$, and
$$Y(A,z)|0\rangle \in V[[z]], \quad Y(A,z)|0\rangle|_{z=0} = A;$$
2. $[T, Y(A,z)] = \partial_z Y(A,z)$;
3. (locality) there exists $N \geq 0$ such that
$$(z-w)^N[Y(A,z), Y(B,w)] = 0.$$

Example 3.2. The **Heisenberg** algebra $\mathcal{H}$ is the central extension

$$0 \to \mathbb{C}\cdot\mathbf{1} \to cH \to \mathbb{C}[t,t^{-1}] \to 0$$

given by the 2-cocycle

$$c(f,g) = -\operatorname{Res}_{t=0} f\,dg.$$

Let $b_n = t^n$. Then the commutation relations are

$$[b_n, b_m] = n\delta_{n,-m}\mathbf{1}, \quad [\mathbf{1}, b_n] = 0.$$

What we will do in most of our examples is turn representations of Lie algebras into VOAs. Let

$$\widetilde{\mathcal{H}} \coloneqq U\mathcal{H}/\langle \mathbf{1} - 1\rangle.$$

Let $\widetilde{\mathcal{H}}_+ \subset \widetilde{\mathcal{H}}$ be the positive subalgebra. Since it is commutative, it has a trivial rep $\mathbb{C}$. The **Fock representation** is

$$\pi \coloneqq \widetilde{\mathcal{H}} \otimes_{\widetilde{\mathcal{H}}_+} \mathbb{C}.$$

By PBW, this has a basis

$$\{\cdots b_{-2}^{e_2} b_{-1}^{e_1} \otimes 1\}.$$

The VOA structure on π is defined as follows.

1. The vacuum is $|0\rangle \coloneqq 1 \otimes 1$.
2. Translation is defined inductively using

$$[T, b_i] = -ib_{i-1}.$$

   This means T behaves as a formal derivative, with

$$T(b_{j_1} \cdots b_{j_k} \otimes 1) = \sum_i j_i b_{j_1} \cdots b_{i-1} \cdots b_{j_k} \otimes 1.$$

3. Vertex operators are defined as follows:

$$Y(b_{-1} \otimes 1, z) \coloneqq b(z) \coloneqq \sum_{n\in\mathbb{Z}} b_n z^{-n-1}$$
$$Y(b_{-k} \otimes 1, z) = \frac{i}{(k-1)!} \partial_z^{k-1} b(z)$$
$$Y(b_{-j_1} \cdots b_{-j_k} \otimes 1, z) = \frac{1}{(j_1-1)! \cdots (j_k-1)!} :\partial_z^{j_1-1} b(z) \cdots \partial_z^{j_k-1} b(z):$$

   where $:A(z)B(w): \coloneqq A(z)_+ B(w) + B(w)A(z)_-$.

Claim. This gives π the structure of a VOA.

Proof. By construction, $[T, b_i] = -ib_{i-1}$ implies

$$[T, b(z)] = \partial_z b(z).$$

So the translation axiom is satisfied by $Y(b_{-1} \otimes 1, z)$. For the general case, use that normal ordering satisfies the Leibniz rule

$$\partial_z :A(z)B(z): = :\partial_z A(z)B(z): + :A(z)\partial_z B(z): .$$

The real content of this claim is in checking the locality axiom.

1. Check that $b(z)$ is local with itself. Recall $\delta(z-w) = \sum_{n\in\mathbb{Z}} z^{-n-1} w^m$. Compute that

$$[b(z), b(w)] = \sum_{n,m} [b_n, b_m] z^{-n-1} w^{-m-1} = \sum_n [b_n, b_{-n}] z^{-n-1} w^{n-1} = \partial_w \delta(z-w).$$

   Locality means this commutator is a sum of delta functions and their derivatives, which is true here.

2. Check that $\partial_z^n b(z)$ is local with $\partial_z^m b(z)$. Start with

$$(z-w)^N [b(z), b(w)] = 0,$$

   which is the definition of locality for $b(z)$ with itself, and differentiate with respect to z. This yields

$$(z-w)^N [\partial_z b(z), b(w)] + n(z-w)^{N-1} [b(z), b(w)] = 0.$$

   Multiply by $(z-w)$ to get rid of the last term and get locality of $\partial_z b(z)$ with $b(z)$. Then induct.

3. Apply induction using Dong's lemma, which says if $A(z), B(z), C(z)$ are mutually local fields then the fields $:A(z)B(z):$ and $C(z)$ are local. $\square$

Remark. We can't avoid normal ordering. One of the consequences of OPE is the following identity:

$$Y(A_n \cdot B, z) = \frac{1}{(-n-1)!} :\partial_z^{-n-1} Y(A,z) Y(B,z): \quad n < 0.$$

In the case of the Heisenberg, note that b_{-1} generates all coefficients of $Y(b_{-1} \otimes 1, z) = b(z)$. Hence the moment we specify $Y(b_{-1} \otimes 1, z)$ we have specified everything. This is why only $b(z)$ and its derivatives occur in $Y(A,z)$.

Example 3.3 (Affine Kac–Moody algebras)**.** Define $\hat{\mathfrak{g}}$ as

$$0 \to \mathbb{C}K \to \hat{\mathfrak{g}} \to L\mathfrak{g} \to 0.$$

Let $\mathbb{C}_K$ be the 1-dimensional rep of the subalgebra $\mathfrak{g}[t] \otimes \mathbb{C}K$. Set $K \cdot v = kv$. The **vacuum rep of level** k is

$$V_k(\mathfrak{g}) \coloneqq \operatorname{Ind}_{\mathfrak{g}[t] \oplus \mathbb{C}K}^{\hat{\mathfrak{g}}} \mathbb{C}_K = U(\hat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t] \otimes \mathbb{C}K)} \mathbb{C}_K.$$

Let $\{J^a\}_{a=1}^{\dim \mathfrak{g}}$ be a basis of $\mathfrak{g}$, and set $J^a_n \coloneqq J^a \otimes t^n$. Then by PBW, a basis of $V_k(\mathfrak{g})$ is

$$\{J^{e_1}_{n_1} \cdots J^{e_m}_{n_m} \otimes 1\}.$$

The VOA structure is defined as follows.

1. The vacuum is $|0\rangle \coloneqq 1 \otimes 1$.

2. Translation is defined inductively by $T|0\rangle = 0$ and $[T, J^a_n] \coloneqq -nJ^a_{n-1}$.

3. Vertex operators are defined as follows:

$$Y(J^a_{-1} \otimes 1, z) \coloneqq J^a(z) \coloneqq \sum_{n \in \mathbb{Z}} J^a_n z^{-n-1}$$
$$Y(J^{a_1}_{-n_1} \cdots J^{a_m}_{-n_m} \otimes 1, z) = \frac{1}{(n_1-1)! \cdots (n_m-1)!} :\partial_z^{n_1-1} J^{a_1}(z) \cdots \partial_z^{n_m-1} J^{a_m}(z): .$$

Checking locality is analogous to the previous case.

Definition 3.4. Let $(V, |0\rangle, T < Y)$ be a vertex algebra. A vector space M is a V**-module** if it is equipped with an action

$$Y_M \colon V \to \operatorname{End} M[[z, z^{-1}]], \quad A \mapsto \sum_{n \in \mathbb{Z}} A^m_{(n)} z^{-n-1}.$$

This action must satisfy the following axioms:

1. $Y_M(|0\rangle, z) = \mathrm{id}_M$;

2. for $A, B \in V$ and $m \in M$, the elements

$$Y_M(A,z) Y_M(B,w) m \in M((z))((w))$$
$$Y_M(Y(A, z-w)B, w) m \in M((w))((z-w))$$

   represent the same element in $M[[z,w]][z^{-1}, w^{-1}, (z-w)^{-1}]$.

Remark. What does this mean? By Taylor expansion, we get a map

$$M((z))((w)) \to M((w))((z-w)), \quad z \mapsto w + (z-w).$$

On one hand,

$$Y_M(A,z)Y_M(B,w) = Y_m(A, w+z-w)Y_m(B,w) = \sum_{n=0}^{\infty} \frac{\partial_w^n Y_M(A,w)}{n!} Y_M(B,w)(z-w)^n.$$

On the other hand,

$$Y_M(Y(A,z-w)B,w) = Y_M(\sum_n A_n \cdot B(z-w)^{-n-1}, w) = \sum_n Y_M(A_nB,w)(z-w)^{-n-1}.$$

If $M = V$, this is an OPE.

Remark. There is a Lie algebra associated to every VOA, such that V-modules are the same as $\mathfrak{g}$-modules. This is as follows. A consequence of OPEs is that for $Y(A,z)$ and $Y(B,z)$,

$$[A_m, B_k] = \sum_{n \geq 0} \binom{m}{n} (A_nB)_{m+k-n}.$$

Hence the span of all Fourier coefficints of all vertex operators form a Lie subalgebra $U'(V) \subset \mathrm{End}(V)$. (That this satisfies the Jacobi identity is automatic from it being a commutator.)

Theorem 3.5. *There is an equivalence of categories*

$$\mathsf{Mod}(V) \xrightarrow{\sim} \left\{ \begin{array}{c} \textit{smooth and coherent modules} \\ \textit{of Lie algebra generated by } U'(V) \end{array} \right\}$$
$$\xrightarrow{\sim} \left\{ \textit{smooth modules of } \widetilde{U}(V) \right\}.$$

Example 3.6 (VOAs associated to 1-dimensional lattices)**.** Let $\mathbb{C}_\lambda$ be the 1-dimensional rep of $\widetilde{\mathcal{H}}$ where $b_0 \left|\lambda\right\rangle = \lambda \left|\lambda\right\rangle$. Let

$$\pi_\lambda \coloneqq \widetilde{\mathcal{H}} \otimes_{\widetilde{\mathcal{H}}_+} \mathbb{C}_\lambda,$$

with basis $\{\cdots b_{-2}^{e_2} b_{-1}^{e_1} \otimes \left|\lambda\right\rangle\}$. This is a locally finite $\widetilde{\mathcal{H}}$-module, and therefore a π-module. Actually one can show it is irreducible. Let $N \in \mathbb{Z}^+$ and set

$$V_{\sqrt{N}\mathbb{Z}} \coloneqq \bigoplus_{m \in \mathbb{Z}} \pi_{m\sqrt{N}}.$$

This turns out to be a VOA.

Theorem 3.7. *For any even N (resp. odd N), the module $V_{\sqrt{N}\mathbb{Z}}$ carries the structure of a VOA (resp. super VOA) such that π_0 is a vertex subalgebra.*

Proof. Once we define $V_\lambda(z) \coloneqq Y(1 \otimes \left|\lambda\right\rangle, z)$ for $\lambda \in \sqrt{N}\mathbb{Z}$, we are done. It looks like

$$V_\lambda(z) \coloneqq S_\lambda z^{\lambda b_0} \exp\left(-\lambda \sum_{n<0} \frac{b_n}{n} z^{-n}\right) \exp\left(-\lambda \sum_{n>0} \frac{b_n}{n} z^{-n}\right). \qquad \square$$

Remark. When $N = 1$, actually there is an identification

$$V_{\mathbb{Z}} \cong \Lambda \coloneqq \mathrm{Cl} \otimes_{\mathrm{Cl}^+} \mathbb{C}$$

where Cl is the Clifford algebra on $\{\psi_n, \psi_n^*\}$. This is the boson-fermion correspondence, and e.g. the Jacobi triple product comes from looking at graded dimension on both sides.

4 Yasha (Oct 15): Minimal models

Recall that $\varphi(z)$ is a **primary field** if it is highest weight in a representation of the Virasoro algebra. In terms of OPEs,

$$T(z)\varphi(w) = \frac{\Delta\varphi(w)}{(z-w)^2} + \frac{\partial\varphi(w)}{z-w} + \text{reg.}$$

In terms of the Virasoro algebra, this is equivalent to

$$L_n\varphi(z) = \begin{cases} 0 & n > 0 \\ \Delta\varphi(z) & n = 0 \end{cases}.$$

Theorem 4.1. *Correlators of descendant fields are determined by correlators of primary fields.*

Proof. Consider a correlator

$$\langle L_{-n}A(z)A_1(z_1)\cdots A_N(z_N)\rangle \quad n \geq 0.$$

We can write

$$L_{-n} = \oint T(\zeta)\zeta^{-n+1}\frac{d\zeta}{2\pi i}.$$

Substituting this into the correlator gives

$$\oint \frac{d\zeta}{2\pi i}(\zeta - z)^{-n+1}\langle T(\zeta)A(z)A_1(z_1)\cdots A_N(z_N)\rangle$$

where the contour is around poles of $A(z)$ and not poles of the other A_i. The stress-energy tensor $T(\zeta)$ decays very quickly as $\zeta \to \infty$, so we can expand the contour to include all other poles, giving

$$-\sum_{i=1}^N \oint \frac{d\zeta}{2\pi i}(z_i - z + \zeta - z_i)^{-n+1}\langle T(\zeta)A(z)A_1(z_1)\cdots A_N(z_N)\rangle.$$

The series expansion of this will have L_n acting on the other fields $A_i(z_i)$:

$$\begin{aligned} &-\sum_{i=1}^N \sum_{k\geq 0} \oint_{C_i} \frac{d\zeta}{2\pi i}\binom{1-n}{k}(z_i - z)^{1-n-k}(\zeta - z_i)^k\langle T(\zeta)A(z)A_1(z_1)\cdots A_N(z_N)\rangle \\ &= -\sum_{i=1}^N \sum_{k\geq 0}\binom{1-n}{k}(z_i - z)^{1-n-k}\langle A(z)A_1(z_1)\cdots L_{k-1}A_i(z_i)\cdots A_N(z_N)\rangle. \end{aligned}$$

If all fields $A_1, \ldots, A_N$ are primary, then only $k = 0$ and $k = 1$ act non-trivially. Hence there are two terms

$$L_{-n}A(z)\phi_1(z_1)\cdots\phi_N(z_N) = \sum_{i=1}^N \left(\frac{(n-1)\Delta_i}{(z_i - z)^n} - \frac{1}{(z_i - z)^{n-1}}\frac{\partial}{\partial z_i}\right)\langle A(z)A_1(z_1)\cdots A_N(z_N)\rangle.$$

□

In fact, the contribution of descendants to OPEs (three-point correlators) is also fully determined by the contribution of primary fields. Suppose we have two primary fields $\varphi_n(z)$ and $\varphi_m(w)$. Then we can always expand

$$\varphi_n(z)\varphi_m(0) = \sum_{p,\lambda} C_{nm}^{p\lambda} z^{\Delta_p - \Delta_n - \Delta_m + |\lambda|}\varphi_p^\lambda(0)$$

where λ encodes the descendant $\varphi_p^\lambda := L_{-\lambda_1}\cdots L_{-\lambda_n}\varphi_p$. The claim is that it suffices to know $C_{nm}^{p\lambda}$ only for $\lambda = \emptyset$, and all the other coefficients are determined by conformal symmetry. Rewrite the expansion as

$$\varphi_n(z)\varphi_m(0) = \sum_{p,\lambda} C_{nm}^q z^{\Delta_p - \Delta_n - \Delta_m}\Psi_p(z|0) \quad \Psi_p(z|0) := \sum_p \beta_{nm}^{p\lambda} z^{|\lambda|}\varphi_p^\lambda(0).$$

How do we determine the β coefficients? Assume for simplicity that $\Delta_p = \Delta_n = \Delta_m = \Delta$. Then

$$\varphi_\Delta(z)\,|\Delta\rangle = \sum C_{\Delta\Delta}^{\Delta\rho}\varphi_\Delta(z)\,|\Delta_p\rangle\,, \quad \varphi_\Delta(z) = \sum z^{|\lambda|}\beta^\lambda L_{-\lambda_1}\cdots L_{-\lambda_k}.$$

Apply L_n to both sides, for $n > 0$, to get

$$\begin{aligned} L_n\varphi_1(z)\varphi_2(0) &= \oint d\zeta T(\zeta)\zeta^{-n+1}\varphi_1(z)\varphi_2(0) \\ &= (z^{n+1}\frac{\partial}{\partial z} + (n+1)z^n\Delta_1)\varphi_1(z)\varphi_2(0) + \varphi_1(z)L_k\varphi_2(0). \end{aligned}$$

(The two terms come from L_1 and L_0 respectively.) The second term is zero, because φ_2 is primary.

Now suppose we want to compute four-point correlators

$$\langle\varphi_k(\zeta_1)\varphi_\ell(\zeta_2)\varphi_n(\zeta_3)\varphi_m(\zeta_4)\rangle = \langle k|\varphi_\ell(1)\varphi_n(x)|m\rangle = G_{nm}^{lk}(x).$$

Here we used an automorphism to set $z_1 = \infty$, $z_2 = 1$, $z_3 = x$ and $z_4 = 0$. By gluing,

$$G_{nm}^{lk}(x) = \sum_p C_{nm}^p C_{klp} A_{nm}^{lk}(p,x).$$

This function

$$A_{nm}^{lk}(p,x) = (C_{kl}^p)^{-1}\langle k|\varphi_\ell(1)\varphi_p(x)|0\rangle$$

is called a conformal block.

In minimal models, we consider degenerate representations of the Virasoro algebra $\mathrm{Vir} = \bigoplus_i \mathbb{C}L_i \oplus \mathbb{C}c$.

Theorem 4.2 (Kac, Feigin–Fuchs)**.** *The Verma module $|\Delta, c\rangle$ is irreducible for generic Δ and c. Set*

$$\begin{aligned} \alpha_\pm &\coloneqq \frac{\sqrt{1-c}\pm\sqrt{25-c}}{\sqrt{24}} \\ \Delta_0 &\coloneqq (1/4)(\alpha_+ + \alpha_-)^2. \end{aligned}$$

For fixed c, there is a singular vector at the following levels:

$$\Delta_{mn} \coloneqq \Delta_0 + \frac{1}{4}(m\alpha_+ + n\alpha_-), \quad m,n \in \mathbb{Z}_{>0}.$$

Example 4.3 (Level 0)**.** The simplest example is if $L_{-1}\,|\Delta\rangle$ is a singular vector. Then we have

$$L_1 L_{-1}\,|\Delta\rangle = 0.$$

But this is equal to

$$[L_1, L_{-1}]\,|\Delta\rangle = 2L_0\,|\Delta\rangle = 2\Delta\,|\Delta\rangle\,.$$

Hence this is only possible when $\Delta = 0$. This corresponds to $m = n = 0$.

Example 4.4 (Level 2)**.** The next simplest example is that

$$v = (L_{-2} + aL_{-1}^2)\,|\Delta\rangle$$

is a singular vector. Then

$$\begin{aligned} 0 = L_1 v &= [L_1, L_{-2}]\,|\Delta\rangle + (a[L_1, L_{-1}]L_{-1} + aL_{-1}[L_1, L_{-1}])\,|\Delta\rangle \\ &= 3L_{-1}\,|\Delta\rangle + 2aL_0L_{-1}\,|\Delta\rangle + 2aL_{-1}L_0\,|\Delta\rangle \\ &= (3 + 2a(2\Delta+1))\,|\Delta\rangle\,. \end{aligned}$$

Hence

$$a = -\frac{3}{2(2\Delta+1)}.$$

By similar calculations,

$$0 = L_2 v = \left(4\Delta + \frac{c}{2} - \frac{9\Delta}{2\Delta+1}\right)|\Delta\rangle .$$

This constrains c.

Let

$$\varphi_{12}^{(2)} = \left(L_{-2} - \frac{3}{2(2\Delta+1)}L_{-1}\right)\phi_{12} = 0.$$

This vector has norm zero and therefore we require it to really be zero. Transforming it into an operator gives

$$-\frac{3}{2(2\Delta+1)}\frac{\partial^2}{\partial z^2} + \sum_{i=1}^{N}\left(\frac{\Delta_i}{(z_i-z)^2} - \frac{1}{z_i-z}\frac{\partial}{\partial z_i}\right)\langle\phi_{12}(z)\phi_1(z_1)\cdots\phi_N(z_N)\rangle = 0.$$

The general OPE looks like

$$\phi_{12}(z)\phi_\Delta(z_1) = \sum_{\Delta'} C_{12,\Delta}^{\Delta'}(z-z_i)^{\Delta'-\Delta_{12}-\Delta}\phi_{\Delta'}(z_1) + \cdots .$$

Applying the operator to this, we get a constraint

$$\frac{3\mathcal{H}(\mathcal{H}-1)}{2(2\Delta_{12}+1)} + \Delta - \mathcal{H} = 0$$

where $\mathcal{H} \coloneqq \Delta' - \Delta_{12} - \Delta$. The two solutions are

$$\Delta' = \Delta(\alpha - \alpha_-), \Delta(\alpha + \alpha_-).$$

Hence we have shown

$$\phi_{12}\phi_{(\alpha)} = [\phi_{\alpha-\alpha_-}] + [\phi_{\alpha+\alpha_-}].$$

Here we are using *fusion ring* notation, where we forget about coefficients and just remember whether or not a field appears in the OPE. Similarly,

$$\phi_{21}\phi_{(\alpha)} = [\phi_{\alpha-\alpha_+}] + [\phi_{\alpha+\alpha_+}].$$

In particular, applying both rules,

$$[\phi_{02}] + [\phi_{22}] = \phi_{12}\phi_{21} = [\phi_{20}] + [\phi_{22}].$$

Hence it must actually just be $[\phi_{22}]$. As another example,

$$\phi_{12}\phi_{12} = [\phi_{11}] + [\phi_{13}].$$

But $[\phi_{11}]$ is the identity operator. So we now know how to apply ϕ_{13}:

$$\phi_{13}\phi_\alpha = [\phi_{\alpha+2\alpha_-}] + [\phi_\alpha] + [\phi_{\alpha+2\alpha_-}].$$

In general,

$$\phi_{m_1n_1}\phi_{m_2n_2} = \sum_{\ell=0}^{\ell_0}\sum_{k=0}^{k_0}\phi_{m_0+2\ell,n_0+2k}$$

where

$$m_0 = |m_1 - m_2| + 1$$
$$n_0 = |n_1 - n_2| + 1$$
$$\ell_0 = \min(m_1, m_2) - 1$$
$$k_0 = \min(n_1, n_2) - 1.$$

This corresponds to multiplication in $\mathrm{SL}_q(2)$.

Now let's consider some minimal models. For minimal models, pick two integers $p \neq q$. Suppose we have

$$\alpha_+ \coloneqq -\sqrt{\frac{q}{p}}, \quad \alpha_- \coloneqq \sqrt{\frac{p}{q}}.$$

Let $m < p$ and $n < q$ and consider $|\Delta_{mn}\rangle$. Note that

$$\Delta_{p+m,q+n} = \Delta_{mn}, \quad \Delta_{p-m,q-n} = \Delta_{mn}.$$

The central charge is

$$c = 1 - \frac{6(p-q)}{4pq}.$$

We have

$$\mathcal{M}(p/q) = \frac{1}{2} \bigoplus [\phi].$$

For minimal models it is possible to determine all the structure constants, by crossing symmetry. The four-point function turns out to be a hypergeometric function in x. We know how to analytically continue it, to get it in $1/x$. This gives explicit constraints on structure constants. For example,

$$\mathcal{M}\left(\frac{2}{5}\right) = \mathbb{C}[\phi_{11}] + \mathbb{C}[\phi_{12}],$$

since $\phi_{12} = \phi_{13}$ and $\phi_{11} = \phi_{14}$. Hence

$$\phi_{12}\phi_{12} = C\phi_{12} + \cdots$$

for some constant C. From the explicit constraints, we can compute

$$C = \frac{i}{5} \frac{\Gamma^2(1/5)}{\Gamma(4/5)\Gamma(3/5)} \sqrt{\frac{\sqrt{5}-1}{2}}.$$

Example 4.5. Consider $\mathcal{M}(p/p+1)$. These are called *unitary* minimal models. It is conjectured that these are the only minimal models with *real* structure constants. In $\mathcal{M}(3/4)$, we have

21 ϵ	22 σ	23 τ
11 I	12 σ	13 ϵ

Then we have

$$\epsilon\epsilon = \phi_{21}\phi_{21} = [I] + [\phi_{31}].$$

But we can also write it as

$$\epsilon\epsilon = \phi_{13}\phi_{13} = [I] + [\phi_{13}] + [\phi_{15}].$$

The dimension of ϕ_{31} is $\Delta = 5/3$, whereas for ϕ_{15} it is $\Delta = 5/2$. Hence they are different fields, and so

$$\epsilon\epsilon = [I].$$

Similarly one can compute (in the fusion ring)

$$\epsilon\sigma = \sigma, \quad \sigma\sigma = I + \epsilon.$$

When we pair with the antiholomorphic part, this example is actually the Ising model (at criticality). For $\phi_{21} = \phi_{13} = \epsilon$, we have $\Delta = 1/2$. Take

$$\psi \in V_{12} \otimes \overline{V}_{11}, \quad \overline{\psi} \in V_{11} \otimes \overline{V}_{12}.$$

Then

$$\psi(z)\psi(0) = \frac{I}{z} + \text{reg.}$$
$$\overline{\psi}(z)\overline{\psi}(0) = \frac{I}{z} + \text{reg.}$$
$$\psi(z)\overline{\psi}(0) = i\epsilon + \text{reg.}$$

We see a free fermion subalgebra $[I] \oplus [\psi] \oplus [\overline{\psi}] \oplus [\epsilon]$, with Hamiltonian

$$H = \frac{1}{2}\int (\psi\overline{\partial}\psi + \overline{\psi}\partial\psi)\, d^2x$$

and $T = (-1/2) :\psi\partial\psi:$. Analogously, $\phi_{21} = \phi_{22}$ has $\Delta = 1/16$. Take $V_{21} \otimes V_{21}$ and the subalgebra

$$\mathcal{A}_{IM} \coloneqq [I] \oplus [\sigma] \oplus [\mu] \oplus [\epsilon] \oplus [\psi] \oplus [\overline{\psi}]$$

where

$$\psi(z)\sigma(0) = z^{-1/2}\mu(0) + \cdots .$$

This space corresponds to the Ising model, where σ is the parameter of order and μ is the parameter of disorder. There is some duality between them at high temperatures.

5 Guillaume (Oct 22): A probabilistic approach to Liouville CFT

We'll start with some motivation. Liouville field theory first appeared in Polyakov's 1981 paper "Quantum geometry of bosonic strings". The idea is that when we have a quantum theory, to model a particle going from point A to point B, we sum over all paths connecting the two points. In string theory, points are replaced by loops. and therefore paths by surfaces connecting the loops. In this talk, the surface will always be the Riemann sphere S^2. Let $\mathcal{M}$ be the space of all Riemannian metrics on S^2. Polyakov tried to understand what the canonical uniform measure on $\mathcal{M}$. This is highly non-trivial because it is an infinite-dimensional, highly non-linear space. In particular we are interested in quantities

$$\int_{\mathcal{M}} Dg\, F[g]$$

for some formal measure Dg and functional $F[g]$. Recall the uniformization theorem from Riemannian geometry, which says that

$$\mathcal{M} = \{e^{\phi} g : \phi\colon S^2 \to \mathbb{R}\}$$

for some fixed metric g on S^2. Physicists understood that choosing a uniform measure on $\mathcal{M}$ is essentially choosing a measure on ϕ given by Liouville field theory.

What is Liouville theory? In the path integral formalism, let Σ be the space of all $X\colon S^2 \to \mathbb{R}$. Define

$$\langle F(\phi)\rangle \coloneqq \frac{1}{Z}\int_{\Sigma} DX\, F(X) e^{-S_L(X)}$$

for some uniform measure DX, and "weight" $e^{S_L(X)}$. The Liouville action is

$$S_L(X) = \frac{1}{4\pi}\int_{S^2} \left(|\partial^g X|^2 + QR_g X + \mu e^{\gamma X}\right)\, d\lambda_g.$$

- The $|\partial^g X|^2$ term is kinetic energy. It is the most basic term we can put into any energy functional.
- The $e^{\gamma X}$ is a non-linear term which makes the whole theory non-trivial. It is the total volume of S^2 using the metric given by X.
- The coupling constant μ is called the **cosmological constant**. The whole theory in the end depends trivially on μ so it doesn't really matter.
- $\gamma \in (0, 2)$. In physics, the notation is $b = \gamma/2$.
- $Q = 2/\gamma + \gamma/2$, and $c_L \coloneqq 1 + 6Q^2$.

Mark some points $z_i \in S^2$, with associated weights $\alpha_i \in \mathbb{R}$. The geometric interpretation is some conical singularities at those points. Then set

$$F(X) \coloneqq \prod_{i=1}^{N} e^{\alpha_i X(z_i)}.$$

For the correlator $\langle F(X) \rangle$ to exist, the **Seiberg bounds** must hold:

$$\sum_i \alpha_i > 2Q, \quad \alpha_i < Q \ \forall i.$$

Hence the minimum number of points for it to be well-defined is three. Note that these $e^{\alpha_i X(z_i)}$ are vertex operators, and primary fields.

In Liouville theory, we can make the path integral formalism rigorous using probability (following David, Kupiainen, Rhodes, Vargas, 2014). Start with

$$\frac{1}{Z} \int_\Sigma DX\, e^{-\frac{1}{4\pi} \int_{S^2} |\partial^g X|^2 \, d\lambda_g} \hat{F}(X).$$

Integrating by parts,

$$\int_2 |\partial^g X| \, d\lambda_g = - \int_{S^2} X \Delta_g X \, d\lambda_g.$$

So the kinetic term will give a Gaussian free field, with covariance given by the Green's function. If we diagonalize $-\Delta_g$ as

$$-\Delta_g \varphi_j(x) = \lambda_j \varphi_j(x) \quad \lambda_j > 0.$$

Then $X = c + \sum_{j \geq 1} c_j \varphi_g(x)$. We think of $c = \int_{S^2} X \, d\lambda_g$, and the φ_j are chosen with $\int_{S^2} \varphi_j(x) \, d\lambda_g = 0$. Then

$$-\frac{1}{4\pi} \int_{S^2} X \Delta_g X \, d\lambda_g = -\frac{1}{4\pi} \sum_{j \geq 1} \lambda_j c_j^2.$$

Hence the correlator becomes

$$\frac{1}{Z} \int_{\mathbb{R}} dc \int_{\mathbb{R}^N} \prod_{j \geq 1} e^{-u_j^2/2} \frac{du_j}{\sqrt{2\pi}} \hat{F}\Bigg(c + \underbrace{\sqrt{2\pi} \sum_{j \geq 1} u_j \frac{\varphi_j}{\sqrt{\lambda_j}}}_{X_{GFF}}\Bigg).$$

Here X_{GFF} is a Gaussian free field. So we *define* the correlator as

$$\int_{\mathbb{R}} dc\, \mathbb{E}[\hat{F}(c + X_{GFF})].$$

The $\hat{F}$ will include the rest of the terms in the Liouville action. In summary,

$$\langle \prod e^{\alpha_i \phi(z_i)} \rangle \coloneqq \int_{\mathbb{R}} dc\, bE \left[\prod_{i=1}^{N} e^{\alpha_i X_{GFF}(z_i) + c} \exp\left(-\frac{1}{4\pi} \int Q R_g (X_{GFF} + c) \, d\lambda_g - \frac{\mu}{4\pi} \int_{S^2} e^{\gamma (X_{GFF} + c)} d\lambda_g \right) \right].$$

Strictly speaking we need to introduce a regularization ϵ and send $\epsilon \to 0$. This integral over the zero mode c can be computed, giving

$$\frac{2\gamma}{\mu}^{-s}\Gamma(s)\prod_{i<j}\frac{1}{|z_i - z_j|^{\alpha_i\alpha_j}}\mathbb{E}\left[\left(\int_{S^2} e^{\gamma X_{GFF}(x)}\prod_{i=1}^N \frac{1}{|x-z_i|^{N_i\gamma}}d\lambda_g(x)\right)^{-s}\right]$$

with $s = (\sum \alpha_i - 2Q)/\gamma$.

Now that this is well-defined, we can ask for the usual structures of CFT (e.g. OPE, BPZ equations) in this language. Given a Mobius map $\psi\colon S^2 \to S^2$, correlators behave as conformal tensors

$$\langle\prod_{i=1}^N e^{\alpha_i\phi(\psi(z_i))}\rangle = \prod_{i=1}^N |\psi'(z_i)|^{-2\Delta_{\alpha_i}}\langle\prod_{i=1}^N e^{\alpha_i\phi(z_i)}\rangle$$

with conformal weight $\Delta_{\alpha_i} = (\alpha_i/2)(Q - \alpha_i/2)$. This is like seeing global conformal invariance. We can also ask for the BPZ equations, which is like seeing local conformal invariance. Let $\chi \coloneqq -\gamma/2$ or $-2/\gamma$. Then

$$\left(\frac{1}{\chi^2}\partial_z^2 + \sum_{i=1}^N \frac{\Delta_{\alpha_i}}{(z_i - z_j)^2} + \sum_{i=1}^N \frac{1}{z - z_i}\partial_{z_i}\right)\langle e^{\chi\phi(z)}\prod_{i=1}^N e^{\alpha_i\phi(z_i)}\rangle = 0.$$

By solving the BPZ equations, we get an explicit formula for 3-point functions called the **DOZZ formula**. Move the points z_1, z_2, z_3 to $0, 1, \infty$ to get

$$\langle\prod_{i=1}^3 e^{\alpha_i\phi(z_i)}\rangle = \frac{1}{|z_1 - z_2|^{\Delta_{12}}|z_1 - z_3|^{\Delta_{13}}|z_2 - z_3|^{\Delta_{23}}}C_\gamma(\alpha_1, \alpha_2, \alpha_3).$$

The DOZZ formula is for C_γ, and is in terms of double gamma functions. It is an analytic expression in the α_i.

Theorem 5.1 (Kupiainen, Rhodes, Vargas, 2017)**.** *The three-point correlator defined via probability is equal to the DOZZ formula.*

6 Ivan (Oct 29): WZW

No notes, sorry!

7 Gus (Nov 05): Free field realizations

Recall last time Ivan discussed WZW theory. The input is a simple Lie algebra $\mathfrak{g}$ with $\mathfrak{g}$-invariant bilinear form $(\cdot,\cdot)$. Then we consider the affine Lie algebra $\hat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}c$, which has commutators

$$[x_n, y_m] = [x, y]_{n+m} + \delta_{n+m,0}nc(x, y).$$

We considered various representations of $\hat{\mathfrak{g}}$. The first kind is Verma modules. If V_λ is a Verma for $\mathfrak{g}$, then there is a corresponding Verma $V_{\lambda,k} \coloneqq \mathrm{Ind}_{\hat{\mathfrak{g}}_+}^{\hat{\mathfrak{g}}}(V_\lambda)$ for $\hat{\mathfrak{g}}$. We always suppose λ is generic, so that $V_{\lambda,k}$ is irreducible. The other kind is evaluation modules. If V is any rep of $\mathfrak{g}$, then $V(z)$ is the rep of $\hat{\mathfrak{g}}$ where x_n acts as $z^n x$.

Using these representations, we considered intertwiners

$$\Phi(z)\colon V_{\lambda_1,k} \to V_{\lambda_2,k} \otimes V_\mu(z).$$

Take such $\Phi_i(z_i)$ for $\lambda_0, \dots, \lambda_{n+1}$ and consider matrix elements

$$\Psi(z_1, \dots, z_n) \coloneqq \langle u_0, \Phi_1(z_1) \cdots \Phi_n(z_n) u_{n+1} \rangle$$

for fixed vectors $u_0 \in V_{\lambda_0,k}$ and $u_{n+1} \in V_{\lambda_{n+1},k}$. These are correlation functions, and satisfy KZ equations

$$(k + h^\vee)\partial_{z_i} \Psi = \sum_{j \neq i} \frac{\Omega_{ij}}{z_i - z_j} \Psi.$$

Today we will focus on how to actually compute these intertwiners Φ explicitly, in a form suitable for calculating matrix elements.

Reminder on OPEs. Given two fields $a(z) = \sum a_n z^{-n-1}$ and $b(z) = \sum b_n z^{-n-1}$, they are *local* if

$$[a(z), b(w)] = \sum_{j=0}^{N-1} c_j(w) \partial_z^j \delta(z - w)$$

is a sum of delta functions and their derivatives. The $c_j(w)$ are *OPE coefficients.* We set

$$a_+(z) \coloneqq \sum_{n<0} a_n z^{-n-1}, \quad a_-(z) \coloneqq \sum_{n \geq 0} a_n z^{-n-1}.$$

This splitting is done so that $(\partial a)_\pm = \partial(a_\pm)$. The *normally ordered* product of fields is

$$\begin{aligned} {:}a(z)b(w){:} &= a_+(z)b(w) + b(w)a_-(z) \\ &= a_+b_+ + a_+b_- + b_+a_- + b_-a_-, \end{aligned}$$

i.e. we put all minus terms *before* plus terms. Then

$$\partial \, {:}ab{:} = {:}\partial a b{:} + {:}a \partial b{:} \, .$$

The only distinction between ${:}a(z)b(w){:}$ and $a(z)b(w)$ is the term b_+a_-, so

$$\begin{aligned} a(z)b(w) &= {:}a(z)b(w){:} + [a_-(z), b(w)] \\ &= {:}a(z)b(w){:} + \iota_{z,w} \sum_{j=0}^{N-1} \frac{c_j(w)}{(z-w)^{j+1}} \end{aligned}$$

where $\iota_{z,w}$ means expansion in $|z| > |w|$. Hence to compute OPEs, we just need the difference between $a(z)b(w)$ and the normal ordered ${:}a(z)b(w){:}$. The notation is

$$a(z)b(w) \sim \sum_{j=0}^{N-1} \frac{c_j(w)}{(z-w)^{j+1}}$$

in this case.

Example 7.1 (Free boson)**.** For the free boson, we have $a(z) = \sum a_n z^{-n-1}$ with $[a_n, a_m] = n\delta_{n+m,0}$. This corresponds to the OPE

$$a(z)a(w) \sim \frac{1}{(z-w)^2}.$$

Equivalently, this says

$$[a(z), a(w)] = \partial \delta(z - w).$$

Example 7.2 (Current algebras)**.** Given a simple $\mathfrak{g}$, for each $x \in \mathfrak{g}$ we produce a field

$$J_x(z) \coloneqq \sum x_n z^{-n-1}.$$

The commutator $[x_n, y_m] = [x, y]_{n+m} + nc(x, y)\delta_{n+m,0}$ yields the OPE

$$J_x(z)J_y(w) \sim \frac{J_{[x,y]}(w)}{z-w} + \frac{c(x,y)}{(z-w)^2}.$$

Now we can talk about free field realizations of $\hat{\mathfrak{g}}$. Some motivation comes from the finite dimensional setting. The *easiest* Lie algebras to handle are the abelian ones; the next easiest, with non-trivial commutators, is the Heisenberg algebra Heis with commutator

$$[\partial, x] = c.$$

In the Heisenberg, $[a, b] =$ scalar for any elements, so it is very easy to compute nested commutators. On the other end of the spectrum, for simple Lie algebras we have $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, giving rise to highly non-trivial terms in the BCH formula.

The difference between simple $\mathfrak{g}$ and the Heisenberg algebra "goes away" if we pass to universal enveloping algebras. Namely, there is an algebra homomorphism

$$U\mathfrak{g} \to U\,\mathrm{Heis}\,.$$

It comes from the action of G on G/B, which gives a homomorphism $\mathfrak{g} \to \mathsf{Vect}(G/B)$. This extends to

$$U\mathfrak{g} \to \mathsf{Diff}(G/B).$$

More generally, we can pick a line bundle $\lambda \in \mathrm{Pic}(G/B) \otimes \mathbb{C}$ and look at differential operators twisted by λ.

Example 7.3. For $\mathfrak{sl}_2$, the flag variety is $G/B = \mathbb{P}^1$. We have

$$\mathrm{Heis}_1 = \mathsf{Diff}(\mathbb{C}) \subset \mathsf{Diff}(\mathbb{P}^1).$$

The homomorphism is

$$e \mapsto \partial, \quad f \mapsto -x^2\partial, \quad h \mapsto -2x\partial,$$

given by restriction to a specific chart $\mathbb{C} \subset \mathbb{P}^1$. In general we can restrict to any chart, which is equivalent to changing a Borel B. More generally, picking a line bundle $\lambda \in \mathbb{C}$ gives

$$e \mapsto \partial, \quad f \mapsto -x^2\partial + \lambda, \quad h \mapsto -2x\partial + \lambda x.$$

We want to affinize this construction. A *free theory* is a collection of pairwise local fields $\{a^j(z)\}$ such that

$$[a^i_\pm(z), a^j_\pm(w)] = 0$$

and all OPE coefficients $c^j(w)$ are just scalars. Roughly this is the affine analogue of $[A, B] =$ scalar. The idea is to build fields with more interesting OPEs in terms of normally ordered products of free fields, in the same way that the example built $\mathfrak{sl}_2$ from Heis_1.

For this purpose, we need some tools for computing products of normally ordered products (of free fields).

1. (Taylor's theorem) Given a field $a(z, w)$ and some cutoff $N \in \mathbb{N}$, there is a Taylor expansion

$$a(z,w) = \sum_{j=0}^{n-1} c^j(w)(z-w)^j + (z-w)^N d^{(N)}(z,w)$$

where $c^j(w) = \partial_z^j a(z,w)|_{z=w}$.

2. (Wick's theorem) If $\{a_1, \dots, a_M\}$ and $\{b_1, \dots, b_N\}$ are two collections of free fields,

$$:a_1(z) \cdots a_M(z)::b_1(w) \cdots b_N(w):$$

is the sum over all possible subset of pairs of "contractions". Formally, it is

$$\sum_{s=0}^{\min(N,M)} \sum_{i_1 < \cdots < i_s} \sum_{j_1 \neq \cdots \neq j_s} \prod_{k=1}^{s} [a_{i_k,-}(z), b_{j_k}(w)] :A(\mathbf{i})B(\mathbf{j}):$$

where $A(\mathbf{i})$ denotes the product of all $a_i(z)$ where i is *not* one of the chosen indices i_k, and similarly for $B(\mathbf{j})$. The proof is roughly that commuting $+$ across $-$ produces terms $[a_-(z), b(w)]$, but because all OPE coefficients are scalar these can be collected in front. Notation:

$$[a_-(z), b(w)] =: \langle a(z)b(w) \rangle.$$

Example 7.4 (Virasoro)**.** Let $a(z)$ be a free boson. Then

$$L(z) := \frac{1}{2} :a(z)^2:$$

satisfies the OPE

$$L(z)L(w) \sim \frac{1}{(z-w)^4} + \frac{L(z)}{(z-w)^2} + \frac{2L'(z)}{z-w},$$

which is exactly the OPE for the Virasoro algebra.

Example 7.5 ($\hat{\mathfrak{sl}_2}$)**.** Take three free fields: a free boson $\alpha(z) = \sum \alpha_n z^{-n-1}$, and a $\beta\gamma$ system $\beta(z) = \sum \beta_n z^{-n-1}$ and $\gamma(z) = \sum \gamma_n z^{-n}$. This indexing of γ_n is done so that

$$[\beta_\pm, \gamma_\pm] = 0.$$

Normalize α so that

$$\alpha(z)\alpha(w) \sim \frac{2}{(z-w)^2}$$

because we want to think of it as the root α in $\mathfrak{sl}_2$. We also set

$$\beta(z)\gamma(w) \sim \frac{1}{z-w}$$

and all other OPEs are trivial. Unpacking,

$$[\beta_n, \gamma_m] = \delta_{n+m,0}.$$

For $\lambda \in \mathbb{C}$, consider the rep $\mathcal{H}_\lambda$ of $\{\alpha_n, \beta_n, \gamma_n\}$ generated by a vacuum vector v satisfying

$$\alpha_0 v = \lambda v, \quad \beta_0 v = 0, \quad \alpha_n v = \beta_n v = \gamma_n v = 0 \ \forall n > 0.$$

Now define

$$\begin{aligned} J_e(z) &= \beta(z) \\ J_h(z) &= -2 :\gamma(z)\beta(z): + \kappa^{1/2}\alpha(z) \\ J_f(z) &= - :\gamma^2(z)\beta(z): + \kappa^{1/2}\alpha(z)\gamma(z) + k\gamma'(z) \end{aligned}$$

where $\kappa := k + \hbar^\vee$ with k the level. For $\mathfrak{sl}_2$ we have $\hbar^\vee = 2$.

Theorem 7.6. *This prescription defines a rep of $\hat{\mathfrak{sl}}_2$ on $\mathcal{H}_\lambda(k)$. If λ, k are generic, i.e. $V_{\lambda,k}$ is irreducible, then*

$$\mathcal{H}_{\lambda/\sqrt{\kappa}}(k) \cong V_{\lambda,k}$$

as $\hat{\mathfrak{sl}}_2$-modules.

For special λ, k, it is not always an isomorphism. For example, if $\lambda = 0$ then it is not hard to see $f_0 v = 0$. Note that there is a strong analogy with the finite case.

Why is this important? It lets us write down a formula for the intertwining operator. For $\lambda \in \mathfrak{h}$, consider free fields

$$h^\lambda(z) h^\mu(w) \sim \frac{(\lambda, \mu)}{(z-w)^2}.$$

Construct a *vertex operator*

$$X(\mu, z) \coloneqq \exp\left(\sum_{n<0} \frac{h_n^\mu}{-n} z^n\right) \exp\left(\sum_{n>0} \frac{h_n^\mu}{-n} z^n\right) e^\mu z^{h_0^\mu}$$

where $e^\mu \colon \mathcal{H}_\lambda \to \mathcal{H}_{\lambda+\mu}$ induced from $v_\lambda \mapsto v_{\lambda+\mu}$. So $X(\mu, z)$ has components in $\mathrm{Hom}(\mathcal{H}_\lambda, \mathcal{H}_{\lambda+\mu})$. They satisfy

$$X(\mu, z) X(\nu, w) = (z-w)^{(\mu,\nu)} :X(\mu, z) X(\nu, w):$$

where $(z-w)^{(\mu,\nu)}$ means

$$\exp\left(-(\mu, \nu) \sum_{n \geq 0} \frac{w^n}{n z^n}\right).$$

This is like doing the calculation $e^A B e^{-A} = B + [A, B]$. Hence we can view $X(\mu, z) X(\nu, w)$ *analytically*, since we know how to analytically continue $(z-w)^{(\mu,\nu)}$ and the normally ordered product is holomorphic. To construct

$$\Phi^m(z) \colon V_{\lambda,k} \to V_{\lambda+\mu-2m,k} \otimes V_\mu(z),$$

we think of $V_{\lambda,k} \cong \mathcal{H}_{\lambda/\sqrt{\kappa}}(k)$ and similarly for the other term.

1. ($m = 0$) In the easy case of level zero, we take

$$\Phi^0(z) u \coloneqq X\left(\frac{\mu}{\sqrt{\kappa}}, z\right) \exp\left(-\gamma(z) \otimes e\right) u \otimes v_\mu.$$

   One can check via Wick's theorem that this is an intertwiner for $\hat{\mathfrak{sl}}_2$.

2. ($m > 0$) In this case we need a *screening current*

$$U(t) \coloneqq X\left(\frac{-\alpha}{\sqrt{\kappa}}, t\right) \beta(t).$$

   It satisfies

$$[U(t), J_e(w)] = 0, \quad [J_f(z), U(w)] = \kappa \partial_w \left(\delta(z-w) X\left(\frac{-\alpha}{\sqrt{\kappa}}, w\right)\right).$$

   Then form

$$\Theta(z, t_1, \ldots, t_m) u = X\left(\frac{\mu}{\sqrt{\kappa}}, z\right) \exp\left(-\gamma(z) \otimes e\right) U(t_1) \cdots U(t_m) u \otimes v_\mu.$$

   This can be analytically continued to cycles $C \subset (\mathbb{C}^\times)^m$, in the t variables. Then

$$\Phi^m(z) \coloneqq \int_C \Theta(z, t)\, d\mathbf{t}$$

   is an intertwiner, because its commutator with J_f is a total t-derivative.

8 Sam (Nov 07): Representations of quantum affine algebras and qKZ equations

Our goal will be to understand the second column of the following table:

$U\widetilde{\mathfrak{g}}$	$U_q(\widetilde{\mathfrak{g}})$
correlators $\Phi\colon V_{\lambda,k} \to V_{\mu,k} \otimes V(z)$	correlators
currents $J_a(z)$	quantum currents $L_a(z)$
KZ equation	qKZ equation
$\frac{\Omega_{ij}}{z_i - z_j}$ satisfies CYB	R satisfies YB.

Then we'll hopefully have enough time to see this for the simplest case $\mathfrak{sl}_2$.

Definition 8.1. Let $\mathfrak{g}$ be finite dimensional with Cartan matrix $A = (a_{ij})$. The **quantum affine algebra** $U_q(\hat{\mathfrak{g}})$ is a Hopf algebra over $\mathbb{C}[[q-1]]$, with generators e_i, f_i, q^h where $h \in \mathfrak{h}$ and relations:

1. $[q^{h_1}, q^{h_2}] = 0$;
2. $q^h e_i = q^{e_i(h)} e_i q^h$ and similarly $q^h f_j = q^{-f_j(h)} f_j q^h$;
3. $[e_i, f_j] = 0$ unless $i = j$ in which case
$$[e_i, f_i] = \frac{q^{d_i h_i} - q^{-d_i h_i}}{q - q^{-1}};$$
4. q-Serre relations.

Since this is a Hopf algebra we have the coproduct Δ, counit ϵ, and antipode γ. For example

$$\Delta(e_i) = e_i \otimes q^{h_i} + 1 \otimes e_i.$$

Definition 8.2. Let $U_q(\widetilde{\mathfrak{g}})$ be the algebra generated by $U_q(\hat{\mathfrak{g}})$ and symbols q^{ad} where $a \in \mathbb{C}$. The additional relations arise from writing

$$q^{ad} = \sum_{n>0} \frac{1}{n!}(\tau ad)^n$$

and set

$$[d, e_i] = \delta_{i0} e_i, \quad [d, f_j] = \delta_{j0} f_j, \quad [d, q^h] = 0.$$

We want an analogue of evaluation reps for $U_q(\hat{\mathfrak{g}})$, i.e. some composition

$$U_q(\hat{\mathfrak{g}}) \to U_q(\mathfrak{g}) \xrightarrow{\rho} V.$$

Unfortunately, unless $\mathfrak{g} = \mathfrak{sl}_n$, this first map does not exist. Instead, define

$$\begin{aligned} D_z \colon U_q(\hat{\mathfrak{g}}) &\to U_q(\hat{\mathfrak{g}}) \\ e_0 &\mapsto z e_0 \\ f_0 &\mapsto z^{-1} f_0 \end{aligned}$$

and all other generators map to themselves. Given some representation $\rho\colon U_q(\hat{\mathfrak{g}}) \to \mathrm{End}(V)$, we introduce the z by the twist

$$\rho \circ D_z \colon U_q(\hat{\mathfrak{g}}) \to \mathrm{End}(V(z)).$$

Definition 8.3. A rep V is an **evaluation rep** if:

1. it is finite length over $U_q(\mathfrak{g}) \hookrightarrow U_q(\hat{\mathfrak{g}})$;
2. $V \cong \bigoplus_\lambda V_\lambda^q$ where V_λ^q are highest weight reps.

Construction 1. Given a highest weight rep V_λ^q of $U_q(\mathfrak{g})$, we can induce it up to $U_q(\hat{\mathfrak{g}})$ to get $V_{\lambda,k}^q$.

Construction 2. If V is a $U_q(\hat{\mathfrak{g}})$-module and

$$g\colon L_{\lambda_1}^q \otimes L_{\lambda_2}^q \otimes V$$

is an $U_q(\mathfrak{g})$ intertwiner, we can extend it uniquely to an intertwiner

$$\Phi^g(z)\colon V_{\lambda_1,k}^q \to V_{\lambda_2,k}^q \hat{\otimes} V(z)$$

of $U_q(\hat{\mathfrak{g}})$ intertwiner. To get a $U_q(\widetilde{\mathfrak{g}})$ intertwiner, we need a twist

$$\widetilde{\Phi}^g(z)\colon V_{\lambda_1,k}^q \to V_{\lambda_2,k}^q \otimes z^{-\Delta} V[z^\pm]$$

where $\Delta = \Delta(\lambda_1) - \Delta(\lambda_2)$ with

$$\Delta(\lambda) \coloneqq \frac{\langle \lambda, \lambda+\rho \rangle}{2(k+h^\vee)}.$$

Definition 8.4. A **universal R-matrix** is an invertible

$$R \in U_q(\widetilde{\mathfrak{g}}) \hat{\otimes} U_q(\widetilde{\mathfrak{g}})$$

such that:

1. $R\Delta(x) = \Delta^{\mathrm{op}}(x) R$;
2. $(\Delta \otimes \mathrm{id}) R = R_{13} R_{23}$;
3. $(\mathrm{id} \otimes \Delta) R = R_{13} R_{12}$.

One can construct such a thing via the Drinfeld double construction. If H is any Hopf algebra, its double is $D(H) \coloneqq H \otimes H^\vee$. Then

$$R = 1 \otimes \mathrm{id}_H \otimes 1 \in H \otimes H^\vee \otimes H \otimes H^\vee$$

is an R-matrix in $D(H)$. We can apply this to

$$U_q(\widetilde{\mathfrak{g}}) = D(U_q(\widetilde{\mathfrak{b}}_+))/U_q(\mathfrak{h})$$

to get the universal R-matrix. The resulting formula is

$$\widetilde{R} = q^{c\otimes d + d \otimes c + \sum e_i \otimes f_i} \sum a_j \otimes a^j$$

where a_j is a basis for $U_q(\widetilde{\mathfrak{g}})$ as a module over the base ring. We'll actually use the R-matrix for $U_q(\hat{\mathfrak{g}})$

$$R \coloneqq q^{-c\otimes d - d\otimes c} \widetilde{R}.$$

We also need to add the spectral parameter z, by

$$R(z) \coloneqq (D_z \otimes \mathrm{id}) R = (\mathrm{id} \otimes D_{z^{-1}}) R.$$

These R-matrices satisfy the Yang–Baxter (YB) equation. (Aside: this means that given any rep W, the braid group B_n acts on $W^{\otimes n}$.)

Definition 8.5. Let $R^{\mathrm{op}} \coloneqq \mathrm{flip} \circ R$. The **q-currents** are

$$L^{+,V}(z) \coloneqq (\mathrm{id} \otimes \rho_V) R^{\mathrm{op}}(z) \in U_q(\hat{\mathfrak{g}}) \hat{\otimes} \operatorname{End}(V)[[z]]$$
$$L^{-,V}(z) \coloneqq (\mathrm{id} \otimes \rho_V) R^{-1}(z^{-1}) \in U_q(\hat{\mathfrak{g}}) \hat{\otimes} \operatorname{End}(V)[[z^{-1}]].$$

These satisfy

$$L^\pm(z) = 1 \otimes 1 + (q - q^{-1}) \sum_{a \in B} \widetilde{J}_a^\pm(z) \otimes \rho_V(a) + O((q-q^{-1})^2).$$

Definition 8.6 (q-Sugawara)**.** We need an action of q^d in $V^q_{\lambda,k}$ when $k \neq -h^\vee$. Use that there is an adjoint action

$$\mathrm{Ad}_{q^d} \colon U_q(\hat{\mathfrak{g}}) \to U_q(\hat{\mathfrak{g}})$$

Let $\widetilde{\rho} \coloneqq \rho + h^\vee d$ and $\widetilde{\lambda} \coloneqq \lambda + dk - \Delta c$ such that

$$\langle \widetilde{\lambda}, \widetilde{\lambda} + \widetilde{\rho} \rangle = 0.$$

Recall there is a quantum Casimir C in $U_q(\widetilde{\mathfrak{g}})$ which acts by $q^{\langle \widetilde{\lambda}, \widetilde{\lambda}+\rho \rangle}$ in $V_{\widetilde{\lambda}}$. If we pick $\widetilde{\lambda}$ as above, then

$$1 = C^{-1} = q^{-2\hat{\rho}} q^{-2kd} \,\mathrm{mult}\left((\gamma \circ \mathrm{Ad}_{q^{2kd}}) \otimes 1 R^{\mathrm{op}} \right).$$

Hence we can define an action of

$$q^{2(k+h^\vee)d} = \mathrm{mult}\left((\gamma \circ \mathrm{Ad}_{q^{2kd}}) \otimes 1 R^{\mathrm{op}} \right)$$

on $V^q_{\lambda,k}$.

Finally we can talk about correlators and qKZ equations. As usual we define correlators

$$\Psi \coloneqq \langle u_0 \widetilde{\Phi}^g(z_1) \cdots \widetilde{\Phi}^g(z_N) u_{N+1} \rangle$$

where u_0 is lowest weight and u_{N+1} is highest weight. If we consider

$$\Psi(z_1, \ldots, q^{2(k+h^\vee)} z_j, \ldots, z_N)$$

this is like acting by d in the j-th insertion. By our construction of what this means, it is the same as inserting L^+ on one side and L^- on the other. Because the $\widetilde{\Phi}$ are intertwining operators, these $L^\pm$ pass through to either side, moduli picking up an R-matrix every time. Finally, the action of $L^\pm$ on lowest and highest weight vectors yields q^{λ_0} and q^{λ_N}. Putting this all together, we get

$$\Psi(z_1, \ldots, pz_j, \ldots, z_N) = q^{\lambda_0 + \lambda_N + 2\rho}|_{V_j} R^{j,1}\left(\frac{pz_j}{z_1}\right) R^{j,j-1}\left(\frac{pz_j}{z_{j-1}}\right) (R^{-1})^{j,j+1}\left(\frac{z_{j+1}}{z_j}\right) \cdots \Psi(z_1, \ldots, z_N),$$

called the **qKZ equation**. They form a *consistent* system, i.e. if we write

$$\Psi(z_1, \ldots, pz_j, \ldots, z_N) = A_j \Psi(z_1, \ldots, z_N),$$

then we have

$$A_i(z_1, \ldots, pz_j, \ldots, z_N) A_j = A_j(\cdots) A_i.$$

9 Henry (Nov 12): An overview of AGT

First I'll say some words about what I understand of the physics behind the AGT correspondence. Then we'll do one explicit calculation to show that the existence of some formula like the one given by AGT should not be too surprising.

Consider 6d $\mathcal{N} = (2,0)$ SCFT. As a theory on $\mathbb{R}^6$, it is indexed by an ADE Lie algebra $\mathfrak{g}$, because one way it arises is as the compactification of type IIB strings on $\mathbb{R}^6 \times \widetilde{\mathbb{C}^2/\Gamma}$ where *both* the volume of cycles on $\widetilde{\mathbb{C}^2/\Gamma}$ and the string length ℓ tend to zero. The type IIB string theory essentially decouples into type IIB supergravity, and the desired 6d theory near $\mathbb{R}^6 \times \{\text{singularity}\}$. Without the limit $\ell \to 0$, we would get *little string theory* instead of 6d $\mathcal{N} = (2,0)$ SCFT. Mathematically, this is the distinction between a q-deformation vs the $q = 1$ case.

Now imagine instead of $\mathbb{R}^6 \times K3$, we take $\mathbb{R}^4 \times C \times K3$. Compactifying on C gives some 4d theory. On the other hand, compactifying on $\mathbb{R}^4$ gives some CFT on C. The technicality here is that $C \times K3$ is not CY3 and therefore destroys supersymmetry, and the 4d theory needs to be topologically twisted. Mathematically, there is a construction of a family $\mathcal{X} \to C$ of CY3s $X \to C$ fibered over C, parameterized by the Hitchin base $\Gamma(C, K_C \times_{\mathbb{G}_m} \mathfrak{h}/W)$, so that

$$\begin{pmatrix} \text{type IIB string} \\ \text{on the CY3 } X \end{pmatrix} \approx \begin{pmatrix} \text{6d } \mathcal{N} = (2,0) \text{ theory} \\ \text{on the curve } C \end{pmatrix},$$

for the purposes of looking at the 4d gauge theory and 2d CFT. Since X is now CY3, the resulting 4d gauge theory is $\mathcal{N} = 2$, and in fact is $\mathcal{N} = 2$ super Yang–Mills. It turns out the Hitchin base is precisely the Coulomb branch of the 4d theory, i.e. the invariant polynomials $\phi_k(z)$ are the vevs.

In general, when compactifies the 6d theory on C, there are lots of non-trivial fields in the 4d theory. To control what fields arise, we add punctures to C. Punctures control divergences in $\phi_k(z)$, which therefore restrict the vevs. One can work out some rules determining a map

$$G\colon (C, \{p_1, \ldots, p_k\}) \to \text{4d gauge theory}$$

such that G behaves nicely under degenerations of C. Then any physical quantity associated to the 4d theory gives an object in the 2d CFT, by pre-composition with G. The AGT correspondence is exactly this dictionary. For example,

1. the Nekrasov partition function

   $$Z_{\text{inst}}(\mathbb{R}^4) \coloneqq Z(Q, \mathbf{u}, t_1, t_2) \coloneqq \sum_{n \geq 0} Q^n \int_{\mathcal{M}(r,n)} 1$$

   yields a conformal block.

Recall that the Nekrasov partition function is one half of the 4d partition function for S^4, which is, schematically,

$$Z(S^4) = \int \mu(\mathbf{u}) Z(\mathbf{u}) \overline{Z(\mathbf{u})},$$

where $\mu(\mathbf{u})$ is some measure. This is very similar to how in 2d CFT, correlators are built from conformal blocks as, schematically,

$$\langle \prod_i \phi_i \rangle = \sum_{\text{primary } \phi} \mu(\phi) \mathcal{F}(\phi) \overline{\mathcal{F}(\phi)}.$$

One can match this exactly, to get the full dictionary

1'. the full partition function $Z(S^4)$ (with insertions) yields correlators in the CFT.

Another aspect of the dictionary concerns the algebra of BPS states. Mathematically, this BPS algebra is the cohomology $\mathcal{H} \coloneqq \bigoplus_n H^*(\mathcal{M}(r,n))$. On the CFT side,

2. the BPS algebra yields the vertex algebra of the CFT.

In particular, this means that whatever symmetries act on CFT are also act on the BPS algebra. The minimum is that $\mathcal{H}$ is a module for the Virasoro algebra.

Now let's look at the mathematics, on the CFT side. Recall that the Hilbert space of a CFT is a rep of the Virasoro algebra, and splits into highest weight irreps:

$$\mathcal{H} = \bigoplus_\alpha \mathcal{H}_\alpha.$$

In other words, any state can be written in the form

$$L_{-i_1} L_{-i_2} \cdots L_{-i_m} |\alpha\rangle$$

where $|\alpha\rangle$ is a primary field of weight α. Sorting $i_1 < i_2 < \cdots < i_m$, since they all commute, these indices form a Young diagram Y. Hence states $v_{\hat{\alpha}}$ are indexed by $\hat{\alpha} = (\alpha, Y)$, where

- α is a highest weight, and
- Y is a Young diagram, encoding $L_{-Y_1} \cdots L_{-Y_k} |\alpha\rangle$.

The main objects we want to compute in a CFT are the *correlation functions* of vertex operators. These correspond to n-point functions where we insert states at various points. In particular, we care about 2- and 3-point correlators. All other correlators arise from gluing 2- and 3-point ones.

- There is an inner product
$$\mathcal{K}_{\hat{\alpha},\hat{\beta}} \coloneqq \langle V_{\hat{\alpha}} | V_{\hat{\beta}} \rangle \propto \langle V_{\hat{\alpha}}(0) V_{\hat{\beta}}(\infty) \rangle.$$
(The rhs scales as $z^{-2\Delta}$ as $z \to \infty$, so there is a scaling factor involved to make this well-defined). The most important property is that
$$\langle L_{-n} V_{\hat{\alpha}} | V_{\hat{\beta}} \rangle = \langle V_{\hat{\alpha}} | L_n V_{\hat{\beta}} \rangle.$$
This comes from a re-expansion of $T(z) = T(1/\xi)$.
- There are structure constants
$$V_{\hat{\alpha}}(z) V_{\hat{\beta}}(w) = \sum_{\hat{\gamma}} \mathcal{C}^{\hat{\gamma}}_{\hat{\alpha}\hat{\beta}} \frac{V_{\hat{\gamma}}(w)}{(z-w)^{-\Delta_{\hat{\gamma}} + \Delta_{\hat{\alpha}} + \Delta_{\hat{\beta}}}}$$
from 3-point functions. The 3-point functions are particularly important, because they determine the OPE of two operators.

In general, note that specializing to $z_1, z_2, z_3 = 0, 1, \infty$ removes the annoying $z_i - z_j$ factors. We will do this starting soon.

By repeated applications of the Ward identity, via moving contours around, one shows that all correlators of descendant fields are determined by correlators of primary fields, so we only need to compute objects like
$$V_\alpha(z) V_\beta(w) = \sum_{\hat{\gamma}} \mathcal{C}^{\hat{\gamma}}_{\alpha\beta} \frac{V_{\hat{\gamma}}(w)}{(z-w)^{-\Delta_{\hat{\gamma}} + \Delta_\alpha + \Delta_\beta}}.$$
In fact all $\mathcal{C}^{\hat{\gamma}}_{\alpha\beta}$ can be determined in terms of just $\mathcal{C}^{\gamma}_{\alpha\beta}$, by e.g. relations like
$$L_m V_{\hat{\alpha}}(z) V_\beta(w) = (z^{n+1}\partial_z + (n+1) z^n \Delta_{\hat{\alpha}}) V_{\hat{\alpha}}(z) V_\beta(w) + V_{\hat{\alpha}}(z) L_m V_\beta(w) \tag{1}$$
for $m > 0$, where the second term is zero because V_β is primary. Then solving the differential equation yields $\mathcal{C}^{\hat{\gamma}}_{\alpha\beta}$ in terms of 3-point correlators of just primaries.

Hence the general 2-point and 3-point correlators factor into two pieces: structure constants of the CFT itself, and quantities coming *purely* from the rep theory of the Virasoro. Namely, the only data *specific* to the CFT are:

1. the constants K_α in the two-point functions
$$\mathcal{K}_{\hat{\alpha},\hat{\alpha}'} = K_\alpha \delta_{\alpha,\alpha'} \delta_{|Y|,|Y'|} S_\Delta(Y, Y');$$
2. the constants $C^{\beta}_{\alpha\alpha'}$ in the three-point functions
$$\mathcal{C}^{\hat{\alpha}''}_{\hat{\alpha}\hat{\alpha}'} = C^{\alpha''}_{\alpha\alpha'} \beta^{\Delta''}_{\Delta\Delta'}(Y, Y'; Y'').$$

The tensors $S_\Delta(Y, Y')$ and $\beta^{\Delta''}_{\Delta\Delta'}(Y, Y'; Y'')$ arise purely from the rep theory of the Virasoro. Here Δ is the conformal dimension associated to the highest weight α irrep. We can use these 2- and 3-point tensors (and the structure constants K_α and $C^{\alpha''}_{\alpha\alpha'}$ to construct n-point correlators by gluing.

Example 9.1 (4-point block)**.** For 4-point functions, by repeated OPE

$$\langle V_{\hat{\alpha}_1}(z_1)V_{\hat{\alpha}_2}(z_2)V_{\hat{\alpha}_3}(z_3)V_{\hat{\alpha}_4}(z_4)\rangle = \sum_{\hat{\alpha}_{12},\hat{\alpha}_{34}} \frac{\mathcal{C}^{\hat{\alpha}_{12}}_{\hat{\alpha}_1\hat{\alpha}_2}\mathcal{C}^{\hat{\alpha}_{34}}_{\hat{\alpha}_3\hat{\alpha}_4}}{z_{12}^{\Delta}z_{34}^{\Delta}}\langle V_{\hat{\alpha}_{12}}(z_2)V_{\hat{\alpha}_{34}}(z_4)\rangle.$$

Plugging in $z_1, z_2, z_3, z_4 = 1, \infty, x, 0$ turns the 2-point correlator into the Shapovalov form, and using only primaries gives

$$x^{\cdots}C^{\alpha}_{\alpha_1\alpha_2}K_\alpha C^{\alpha}_{\alpha_3\alpha_4}\mathcal{B}_\Delta(\Delta_1,\Delta_2;\Delta_3,\Delta_4|x)$$

where $\mathcal{B}$ is the 4**-point conformal block** built from the 2- and 3-point rep theoretic objects S and β:

$$\mathcal{B}_\Delta = \sum_k x^k\mathcal{B}^{(k)}_\Delta = \sum_{|Y|=|Y'|} x^{|Y|}\mathcal{B}^{Y,Y'}_\Delta$$

where

$$\mathcal{B}^{Y,Y'}_\Delta = \beta^{\Delta}_{\Delta_1\Delta_2}(Y)S_\Delta(Y,Y')\beta^{\Delta}_{\Delta_3\Delta_4}(Y').$$

We can compute the 4-point conformal block for a free field, where all correlators can be written in closed form. Recall that the free boson has action $\int d^2z\,\partial\phi\overline{\partial\phi}$. To get the propagator, consider

$$0 = \frac{\delta}{\delta\phi(z)}\int D\phi\, e^{-S}\phi(w) = \int D\phi\, e^{-S}(-\partial^2\phi(z)\phi(w) + \delta(z-w)).$$

Solve this ODE using $\partial^2\ln(z-w)^2 \propto \delta(z-w)$ to get things like

$$\langle\partial\phi(z_1)\phi(z_2)\rangle = \frac{1}{z_1-z_2}, \quad \langle\partial\phi(z_1)\partial\phi(z_2)\rangle = \frac{1}{(z_1-z_2)^2}.$$

In this theory, $T(z) :=:\partial^2\phi(z):$ is the stress-energy tensor. We can use it to verify things are primary fields.

Proposition 9.2. $:e^{\alpha\phi}:$ *is a primary field of weight* $-\alpha^2$.

To compute conformal blocks, note that the OPE of primaries in this theory is very simple:

$$\prod_i :e^{\alpha_i\phi(z_i)}: = \prod_{i<j}(z_i-z_j)^{-2\alpha_i\alpha_j} :\prod_i e^{\alpha_i\phi(z_i)}: .$$

This is a nice combinatorial exercise. Taking correlators gives

$$\langle\prod_i :e^{\alpha_i\phi(z_i)}:\rangle = \prod_{i<j}(z_i-z_j)^{-2\alpha_i\alpha_j}\delta(\sum_i\alpha_i).$$

The δ function comes from the same place as $\delta(z-w)$ in the propagator; it is a *selection rule* for correlators. Specifically, the 4-point correlator is

$$\langle\prod_{i=1}^4 :e^{\alpha_i\phi(z_i)}:\rangle = \prod_{i<j}(z_i-z_j)^{-2\alpha_i\alpha_j}$$

when $\alpha_1+\alpha_2+\alpha_3+\alpha_4 = 0$. Plug in $z_1, z_2, z_3, z_4 = 1, \infty, x, 0$ to get the 4-point conformal block

$$\mathcal{B}_\Delta(x) = x^{-2\alpha_3\alpha_4}(1-x)^{-2\alpha_1\alpha_3} = x^{-2\alpha_3\alpha_4}(1+2\alpha_1\alpha_3x+\cdots).$$

The intermediate state is $:e^{\alpha\phi}:$ where

$$\alpha = \alpha_1+\alpha_2 = -\alpha_3-\alpha_4$$

and therefore has weight $\Delta = -\alpha^2$.

Set $\sqrt{2}\alpha_1 = m_1$ and $\sqrt{2}\alpha_3 = m_2$ for normalization. One can expand in x to get

$$(1-x)^{-m_1 m_2} = \sum_k x^k \mathcal{B}_\Delta^{(k)} = \sum_{k\geq 0} \frac{x^k}{k!} \frac{\Gamma(k+m_1m_2)}{\Gamma(m_1m_2)}.$$

From our general theory above,

$$(1-x)^{-m_1m_2} = \sum_{|Y|=|Y'|} x^{|Y|} \mathcal{B}_\Delta^{Y,Y'}.$$

For this specific function, there is actually a way to rewrite it in terms of a *single* sum over partitions, thereby simplifying the double sum.

Proposition 9.3.

$$(1-x)^{-m_1m_2} = \sum_Y x^{|Y|} \prod_{\square\in Y} \frac{(m_1 + j(\square) - i(\square))(m_2 + j(\square) - i(\square))}{h(\square)^2},$$

where $h(\square)$ is the hook length.

Proof. It suffices to prove this for integers m_1, m_2 and analytically continue. Recall the Cauchy formula

$$\prod_{i,j} \frac{1}{1-t_is_j} = \sum_Y s_Y(t)s_Y(s).$$

Plug in $t = (\sqrt{x}, \ldots, \sqrt{x}, 0, 0, \ldots)$ where there are m_1 non-zero entries, and similarly for s and m_2. Then the lhs is $(1-x)^{-m_1m_2}$. For the rhs, compute

$$s_Y(t) = x^{|Y|/2} \#\mathrm{SSYT}(Y;r) = x^{|Y|/2} \prod_{\square\in Y} \frac{m_1 + j(\square) - i(\square)}{h(\square)},$$

where $\mathrm{SSYT}(Y;r)$ is semistandard Young tableaux of shape Y with labels from 1 to r. (Equivalently, it is a basis of the rep L_Y of SL_r.) The desired formula follows. □

This formula is *exactly* the localization formula for

$$\sum_{n\geq 0} x^n \int_{\mathrm{Hilb}(\mathbb{C}^2,n)} \mathrm{ch}(L_{m_1}^{[n]})\,\mathrm{ch}(L_{m_2}^{[n]})$$

where L_m is the trivial line bundle of weight m on $\mathbb{C}^2$, and $L^{[n]}$ is the induced bundle on $\mathrm{Hilb}(\mathbb{C}^2, n)$. In physics-speak, this is the *Nekrasov partition function* for $U(1)$ gauge theory with two fundamental hypermultiplets of mass m_1 and m_2.

The more famous and much less trivial check of AGT is to show that 4-point functions in Liouville theory matches with the Nekrasov partition function for SU(2) gauge theory and four matter hypermultiplets. Computationally, this can be checked by:

1. computing conformal blocks using the S and β tensors;
2. computing Nekrasov's partition function by localization.

Liouville theory fits into a hierarchy of CFTs called the A_{N-1} Toda theories; Liouville theory is when $N = 1$. Such CFTs have an *extended* $\mathcal{W}$-symmetry when $N \geq 2$. In addition to the stress-energy tensor T, there are additional (holomorphic) symmetry currents $\mathcal{W}^{(k)}$, forming a $\mathcal{W}$-algebra.

Theorem 9.4 (Maulik–Okounkov, Schiffmann–Vasserot)**.** *$\bigoplus_n H^*(\mathcal{M}(r,n))$ carries a $\mathcal{W}$-algebra action.*

10 Shuai (Nov 26): Analogies between conformal field theory and number theory

The goal today is to show how some kind of "S-duality" implies the quadratic reciprocity law, under the assumption that some kind of gauge theory exists over number fields. For the analogy today, we'll think of a number field K in terms of the "Riemann surface" $\operatorname{Spec} \mathcal{O}_K$. It includes some special points, corresponding to infinite places, and other points are finite places. There is a projection $\pi\colon \operatorname{Spec} \mathcal{O}_K \to \operatorname{Spec} \mathbb{Z}$.

In general, we want to explore the similarity between conformal field theory and this picture. Since the theory is conformal, there is no difference between a puncture and an S^1, so we think of a punctured Riemann surface. Then we look at integrals like

$$\int_{\mathcal{A}/\mathcal{G}} f(x) e^{CS(A)} \prod_{i=1}^n \mathcal{O}_{p_i}\, dx,$$

where $\mathcal{A}/\mathcal{G}$ is the moduli of connections up to gauge equivalence, and $\mathcal{O}_{p_i}$ are local operators. We'll explain what the action $e^{CS(A)}$ should correspond to in number theory, but most importantly we'll identify what should correspond to local operators, e.g. Wilson loops or t'Hooft operators.

First we'll consider the path integral without local operator insertions. In number theory, local information is encoded in the adeles

$$A_K \coloneqq \prod_{v \text{ primes}}' K_V$$

where $\prod'$ is the *restricted* product, namely where only finitely many entries are allowed to lie outside the corresponding ring of integers. For example, $A_\mathbb{Q}$ consists of sequences $(x_\infty, x_2, x_3, x_5, \ldots)$. Also we have the ideles

$$I_K \coloneqq \prod{}' \mathrm{GL}(1, K_V) = \prod{}' K_V^\times.$$

The conformal symmetry becomes the global action by $\mathrm{GL}(1, K) = K^\times$. So in general we should consider $C_K \coloneqq I_K/K^\times$, called the **idele class group**. This is the analogue of the state space $\mathcal{A}/\mathcal{G}$ in CFT. Luckily, in number theory we have the Haar measure $d^\times x$ on C_K.

Now we should explain the exponential action $e^{CS(A)}$. This should be viewed as some unitary representation of the time-evolution operator. In other words, on the number theory side it should be the character of something. In number theory we do have characters of A_K, but more importantly we have characters of the ideles I_K. For a given number field, all these characters are of the form

$$I_K \xrightarrow{w\omega_s} \mathbb{C}^\times$$

where $\omega_s = |x|^s$ is a **quasi-character**. We will also insert functions $f(x)$. These will be functions $f\colon A_K \to \mathbb{C}$, but they should be $K^\times$-invariant. The naive way to construct such things is to average

$$\sum_{\alpha \in K^\times} f(\alpha x), \quad f \in S(A_K)$$

where $S(A_K)$ is all Schwartz functions on A_K, for convergence. Hence on the number theory side, the analogue of the path integral is

$$\int_{I_K/K^\times} \sum_{\alpha \in K^\times} f(\alpha x)(w\omega_s)(x)\, d^\times x.$$

From number theory, this is equal to

$$\int_{I_K} f(x)(w\omega_s)(x)\, d^\times x = Z(s; w, f),$$

called the **global zeta function**. One property of the global zeta function which will help us later "prove" the quadratic reciprocity law is the functional equation

$$Z(1-s, w^{-1}) \circ \hat{(\cdot)} = Z(s, w),$$

where $\hat{(\cdot)}$ is Fourier transform.

The quantum equation of motion comes from the variation of the path integral with respect to $\alpha \in I_K$:

$$\int_{I_K} f(\alpha x)(w\omega_s)(\alpha x)\, d^\times x = w\omega_s(\alpha) \int_{I_K} f(x)(w\omega_s)(x)\, d^\times x.$$

Now we can discuss insertions. A t'Hooft operator is a singularity, where we care about the monodromy around it. For every prime p we want to introduce a t'Hooft operator. How can we find something that only has a singularity at one prime p? Naively we can take field extensions $\mathbb{Q}(\sqrt{p})/\mathbb{Q}$. In number theory we know that when $p \equiv 1 \bmod 4$, this extension only ramifies at p. Then we want to consider the "monodromy of q" for other primes $q \neq p$. In number theory we can look at

$$\left(\frac{p}{q}\right) := \chi_p(\mathrm{Frob}_q).$$

This is the Legendre symbol. Then it becomes reasonable to consider

$$\omega_p(x) := \prod_{q \neq p} \left(\frac{p}{q}\right)^{-\nu_q(x)}.$$

These are the appropriate analogues of t'Hooft operators. However, for some primes, e.g. $p \equiv 3 \bmod 4$, there are actually no real quadratic extensions which only ramifies at p. We have to take a different extension, namely $\mathbb{Q}(\sqrt{-p})/\mathbb{Q}$. In general, it is necessary to insert an appropriate combination of t'Hooft operators, one for each ramification point. Ramification points are determined by the discriminant of the number field.

In the abelian case, Wilson loops depend on the choice of a rep $A_K \to \mathbb{C}^\times$. It turns out we should look for those with norm 1 and only possibly positive valuation at p or ∞. There aren't many choices: they are all of the form

$$\alpha_p = (p, 1, \ldots, 1, p, 1, \ldots)$$

where the p occur at x_∞ and x_p.

Now we can prove quadratic reciprocity via S-duality. On one side, we want to consider correlators like

$$\langle W_p \rangle_q := \frac{\int f(\alpha_p x)(w_q\omega_s)(x)\, dx}{\int f(x)(w_q\omega_s)(x)\, dx}$$

where we have a t'Hooft operator at q and W_p is a Wilson loop at p. Applying the quantum equation of motion, this correlator becomes

$$(w\omega_s)(\alpha_p)^{-1}.$$

The norm 1 condition says

$$(w\omega_s)(\alpha_p)^{-1} = \omega_q(\alpha_p)^{-1} = \left(\frac{q}{p}\right).$$

S-duality says that an appropriate interchange of t'Hooft operators and Wilson loops produces equivalent theories. On the S-dual side, the appropriate correlator is

$$\langle W_q^{-1} \rangle_{p^{-1}} = \left(\frac{p}{q}\right).$$

This whole argument is only for primes $p, q \equiv 1 \bmod 4$, because otherwise we necessarily require insertion at other primes.

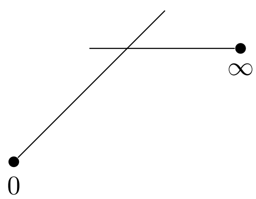
11 Andrei (Dec 03): qKZ equations and their role in enumerative geometry

Let X be a Nakajima quiver variety, but this should be a case in a much bigger universe, where X is some critical locus of a potential W on $V /\!\!/ G$ and V has some self-duality properties. A typical example could be

$$V = (\text{symplectic rep}) \oplus (\text{adjoint reps}).$$

Nakajima quiver varieties are of this kind.

We want to do K-theoretic counts of quasimaps $\mathbb{P}^1 \to X$. Take two marked points $0, \infty \in \mathbb{P}^1$. The quasimaps we consider are non-singular at ∞, and at 0 we can put any kind of insertion. All these counts are deformation-invariant. Degenerating the $\mathbb{P}^1$ into a node does not change counts, and this degeneration can be done equivariantly.



From the moduli of quasimaps $f\colon \mathbb{P}^1 \to X$, there is an evaluation map to X by taking $f(\infty)$. The count involving *relative* quasimaps is a count over a proper moduli space, and therefore results in a rational function of relevant variables. In the degeneration of the $\mathbb{P}^1$, the first piece will be such a rational function, and the second piece looks like some q-hypergeometric series. For example, when $X = T^*\mathbb{P}^N$, it looks like

$$\sum_{d \geq 0} z^d \prod \prod_{i=0}^{N} \frac{(\hbar a_i/a_k)_d}{(q a_i/a_k)_d}$$

where $\operatorname{diag}(a_0, \ldots, a_N) \in \mathrm{GL}(N+1)$ acts on $\mathbb{P}^N$, and $\hbar$ scales cotangent fibers. Since it is of q-hypergeometric form, it should satisfy q-difference equations in all variables. Hence, in general, the object

$$\Psi := \left(\begin{matrix} \text{relative} & & \text{nonsing} \\ \bullet & \text{———} & \bullet \\ 0 & & \infty \end{matrix} \right) \in K_{\mathrm{Aut}}(X)^{\otimes 2}[q^{\pm 1}]_{\mathrm{loc}}[[z]]$$

satisfies a q-difference equation in every variable except q.

1. The shift $z \mapsto q^\lambda z$ means an insertion of the kind $\det(\mathcal{V}_0)$ at 0. Since $z \in \operatorname{Pic}(X) \otimes \mathbb{C}^\times$, we need $\lambda \in \operatorname{Pic}(X)$.
2. The shift $a \mapsto q^\sigma a$ where $\sigma\colon \mathbb{C}^\times \to \operatorname{Aut}(X)$ means to use σ as a non-trivial clutching function for an X-bundle over $\mathbb{P}^1$.

The conclusion is that Ψ satisfies q-difference equations

$$\Psi(q^\lambda z, a) = M_\lambda \Psi$$
$$\Psi(z, q^\sigma a) = S_\sigma \Psi.$$

In general, we know these counts in limits $z \to 0$ or $a \to \infty$, which correspond to a classical computation and a computation on fixed loci respectively. The condition that an operator commutes with a q-difference operator is itself a q-difference equation, and therefore with an initial condition like this, the desired operator is uniquely specified. This is how one determines M_λ from S_σ.

In the situation where X is a Nakajima quiver variety, these operators come from quantum loop groups. These are Hopf algebra deformations of $U(\mathfrak{g}[t^\pm])$, which preserve the loop rotation $t \mapsto ct$ for $c \in \mathbb{C}^\times$. In the undeformed situation, Lie algebras act symmetrically on tensor products, so given two reps there is an isomorphism

$$\mathbb{M}_1(u_1) \otimes \mathbb{M}_2(u_2) \cong \mathbb{M}_2(u_2) \otimes \mathbb{M}_1(u_1).$$

After deformation, for specific values of u_1 and u_2, this will not be an isomorphism. There is an intertwining operator called the R-matrix:

$$R(u_1/u_2)\colon \mathbb{M}_1(u_1) \otimes \mathbb{M}_2(u_2) \xrightarrow{\sim} \mathbb{M}_2(u_2) \otimes \mathbb{M}_1(u_1).$$

The Cartan $\mathfrak{h} \subset \mathfrak{g}$ stays group-like. In fact the variables z will be $z = e^h$ for $h \in \mathfrak{h}$. Group-like means they act on the tensor product by $z \otimes z$. So the R-matrix commutes with these $z \otimes z$. It should also commute with $T \otimes T$, where $T(u) \coloneqq uq$. This gives a commuting set of q-difference operators in

$$\mathbb{M}_1^{(u_1)} \otimes \mathbb{M}_2^{(u_2)} \otimes \cdots \otimes \mathbb{M}_n^{(u_n)}.$$

This really means we have *functions* of $u_1, \ldots, u_n$ taking values in the tensor product $\mathbb{M}_1 \otimes \cdots \otimes \mathbb{M}_n$. Now consider a collection

$$M_1, \quad M_2, \quad \cdots, \quad M_n, \quad M_{n+1} \cong M_1$$

but where the last isomorphism is via zT, not the trivial one. If we think of these M as placed around a cylinder, now there are braids which "go around" the cylinder and pick up zT. This yields an action of the *affine* Weyl group of type A_n, which is $S(n) \ltimes \mathbb{Z}^n$. It is the $\mathbb{Z}^n$ which gives the commuting difference operators. This lattice part looks like a single strand which wraps around the whole cylinder, with all other strands trivial. Such difference operators are qKZ.

Now one can ask: are there other difference operators which commute with these qKZ? These qKZ are in variables $u_1, \ldots, u_n$ but also involve the z variables in the Cartan torus of $U_\hbar(\hat{\mathfrak{g}})$, and these difference operators should be in z. The answer is yes, and they form a kind of "quantum affine Weyl group". If we decompose $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus \mathfrak{g}_\alpha$ where α are roots, then

$$\hat{\mathfrak{g}} = \hat{\mathfrak{f}} \oplus \bigoplus_{\substack{\alpha \in \text{roots} \\ m \in \mathbb{Z}}} t^m \mathfrak{g}_\alpha.$$

Then α is an eigenvalue of z, and m is an eigenvalue of T. Hence if we visualize a lattice containing $(\log T, \log z)$, the intersection of root hyperplanes on this lattice with the hyperplane consisting of just $\log z$ results in some kind of locally periodic arrangement of hyperplanes, of the form

$$\langle \alpha, \log z \rangle + m = 0.$$

The choice of a factorization of longest element in the affine Weyl group amounts to a choice of monotone path from $-\infty$ to ∞ on this arrangement. This path crosses each hyperplane in specific alcoves, and so define specific *walls* w. This determines rank-1 subalgebras

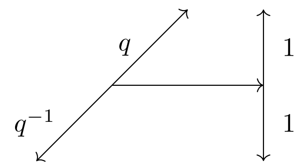
$$U_\hbar(\mathfrak{g}_w) \subset U_\hbar(\hat{\mathfrak{g}}).$$

In these rank-1 subalgebras are the desired operators B_w.

One shows that actions of elements

$$q^\sigma \coloneqq \operatorname{diag}(q, q, \ldots, 1, 1, \ldots) \in \mathrm{GL}(W_i)$$

in automorphisms of framing spaces are *minuscule*. It turns out that whenever σ is minuscule, S_σ is always qKZ. The basic argument for this is as follows. Recall S_σ counts *twisted* quasimaps in a geometry like



If we pick a particular basis of insertions in $K(X) \otimes K(X)$ such that they are both repelling for the q-action, then only constant curves contribute to the count in this geometry. In other words, in this special basis, this operator is just $z^{\deg}$.

One aspect of the relationship with W-algebras and CFT comes from considering the vertex

$$\mathsf{QM}\left(\overline{\qquad\qquad}\!\!\bullet_{\!ns} \;\to X \right) \in K[X][q^{\pm}]_{\mathrm{loc}}[[z]].$$

Since X is a GIT quotient, the quasimap fixed points themselves are GIT quotients by G with some stability condition. Taking Euler characteristic (of some virtual sheaf), this should be expressible as a contour integral

$$\int \chi(\text{on prequotient}) \prod \frac{\Gamma_q(\cdots)}{\Gamma_q(\cdots)}.$$

Heuristically, the extra Γ_q terms come from the $G[[t]]$ in

$$\mathsf{QM}(X) \approx \mathsf{QM}(\text{prequotient}) /\!\!/ G[[t]].$$

We don't have a definition of what a q-deformed W-algebra, but whatever the definition is, this integral should be its conformal block. The χ term should be a q-deformed vertex operator of some sort. There *is* however a well-defined notion of q-deformed screening operators, and the Γ_q terms should be screening operators.