# Notes for Topics in Representation Theory
# Instructor: Andrei Okounkov

Davis Lazowski and Henry Liu

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**Abstract**

These are live-texed notes for the Spring 2020 offering of MATH GR6250 Topics in Representation Theory, on equivariant K-theory. (Notes for some lectures were graciously supplied by Davis Lazowski.) Let me know when you find errors or typos. I'm sure there are plenty.

## Contents



## 1 (Jan 22) An introduction to an introduction

Enumerative geometry asks questions like: how many lines intersect four given lines in $\mathbb{P}^3$? The answer is 2. Traditionally, the way to get this answer is by setting up a moduli space of all lines, and then by doing intersection theory in this moduli space. A line in $\mathbb{P}(\mathbb{C}^4)$ is a plane in $\mathbb{C}^4$, so the moduli space of lines is $\mathrm{Gr}(2,4)$. The condition of passing through a given line is a *unique* divisor class $D$, consisting of the subvariety of all lines meeting a given line. Hence the intersection-theoretic number we want to compute is $D^4$. To do this computation requires using the $\mathrm{GL}(4)$ action on $\mathrm{Gr}(2,4)$ to get the Bruhat decomposition

$$\mathrm{Gr}(2,4) = \bigsqcup (\text{Bruhat cells}).$$

Each cell corresponds to a Young diagram in a $k \times (n-k)$ rectangle. Multiplication by a divisor corresponds to adding a box. Hence $D^4$ corresponds to all ways to add four boxes inside a $2 \times 2$ rectangle. The result is, as expected, 2.

To do this computation in K-theory means to look at $K(\mathrm{Gr}(2,4))$ instead of $H^*(\mathrm{Gr}(2,4))$. Elements of $K(X)$ in general are coherent sheaves on $X$, up to the equivalence relation $[\mathcal{E}_2] = [\mathcal{E}_1] + [\mathcal{E}_3]$ defined by short exact sequences

$$0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0.$$

The analogous computation, in K-theory, of lines meeting four given lines, is given by taking the *structure sheaf* $\mathcal{O}_D$ instead of the divisor $D$, and computing $\mathcal{O}_D^{\otimes 4}$. To get an actual number, we apply $\chi$. This K-theoretic invariant still turns out to be

$$\chi(\mathcal{O}_D^{\otimes 4}) = 2.$$

K-theory is richer than cohomology. One reason is that it is a *vector space*, and vector spaces admit group actions. One can ask why we stay in K-theory instead of going to the full-fledged derived category $D^b\mathsf{Coh}(X)$. The answer is that we want to preserve the crucial property of *deformation invariance*.

Why do we care about such enumerative problems in 2020? Often we want to study curves in varieties, e.g. Gromov–Witten theory, subject to some intersection-theoretic constraints. There is often some moduli $\mathcal{M}$ of maps $f\colon C \to X$, where these constraints become cohomology/K-theory classes, and one ends up computing invariants like

$$\chi(\mathcal{M}, \mathcal{O}_{\mathcal{M}}^{\mathrm{vir}} \otimes (\text{constraints}))$$

where $\mathcal{O}_{\mathcal{M}}^{\mathrm{vir}}$ is some *improved* version of the ordinary structure sheaf. For $\mathrm{Gr}(2,4)$, which is an amazingly nice moduli space, we don't see this improvement because the improved version is the same as the original structure sheaf.

Physically, these invariants are also thought of as indices in some $2+1$ dimensional QFTs. What does this mean? In classical mechanics, if we have some physical system then there is an associated configuration space $(M, g)$ which also has a metric measuring the energy. There is also a potential, which is a function $V\colon M \to k$. The corresponding quantum-mechanical problem has an operator $E \coloneqq -\Delta_g + V$ and one would like to find its spectrum. In particular we would like to know its eigenspaces, as representations of $\mathrm{Aut}(M, g, V)$. The typical example is a hydrogen atom, or really any kind of particle in a rotationally-invariant system.

In general, this problem is very hard. The first and biggest simplification we make is to introduce supersymmetry (SUSY). We say the problem is *supersymmetric* if the Hilbert space decomposes as

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1,$$

and there is an operator $Q, Q^*\colon \mathcal{H}_i \to \mathcal{H}_{i+1}$ such that

$$Q^2 = (Q^*)^2 = 0, \quad E = \{Q, Q^*\} \coloneqq QQ^* + Q^*Q.$$

This is like a *super* Lie algebra of symmetries. A super Lie algebra is like a Lie algebra with $\mathbb{Z}/2$-grading, but where the *anti-commutator* $\{-,-\}$ is used for odd degree elements.

How do we incorporate odd degrees of freedom into the configuration space? We need to add "odd directions" to the moduli space, and then ask what should happen when we take functions on such odd directions. The simplest example is when $M$ is a vector space. Then functions on odd coordinates are just *dual* vectors, and they have to anti-commute. Hence

$$\begin{aligned} \mathcal{H}_0 &= \Gamma(\wedge^{\mathrm{even}} T^*M) \\ \mathcal{H}_1 &= \Gamma(\wedge^{\mathrm{odd}} T^*M), \end{aligned}$$

and we can take $Q = d$ and $Q^* = d^*$. Here the adjoint is with respect to the Riemannian metric $g$. Then

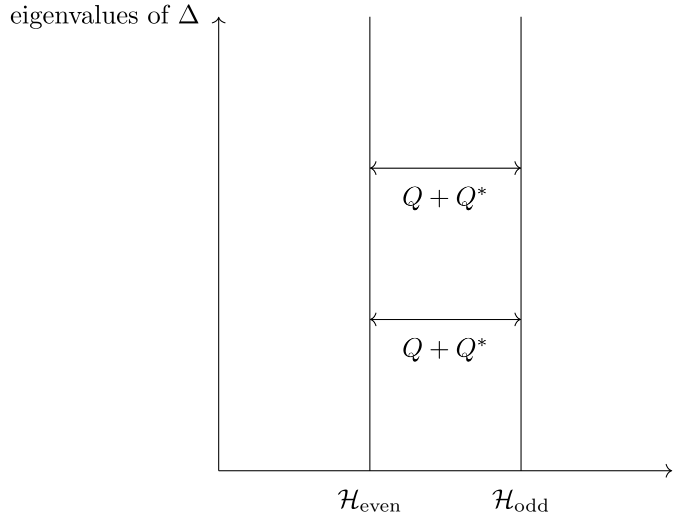
$$E = \{Q, Q^*\} = dd^* + d^*d = -\Delta_g$$

on $\Omega^*(M)$. So now we have to compute not just eigenvalues (and associated eigenspaces) on *functions*, but also on differential forms.

One eigenvalue is easy. We know that $-\Delta \geq 0$. In other words,

$$(-\Delta v, v) = \{Q^* v\|^2 + \|Qv\|^2 \geq 0.$$

If $\Delta v = 0$, then $Q^* v = Qv = 0$. In our simple example, this means we want forms $\omega$ such that $d\omega = 0$, and also that $d^*\omega = 0$. To be killed by $d^*$ means it is orthogonal to anything in the image of $d$. Hence $\omega$ must be a harmonic representative of a cohomology class. The picture to have is the following.




- All non-zero eigenvalues of $\{Q, Q^*\}$ pair up between $\mathcal{H}_{\text{even}}$ and $\mathcal{H}_{\text{odd}}$.
- $\mathcal{H}_{\text{even}} \ominus \mathcal{H}_{\text{even}} = (\ker \Delta)_{\text{even}} \ominus (\ker \Delta)_{\text{odd}}$, which is now a *finite-dimensional* (virtual) vector space. In the de Rham example, this is $H^{\text{even}}_{\text{dR}}(M) - H^{\text{odd}}_{\text{dR}}(M)$. Importantly, this object is *invariant* under deformations.

Why is the index deformation-invariant? Consider again the example where $(M, g)$ is a Riemannian manifold, and $Q = d\colon \Omega^i \to \Omega^{i+1}$. Introduce a *non-trivial* local system $\mathcal{L}$, and instead of $d$ we take $d_{\mathcal{L}}$. We can also deform $d_{\mathcal{L}}$ by some operator that raises degree, like

$$d_{\mathcal{L}} + dW \wedge .$$

The corresponding energy operator is

$$-\Delta = \{Q, Q^*\} = \Delta_{\mathcal{L}} + (\text{some dependence on } W'') + \|dW\|^2.$$

We can either deform $\mathcal{L}$ or $W$. If we move $\mathcal{L}$ around, the *individual* cohomologies *will* change but the Euler characteristic will not. If we scale $W$ by a large $N$, then the dominant contribution is from $N^2\|dW\|^2$. All eigenfunctions will therefore be very localized around the critical locus $dW = 0$. (Good reading for checking these claims is Witten's "Supersymmetry and Morse theory" paper.)

We are interested in a two-fold generalization of this setup: extended SUSY, and *field* theories in $2+1$ dimensions. What is extended SUSY? Suppose $M$ is a Kähler manifold. Then we can get *more* of these $Q$ operators. As before, we have $-\Delta = dd^* + d^*d$, but since $d = \partial + \overline{\partial}$,

$$\begin{aligned} -\Delta &= 2(\partial\partial^* + \partial^*\partial) \\ &= 2(\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}). \end{aligned}$$

Setting $Q_1 \coloneqq \partial$ and $Q_2 = \overline{\partial}$ yields

$$\{Q_i^*, Q_j\} = \frac{1}{2}\delta_{ij}E.$$

So now there are *two* $Q$'s. (If we specialize further to *hyperkähler* manifolds, we can get even more.) Choose

$$Q \coloneqq \overline{\partial}\colon \Omega^{0,i}(M) \otimes \mathcal{L} \to \Omega^{0,i+1}(M) \otimes \mathcal{L}.$$

where instead of a *flat* bundle $\mathcal{L}$ we can put an arbitrary *holomorphic* bundle. Then, by Dolbeault's theorem, the index is

$$H^{\text{even}}(M, \mathcal{L}) \ominus H^{\text{odd}}(M, \mathcal{L}).$$

As before, we can deform this construction by tensoring with a Koszul complex and adding a section $+s\wedge$ for $s \in \Gamma(V)$. We can even tensor with a Chevalley–Eilenberg complex to introduce an additional Lie algebra. The effect is that $-\Delta$ picks up extra terms $\|s\|^2$. In the end, such a complex still computes the Euler characteristic of some coherent sheaf, but maybe on a stack instead of an algebraic variety.

What about "field theory in $2+1$ dimensions"? Let $C$ be a Riemann surface, which is the 2-dimensional space. (Time is the $+1$.) In field theory, the configuration space is

$$M = (\text{all functions from } C \text{ to } X).$$

One can imagine a little magnet at each point in space, pointing in some arbitrary direction, which specifies an *arbitrary* function. Given a function $f$, we can put a PDE on $f$. The complex we discussed will then sit on the *zero locus* of this PDE, which is finite-dimensional.

The point of this entire discussion is to show that all this physics reduces to something very mathematical: the Euler characteristic of some coherent sheaf on an algebraic variety/stack. But this finite-dimensional computation does some computation in an originally infinite-dimensional setting.

## 2 (Jan 27) Equivariant K-theory

Let $X$ be an algebraic variety (or scheme) with a group $G$ acting on it. Associated to this data is the K-group $K(X)$ and the $G$-equivariant K-group $K_G(X)$. Recall that given $X$, we can talk about the abelian category $\mathsf{Coh}(X)$ of coherent sheaves on $X$. If $X = \operatorname{Spec}(R)$ is affine, then this is just finitely-generated $R$-modules. Inside $\mathsf{Coh}(X)$ sits $\mathsf{Perf}(X)$, the category of *locally free* sheaves. Again, in the affine case, this is finitely-generated *projective* $R$-modules.

The K-group is generated by symbols $[\mathcal{E}]$ where $\mathcal{E} \in \mathsf{Coh}(X)$ is a coherent sheaf. If there is a short exact sequence $0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$, then there is a corresponding relation

$$[\mathcal{E}] = [\mathcal{E}_1] + [\mathcal{E}_2]$$

in the K-group. From this definition it is clear that there is a map

$$K_{\mathsf{Perf}}(X) \to K(X)$$

where $K_{\mathsf{Perf}}$ means the K-group of the category $\mathsf{Perf}(X)$.

At this point we may as well assume that $X$ is reduced. Otherwise, if $R$ has a nilpotent ideal $I$, then there is a (finite) filtration $M \supset IM \supset I^2M \supset \cdots$ for *any* $R$-module $M$. Then in the K-group, $M$ is a sum of successive quotients, and these quotients are modules for $R/I$. So wlog we assume $X$ is reduced.

If $X$ is smooth, then every coherent sheaf $\mathcal{E}$ admits a finite locally free resolution. Assuming $X$ is quasi-projective (which we will), this is fairly easy to see: we can twist by an ample line bundle until there is a section corresponding to a map $\mathcal{F}_0 \to \mathcal{E}$, and then one can repeat with the kernel. Hilbert's syzygy theorem guarantees this process terminates in finitely many steps. Hence for smooth $X$,

$$K_{\mathsf{Perf}}(X) \simeq K(X).$$

We would like a map $f\colon X \to Y$ to induce a map between K-groups. Given a sheaf $\mathcal{E}$ on $X$, it has a pushforward $f_*\mathcal{E}$. But $f_*$ is not an exact functor, so to avoid losing information in K-theory we define

$$f_*[\mathcal{E}] \coloneqq \sum (-1)^i [R^i f_* \mathcal{E}].$$

(In the case of pushforward to a point, this is just Euler characteristic, which is what last lecture we said we wanted to compute in the end.)

There is also a pullback map. Given $\mathcal{F}$ on $Y$, there is the pullback $\mathcal{O}_X \otimes_{f^{\text{sheaf},*}\mathcal{O}_Y} f^{\text{sheaf},*}\mathcal{F}$. Again, $\mathcal{O}_X \otimes -$ is not exact, so in K-theory we have to define

$$f^*[\mathcal{F}] \coloneqq \sum (-1)^i \operatorname{Tor}^i(\mathcal{O}_X, f^{\text{sheaf},*}\mathcal{F}).$$

Note that we need some criteria on the map $f$ in order for pushforwards/pullbacks to exist. For pushforwards, for the sheaves $R^i f_*\mathcal{E}$ to be coherent we need $f$ to be proper. For pullbacks, the sum must be finite, so e.g. $\mathcal{F}$ should be locally free, or $f$ should have finite Tor dimension. This is one reason it is nice to assume the target $Y$ is smooth, because then we can pull back *any* coherent sheaf. More generally, the way to think is that $K(Y)$ is *covariant* while $K_{\mathsf{Perf}}(X)$ is *contravariant*, so when the two coincide we get both properties.

If $f\colon X \to Y$ is a closed embedding, then pushforward has no higher cohomology. (It is just extension by zero.) Pullback is more complicated: we have to resolve the sheaf by locally free sheaves and then pull back. We'll see the primary example in a moment.

The tensor $\mathcal{E} \otimes \mathcal{F}$ of two sheaves requires infinitely many Tors when the space is singular. But if one of the sheaves is locally free, there are only finitely many Tors. Hence there is a map

$$K(X) \otimes K_{\mathsf{Perf}}(X) \to K(X).$$

When $G$ acts on $X$, there is a category $\mathsf{Coh}_G(X)$ of $G$-*equivariant* coherent sheaves. Let's first talk about locally free sheaves, which are the same as vector bundles. A vector bundle $V$ is $G$-equivariant if there is a $G$-action on the total space which is linear on fibers. One way to phrase this is to take the action $a\colon G \times X \to X$, and consider $a^*V$. This is a bundle on $G \times X$, whose fiber over $(g, x)$ is

$$(a^*V)_{(g,x)} = V_{gx} \cong V_x.$$

So $G$-equivariance is an isomorphism

$$a^*V \cong p_2^*V.$$

Note that this definition makes sense for *arbitrary* sheaves. For more details, read the book by Chriss and Ginzburg, chapter 5.

**Definition 2.1.** The **equivariant K-group** of $X$ is

$$K_G(X) \coloneqq K(\mathsf{Coh}_G X).$$

As before, there is a map $K_{G,\mathsf{Perf}}(X) \to K_G(X)$.

**Example 2.2.** Since a $G$-equivariant vector bundle on a point is just a representation of $G$,

$$K_G(\text{pt}) = R(G)$$

is the representation ring of $G$. Explicitly, $R(G) = \mathbb{Z}[V_i]$ where $V_i$ ranges over all irreps.

An interesting question is to examine the forgetful map $K_G(X) \to K(X)$. At the level of reps of $G$, this is like sending a representation to its dimension. So one can ask: is

$$\mathbb{Z} \otimes_{R(G)} K_G(X) \to K(X)$$

an isomorphism? It turns out the answer is yes, for a large class of groups, independently of $X$.

Suppose $G$ acts freely on $X$ with $Y \coloneqq X/G$. There are, of course, all kinds of algebraic quotients, but all of them should satisfy

$$\mathsf{Coh}_G(X) \simeq \mathsf{Coh}(Y).$$

If $\pi\colon X \to Y$ is the quotient, then one direction is the pullback $\pi^*$. The other direction is $\pi_*$, which produces a $G$-equivariant sheaf with trivial $G$-action, and then take $G$-invariants. Sometimes this is denoted as

$$\pi_{*,G}\mathcal{E} \coloneqq (\pi_*\mathcal{E})^G.$$

In particular, passing to K-theory,

$$K_G(X) = K(Y).$$

We can use this to compute the K-theory of projective space. The first step is to recognize that

$$\mathbb{P}(V) = (V \setminus \{0\})/\operatorname{GL}(1).$$

The second step is to return to studying a closed embedding $Z \to X$ of a closed, $G$-invariant subvariety. Then there are maps

$$\mathsf{Coh}_G(Z) \to \mathsf{Coh}_G(X) \to \mathsf{Coh}_G(X \setminus Z).$$

One says that this is a *Serre subcategory*, which just means that given two objects in two of these categories, the third lies in the third category. Passing to K-theory, we get

$$K_G(Z) \to K_G(X) \to K_G(X \setminus Z) \to 0.$$

That the last map is surjective is just extension by zero. However, the first map is *not* injective; take any interesting hypersurface in projective space.

Applying this to $\mathbb{P}(V)$, we get a sequence

$$K_{\operatorname{GL}(1)}(\mathrm{pt}) \to K_{\operatorname{GL}(1)}(V) \to K_{\operatorname{GL}(1)}(V \setminus \{0\}) \to 0.$$

Here we may as well discuss $\operatorname{GL}(V)$-equivariant K-theory, so this sequence becomes

$$K_G(\mathrm{pt}) \to K_G(V) \to K_G(V \setminus \{0\}) \to 0.$$

where $G \coloneqq \operatorname{GL}(1) \times \operatorname{GL}(V)$. These can be identified with

$$R(G) \to R(G) \to K_{\operatorname{GL}(V)}(\mathbb{P}(V)) \to 0.$$

The second identification is because every module on $V$ has a resolution by free modules, but there aren't many free modules. It remains to compute the cokernel, i.e. compute the pushforward $f_*\mathcal{O}_{\mathrm{pt}}$.

We'll do $f_*\mathcal{O}_{\mathrm{pt}}$ in steps: first for dimension 1, then for dimension 2, then in general. When $V$ is a line with coordinate $x$, we get

$$0 \to \mathcal{O}_V \xrightarrow{\cdot x} \mathcal{O}_V \to \mathcal{O}_0 \to 0.$$

To restore the equivariance, recall that $x$ is an element in $V^*$. This is actually less confusion in dimension 2, so we'll do that case. In dimension 2, when $V$ is a plane with coordinates $x$ and $y$, we get

$$0 \to \mathcal{O}_V \xrightarrow{\begin{pmatrix} y \\ -x \end{pmatrix}} \mathcal{O}_V^{\oplus 2} \xrightarrow{\begin{pmatrix} x & y \end{pmatrix}} \mathcal{O}_V \to \mathcal{O}_0 \to 0.$$

To restore equivariance, since $x, y \in V^*$, it makes sense that $\mathcal{O}_V^{\oplus 2} = \mathcal{O}_V \otimes V^*$, and the map is

$$\mathcal{O}_V \otimes V^* \xrightarrow{\sum x_i \otimes \frac{d}{dx_i}} \mathcal{O}_V \otimes 1.$$

There is a beautiful way to continue this sequence, using the same element $d \coloneqq \sum x_i \otimes d/dx_i$:

$$\cdots \xrightarrow{d} \mathcal{O}_V \otimes \wedge^2 V^* \xrightarrow{d} \mathcal{O}_V \otimes V^* \xrightarrow{d} \mathcal{O}_V \otimes 1 \to \mathcal{O}_0 \to 0.$$

It remains to add in the GL(1) equivariance. It is the *center* of $\mathrm{GL}(V)$, which acts on the defining representation with the defining character. On $V^*$ it acts by the *dual* to the defining character, which we notate as

$$\cdots \xrightarrow{d} \mathcal{O}_V \otimes \wedge^2 V^*(-1) \xrightarrow{d} \mathcal{O}_V \otimes V^*(-1) \xrightarrow{d} \mathcal{O}_V \otimes 1 \to \mathcal{O}_0 \to 0.$$

Let's put all this together. Let $s^{\pm}$ denote the variable for GL(1), and $a_1^{\pm}, \ldots, a_n^{\pm}$ denote the variables for $\mathrm{GL}(V)$. The map $K_G(\mathrm{pt}) \to K_G(V)$ produces the relation which is the image of the generator:

$$1 - s^{-1}V^* + s^{-2} \wedge^2 V^* - \cdots = \prod_{i=1}^n \left(1 - s^{-1}a_i^{-1}\right).$$

We have shown the following.

**Proposition 2.3.** *The equivariant K-ring of* $\mathbb{P}(V)$ *is*

$$K_{\mathrm{GL}(V)}(\mathbb{P}(V)) = \mathbb{Z}[s^{\pm}, (a_1^{\pm}, \ldots, a_n^{\pm})^{S(n)}]/\langle \prod_{i=1}^n \left(1 - s^{-1}a_i^{-1}\right)\rangle.$$

Recall we had a map $\mathsf{Coh}_G(X) \to \mathsf{Coh}(Y \setminus X)$. Multiplying by a character in $\mathsf{Coh}_G(X)$ corresponds to a line bundle on $\mathsf{Coh}(Y \setminus X)$, and one can check that $s$ corresponds to $\mathcal{O}(1)$.

# 3 (Jan 29) Equivariant restriction

If $G$ is a group and $H \subset G$ is a subgroup, then there is a forgetful map $K_G(X) \to K_H(X)$. It is more natural to think about $R(H) \otimes_{R(G)} K_G(X) \to K_H(X)$, and the question for today is whether this map is surjective. The general criterion for surjectivity is that the commutator subgroup $G'$ is simply connected.

In particular, if $H$ is trivial, this is a comparison between equivariant and non-equivariant K-theory. It is not true in general that a group can be made to act on an arbitrary line bundle. In Chriss–Ginzburg, there is the following.

**Proposition 3.1.** *If* $X$ *is normal and* $\mathcal{L} \in \mathrm{Pic}(X)$*, then* $\mathcal{L}^{\otimes n}$ *can be made* $G$*-equivariant for some* $n$*.*

Why do we need to raise $\mathcal{L}$ to some power? Consider $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}(2)$ acting on $\mathbb{P}^1$. As discussed last time, $K(\mathbb{P}^1) = \mathbb{Z}$ with the generator $\mathcal{O}(1)$. However, PGL(2) does not act on $\mathcal{O}(1)$; otherwise it would also act on the two-dimensional rep $H^0(\mathbb{P}^1, \mathcal{O}(1))$, but PGL(2) has no two-dimensional rep. Actually, $K_{\mathrm{Aut}(\mathbb{P}^1)}(\mathbb{P}^1)$ is generated by $\mathcal{O}(2)$.

Why do we need normality? Let $X$ be the nodal cubic, so that $X^{\mathrm{nonsing}} \cong \mathbb{C}^\times$. By normalization, a degree-$d$ line bundle on $X$ pulls back to a degree-$d$ line bundle on $\mathbb{P}^1$. There is only one such bundle: $\mathcal{O}_{\mathbb{P}^1}(d)$. The pre-image of the node is two points on $\mathbb{P}^1$, which we call 0 and $\infty$. To pass back to $X$, we need an identification of fibers at 0 and $\infty$. On sections, this is the condition $f(0) = af(\infty)$ for some $a \in \mathbb{C}^\times$. Hence $\mathrm{Pic}_d(X) = \mathbb{C}^\times$. If we let $X^{\mathrm{nonsing}} = \mathbb{C}^\times$ act on $X$ with weight $q$, then clearly on $\mathrm{Pic}_d(X)$ the action is $a \mapsto q^d a$. It follows that there is *no line bundle* of non-zero degree that can be made equivariant.

Here is another (perhaps more useful) argument. Suppose $\mathrm{Pic}^G_{d>0}(X)$ is non-empty, and pick $\mathcal{L}$ in it. Then $\mathcal{L}$ is ample and some power of it gives an equivariant embedding into $\mathbb{P}(V)$ where $V \coloneqq H^0(\mathcal{L}^{\otimes m})^*$. But note that given a non-fixed point $x \in X$, both limits $\lim_{q\to 0} qx$ and $\lim_{q\to\infty} qx$ are *the same point*, namely the nodal point. This can never happen inside $\mathbb{P}(V)$.

Note that although we have produced many different bundles on $X$, they are actually the same in the K-theory. This is because of excision:

$$K(\mathrm{pt}) \to K(X) \to K(\mathbb{C}^\times) \to 0.$$

*Remark.* A Lie group can always be written as $G \ltimes U$, where $G$ is reductive and $U$ is the unipotent radical. As part of the proof we are about to see, it turns out that

$$K_{G \ltimes U}(X) = K_G(X).$$

**Theorem 3.2.** *If $G$ is reductive with $G'$ simply connected, then there is a surjection*

$$K_G(X) \otimes_{R(G)} R(H) \twoheadrightarrow K_H(X).$$

As an immediate corollary, $K(G) = K_G(G) = K(G/G) = K(\mathrm{pt}) = \mathbb{Z}$, and therefore $\mathrm{Pic}(G) = 0$. Such groups are called **factorial**.

Another interesting case is $X = G/H$. Here $K_G(G/H) = K_{G\times H}(G) = K_H(\mathrm{pt}) = R(H)$. So the theorem implies that

$$R(H) \otimes_{R(G)} R(H) \twoheadrightarrow K_H(G/H).$$

Note that the computation of $K_G(G/H)$ shows more generally that $K_G(G/H \times X) = K_H(X)$. Hence

$$R(H) \otimes_{R(G)} K_G(X) \twoheadrightarrow K_G(G/H \times X).$$

We can rewrite this as

$$K_G(G/H) \otimes_{R(G)} K_G(X) \twoheadrightarrow K_G(G/H \times X).$$

This starts looking like some kind of Kunneth formula. In general, in algebraic K-theory there is no Kunneth formula; $G/H$ is special.

We will prove the theorem by passing from $G$ to $H$ in several stages. First take a reductive subgroup $L \supset H$ of the same *rank* as $G$, by taking $L/H = (\mathbb{C}^\times)^{\mathrm{rank}\, G - \mathrm{rank}\, H}$. Then take $P := L \rtimes U \supset L$. The unipotent radical has a filtration

$$U \supset U_1 \supset \cdots \supset U_{\dim U}, \quad U_i/U_{i+1} = \mathbb{C}.$$

Finally, $G/P$ is projective (this is the hardest part). In summary, there is a chain

$$G \underbrace{\supset}_{\text{projective}} P \underbrace{\supset}_{\mathbb{C}} L \underbrace{\supset}_{\mathbb{C}^\times} H.$$

**Theorem 3.3.** *Let $G$ be simply-connected and semisimple, and $P \subset G$ be a parabolic. The map*

$$K_G(G/P) \otimes_{R(G)} K_G(X) \to K_G(G/P \times X)$$

*is surjective.*

**Proposition 3.4.** *Let $Y$ be any space. For the map*

$$K_G(Y) \otimes_{R(G)} K_G(X) \to K_G(Y \times X),$$

*the following are equivalent*

1. *it is surjective for all $X$;*
2. *it is surjective for $X = Y$;*
3. *the diagonal $\mathcal{O}_\Delta$ is in the image when $X = Y$.*

*Proof.* The forward directions are obvious, so we go backward from (3) to (1). If $X$ is proper, then there is a "multiplication" map

$$K(Y \times X) \otimes K(X) \to K(Y)$$

given by push-pull. Namely, because $p_X : Y \times X \to X$ is flat and $p_Y : Y \times X \to Y$ is proper, we can do

$$(\mathcal{M} \star -): \mathcal{E} \mapsto p_{Y,*}(\mathcal{M} \otimes p_X^* \mathcal{E})$$

for any class $\mathcal{M} \in K(Y \times X)$. Importantly, if $\mathcal{M} = \mathcal{O}_\Delta$ then this is the *identity* map.

If $\mathcal{O}_\Delta$ is in the image, this means we can write it as

$$\mathcal{O}_\Delta = \sum_i \alpha_i \boxtimes \alpha^i.$$

Given any class $\beta \in K_G(Y \times X)$, we therefore have

$$\beta = \Delta \star \beta = p_{13,*}(p_{12}^* \Delta \otimes p_{23}^* \beta).$$

By push-pull, this simplifies to

$$\sum \alpha_i \boxtimes p_{3,*}(\cdots).$$

So we have exhibited $\beta$ as an element of $K_G(Y) \otimes_{R(G)} K_G(X)$. □

In particular, we can apply this proposition to the case $X = \text{pt}$. Then we see that $K(Y)$ is spanned by $\{\alpha_i\}$ or $\{\alpha^i\}$, by the same reasoning:

$$\beta = \Delta \star \beta = \sum \alpha_i p_{Y,*}(\alpha^i \otimes \beta).$$

The term $p_{Y,*}(\alpha^i \otimes \beta)$ we can think of as some bilinear pairing

$$(\alpha^i, \beta)_Y \coloneqq \chi(\alpha^i \otimes \beta).$$

To sum up, we have proved the following.

**Corollary 3.5.** *Let $X$ be smooth and proper. Suppose $\Delta = \sum \alpha_i \boxtimes \alpha^i \in K_G(X \times X)$. Then:*

1. *$K_G(X)$ is spanned by $\{\alpha_i\}$ over $R(G)$;*
2. *$K_G(X)$ is a projective module over $R(G)$;*
3. *the map $\alpha \mapsto (\alpha, -)_X$ is an isomorphism $K_G(X) \to K_G(X)^\vee$.*

For example, the diagonal in $E \times E$ cannot be decomposed like this. Otherwise $\text{Pic}(E)$ would have finite rank over $\mathbb{Z}$.

The simplest varieties with a decomposition of the diagonal like this are $G/P$, and the simplest $G/P$ are when $G = \text{GL}(n)$ and $P$ is the parabolic giving $G/P = \mathbb{P}^{n-1}$. We'll see next time how to explicitly decompose the diagonal $\Delta \subset \mathbb{P}(V) \times \mathbb{P}(V)$.

# 4 (Feb 03) Decomposition of the diagonal

What is the (non-equivariant) K-theory of a smooth genus-$g$ curve? Given a sheaf over an algebraic variety $X$, the first thing one can do is consider the corresponding module over the function field $\mathbb{C}(X)$. In other words, there is a map from $K(X)$ to the K-theory of the generic point:

$$K(X) \to K(\mathsf{Mod}_{\mathbb{C}(X)}).$$

For any given module, this loses the data of finitely many points. So there is an exact sequence

$$\bigoplus_{\substack{p=\text{ points} \\ \text{of codim 1}}} K(p) \to K(X) \to K(\mathsf{Mod}_{\mathbb{C}(X)}) \to 0.$$

Concretely, the last arrow just takes the rank of the sheaf. The first arrow just sends a point $p$ to the structure sheaf $\mathcal{O}_p$.

We can keep extending this sequence to the left. Fact: given $Y = \operatorname{Spec} R$, the higher K-theory $K_1(R)$ is just $\mathrm{GL}(\infty, R)/[\cdot,\cdot]$. Here $\mathrm{GL}(\infty, R)$ means *locally finite* infinite-dimensional matrices. The commutator subgroup inside GL is just SL, so $K_1$ consists of units. In general (not just for curves), $K_1(X) = \mathbb{C}(X)^\times$. There is therefore a map

$$\mathbb{C}(X)^\times \to \bigoplus_p \mathbb{Z}, \quad f \mapsto \operatorname{div} f.$$

If we specialize to curves, we get that

$$K(\text{curve}) = \mathbb{Z} \oplus \mathrm{Cl}(\text{curve}),$$

with a degree map $\deg\colon \mathrm{Cl} \to \mathbb{Z}$. This is because the $K_1$ term mods out by rational equivalence.

In general, $\mathsf{Coh} X$ has a filtration by $F^i \mathsf{Coh} X$ consisting of sheaves with support in codimension $i$. There is a corresponding filtration on K-groups, with maps

$$K_1(F^{i-1}/F^i) \to K_0(F^i/F^{i+1}) \to K_0(F^{i-1}/F^{i+1}) \to K_0(F^{i-1}/F^i) \to 0.$$

Clearly

$$K_0(F^{i-1}/F^i) = \bigoplus_{W \text{ codim } i-1} K_0(\mathbb{C}(W)).$$

The boundary map takes a function $f$ and again sends it to its divisor. Hence we conclude

$$K(F^{i-1}/F^{i+1}) = \bigoplus_{W \text{ codim } i-1} \mathcal{O}_W \oplus \mathrm{CH}^i(X)$$

where $\mathrm{CH}^i$ is the $i$-th Chow group. Namely, given a cycle $Z$, there is a well-defined map to $K_0(F^i/F^{i+1})$ given by $Z \mapsto \mathcal{O}_Z$.

To reiterate slightly differently: given a curve $X$, there is a map $K_{\mathsf{Perf}}(X) \to \mathbb{Z} \oplus \mathrm{Pic}(X)$ given by taking rank and determinant. The determinant of a (perfect) sheaf $\mathcal{E}$ is $\det \mathcal{E} := \wedge^{\mathrm{rank}} \mathcal{E}$. On short exact sequences $0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0$, we have

$$\det \mathcal{E}_2 = \det \mathcal{E}_1 \otimes \det \mathcal{E}_3$$

so it is well-defined in K-theory. Conclusion: if $X$ is a curve of genus $g > 0$, then $K(X)$ is:

1. not finitely generated;
2. the quadratic form
$$K(X) \otimes K(X) \to K(\mathrm{pt}), \quad (\mathcal{E}, \mathcal{F}) \mapsto \chi(\mathcal{E} \otimes \mathcal{F})$$
has kernel $\mathrm{Pic}_0(X)$. This is because Riemann–Roch tells us
$$\chi(\mathcal{E} \otimes \mathcal{F}) = \operatorname{rank} \mathcal{E} \cdot \operatorname{rank} \mathcal{F} \cdot (1-g) + \operatorname{rank} \mathcal{E} \cdot \deg \mathcal{F} + \operatorname{rank} \mathcal{F} \cdot \deg \mathcal{E}.$$

From our discussion last class, it immediately follows that on curves with $g > 0$, the diagonal is *not* decomposable. Otherwise $K(X)$ would be finitely generated. Let's return to proving the theorem from last class.

**Theorem 4.1.** *Suppose $X$ is proper and*

$$K(X) \otimes_{K(pt)} K(X) \twoheadrightarrow K(X \times X).$$

*Then:*

1. *$K(X)$ is a finitely generated projective module over $K(pt)$;*

2. *the pairing* $K(X) \to \mathrm{Hom}(K(X), K(pt))$ *given by* $\mathcal{E} \mapsto \chi(\mathcal{E} \otimes -)$ *is non-degenerate.*

*Proof.* Write $R := K(\mathrm{pt})$ and $M := K(X)$. Let $\mathcal{O}_\Delta = \sum_{i=1}^n \alpha_i \boxtimes \alpha^j$, and recall that convolution with $\mathcal{O}_\Delta$ is the identity:

$$\mathrm{id} = \mathcal{O}_\Delta \star - = \sum_i (\alpha^j, -)\alpha_i.$$

This implies that

$$M \xrightarrow{\beta \mapsto (r_1, \ldots, r_n)} R^n \xrightarrow{(r_1, \ldots, r_n) \mapsto \sum r_i \alpha_i} M$$

is the identity, and therefore this sequence is an inclusion followed by a surjection. Hence $M$ is finitely generated. Note that therefore

$$R^n \to M \to R^n$$

is an idempotent, and $M$ is the image of the idempotent map $p$. So

$$R^n = \operatorname{im} p \oplus \operatorname{im}(1-p) = M \oplus \operatorname{im}(1-p).$$

Hence $M$ is projective. Also, it is clear that if $\beta$ is in the kernel of

$$M \mapsto M^\vee = \mathrm{Hom}(M, R), \quad \beta = \sum_i \alpha_i(\alpha^i, \beta),$$

then $\beta = 0$. Applying $f$ to both sides, any map $f \in \mathrm{Hom}(M, R)$ can be written as

$$f(\beta) = \sum_i f(\alpha_i)(\alpha^i, \beta).$$ □

Recall that our original goal was to determine, if $G$ is a group and $P \subset G$ is a subgroup, whether

$$K_G(G/P \times G/P) \stackrel{?}{=} K_G(G/P) \otimes_{K_G(\mathrm{pt})} K_G(G/P).$$

Actually, we started with an *arbitrary* variety $Y$, and the question of whether

$$K_P(Y) \stackrel{?}{=} K_P(\mathrm{pt}) \otimes_{K_G(\mathrm{pt})} K_G(Y).$$

But via the decomposition of the diagonal, we reduced down to the problem of whether

$$K_P(G/P) \stackrel{?}{=} K_P(\mathrm{pt}) \otimes_{K_G(\mathrm{pt})} K_P(\mathrm{pt}).$$

In principle, this can be analyzed purely Lie-theoretically. There are finitely many $P$-orbits in $G/P$ and one can just figure out what can happen. Of course, as discussed, this statement is equivalent to finding a decomposition of the diagonal for $G/P$. Without loss of generality, we may as well assume $P \subset G$ is a *maximal* parabolic. This is because we can reduce down from $G$ to $P$ in multiple stages, taking a maximal parabolic at each stage.

So we will discuss the decomposition of the diagonal in the specific case when $G = \mathrm{GL}(n)$ and

$$P = \begin{pmatrix} \overbrace{*}^{k} & \overbrace{*}^{n-k} \\ 0 & * \end{pmatrix},$$

so that $G/P = \mathrm{Gr}(k, n) = \{L \subset \mathbb{C}^n : \dim L = k\}$. Let $\mathcal{L}$ denote the tautological bundle, whose fiber over a point $[L]$ is the space $\mathbb{C}^n/L$. Over $\mathrm{Gr}(k, n) \times \mathrm{Gr}(k, n)$, there are two of these bundles $L_1$ and $L_2$ (from each factor), and there is a canonical map

$$\mathcal{L}_1 \to \mathbb{C}^n \to \mathbb{C}^n/\mathcal{L}_2.$$

This is, equivalently, a section $s \in \mathcal{H}om(\mathcal{L}_1, \mathbb{C}^n/\mathcal{L}_2)$. This section has the very nice property that

$$s(L_1, L_2) = 0 \iff L_1 = L_2.$$

One can check that it is a *regular* section. Hence it cuts out the diagonal $\Delta \subset \mathrm{Gr}(k,n) \times \mathrm{Gr}(k,n)$. Write $\mathcal{E} \coloneqq \mathcal{H}om(\mathcal{L}_1, \mathbb{C}^n/\mathcal{L}_2)$ and $X \coloneqq \mathrm{Gr}(k,n)$ for short. Then we have obtained a sequence

$$\mathcal{E}^\vee \xrightarrow{s} \mathcal{O}_{X \times X} \to \mathcal{O}_\Delta \to 0.$$

The way to continue this is with higher exterior powers $\wedge^i \mathcal{E}^\vee$:

$$\cdots \to \wedge^2 \mathcal{E}^\vee \to \mathcal{E}^\vee \xrightarrow{s} \mathcal{O}_{X \times X} \to \mathcal{O}_\Delta \to 0.$$

The higher maps are just contractions with $s$. In general, this construction is known as a **Koszul complex**. It is exact iff the section is regular. One checks now that all bundles $\wedge^i \mathcal{E}^\vee$ sit inside $K(X) \otimes K(X)$, and we are done.

For projective space $\mathbb{P}(\mathbb{C}^n) = \mathrm{Gr}(1,n)$, the tautological bundle is $\mathcal{L} = \mathcal{O}(-1)$. The resulting Koszul complex is exactly the Beilinson sequence

$$\mathcal{O}(i) \boxtimes \Omega^i_{\mathbb{P}^{n-1}}(i) \to \mathcal{O}_\Delta,$$

and we immediately conclude that either $\{\mathcal{O}(i)\}_{i=0}^{n-1}$ or $\{\Omega^i(i)\}_{i=0}^{n-1}$ spans $K(\mathbb{P}^{n-1})$.

More generally, given a universal bundle $V \to X$, we can do a resolution of the diagonal on the total space of the bundle $Y \coloneqq \mathrm{Gr}(k,V)$ by the same argument. It follows that

$$K_G(Y) = \bigoplus_{\substack{\gamma_i \text{ basis} \\ \text{of } K_G(\mathrm{Gr})}} K_G(X)\gamma_i.$$

This is known as the **projective bundle theorem**.

Be careful: on a *vector* bundle $V$, one cannot apply this argument because $V$ itself is not proper! However we can embed it into its projective closure

$$Y \hookrightarrow \mathbb{P}(V \oplus \mathcal{O}_X)$$

Call the total space of the projective bundle $Y$. Inside $Y$ is the complement of $V$, which we call $Z \coloneqq \mathbb{P}(V)$. This yields an exact sequence

$$K(Z) \to K(Y) \to K(V) \to 0.$$

But both $Z$ and $Y$ are projective bundles, with $\operatorname{rank} Y = \operatorname{rank} Z + 1$. Applying the projective bundle theorem to both, this sequence becomes

$$\bigoplus_{i=1}^n \gamma_i K(X) \to \bigoplus_{i=0}^n \gamma_i K(X) \to K(V) \to 0.$$

Hence $K(V) \cong K(X)$, and the isomorphism is just the pullback.

Interestingly, for this argument, we did *not* need the sum $V \oplus \mathcal{O}_X$ to be a *direct* sum! In other words, it suffices to have a map of bundles

$$0 \to V \to W \to \mathcal{O}_X \to 0.$$

Then $\mathbb{P}(W) \setminus \mathbb{P}(V)$ is a principal homogeneous space over $V$. It is important that there is no *canonical* identification between it and $V$. In this situation, the same argument goes through.

## 5 (Feb 05) Localization

Consider a vector bundle $V \to B$. It could be that there is a space $Y \to B$ with an action by $V$ fiber-wise, such that the fibers of $V$ and $Y$ are non-canonically isomorphic. If we consider functions of degree $\leq 1$ on $Y$, there is a short exact sequence

$$0 \to \mathbb{C} \to (\text{such functions}) \to V^* \to 0.$$

Let $W$ denote the space of such functions. This means there is an embedding

$$Y \hookrightarrow \operatorname{Spec} \operatorname{Sym}^\bullet W.$$

Rephrasing, $Y \cong \mathbb{P}(W) \setminus \mathbb{P}(V)$. Hence any affine bundle can be written as some projective bundle minus a certain section.

Let's continue the discussion from last time. We are in the middle of restricting from $G$ to $H$. By the structure theory of Lie groups, we can always find a sequence

$$G = G_0 \supset G_1 \supset G_2 \supset \cdots \supset G_n = H$$

such that $G_i/G_{i+1}$ is either projective, $\mathbb{C}$, or $\mathbb{C}^\times$. In other words, in the restriction process we will have to deal with objects like

$$K_G(G/P \times Y), \quad K_G(\mathbb{C} \times Y), \quad K_G(\mathbb{C}^\times \times Y).$$

We already discussed the $G/P$ case, and the $\mathbb{C}$ case is a special case of the vector bundle theorem from last time. It remains to do $\mathbb{C}^\times$. There is a sequence

$$K_G(Y) \hookrightarrow K_G(\mathbb{C} \times Y) \to K_G(\mathbb{C}^\times \times Y) \to 0.$$

The group $G$ must act by a character $\chi$ on the $\mathbb{C}$ in the middle term. The first map is inclusion of the zero section; the image is described by the Koszul complex, so we describe the map as $1 - \chi^{-1}$. Explicitly, think of $0 \in \mathbb{C}_\chi$ being cut out by the function $\chi$, which has weight $\chi^{-1}$. The Koszul complex is

$$0 \to \chi^{-1}\mathcal{O}_{\mathbb{C}_\chi} \to \mathcal{O}_{\mathbb{C}_\chi} \to \mathcal{O}_0 \to 0$$

and therefore $\mathcal{O}_0 = (1 - \chi^{-1})\mathcal{O}_{\mathbb{C}_\chi}$. It follows that

$$K_G(\mathbb{C}^\times \times Y) = K_G(Y)/\langle 1 - \chi^{-1} \rangle.$$

If we write $G = \mathbb{C}^\times \times H$, then $\chi$ is exactly the coordinate on this $\mathbb{C}^\times$.

To reiterate: if $G \xrightarrow{\chi} \mathbb{C}^\times_\chi$ is a character and we set $H \coloneqq \ker \chi$, then

$$K_H(X) = K_G(\mathbb{C}^\times_\chi \times X) = K_G(X)/\langle \chi = 1 \rangle.$$

This is a very important result that we will use later to understand the structure of $K_G(X)$ more deeply.

This concludes the proof of the following theorem which we started discussing a few classes ago.

**Theorem 5.1.** *If $G'$ is simply connected, then*

$$K_G(X) \otimes_{R(G)} R(H) \cong K_H(X).$$

Consider $\operatorname{Spec} K_G(X)$. This is a scheme over $G/G$, where $G$ acts on itself by conjugation. There is a natural map $H/H \to G/G$, and this theorem says the square

$$\begin{array}{ccc} \operatorname{Spec} K_H(X) & \longrightarrow & \operatorname{Spec} K_G(X) \\ \downarrow & & \downarrow \\ H/H & \longrightarrow & G/G \end{array}$$

is Cartesian. For example, from now on we will basically restrict to the maximal torus $T \subset G$ for computing characters; nothing is lost.

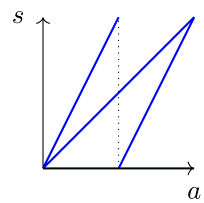
Let $G = \mathbb{C}^\times$ act linearly on a vector space $V$ by something like

$$a \mapsto \begin{pmatrix} 1 & & \\ & a & \\ & & a^2 \end{pmatrix},$$

and so it also acts on $\mathbb{P}(V)$. We can ask for $K_{\mathbb{C}^\times}(\mathbb{P}(V))$, as a module over the base ring $R := K_{\mathbb{C}^\times}(\mathrm{pt}) = \mathbb{Z}[a^{\pm 1}]$. From the Koszul complex computation of a few classes ago,

$$K_{\mathbb{C}^\times}(\mathbb{P}(V)) = R[s^{\pm 1}]/\langle (1 - s^{-1})(1 - s^{-1}a^{-1})(1 - s^{-1}a^{-2})\rangle$$

where $s$ is the class of $\mathcal{O}(1)$. What does it look like as a scheme over $\operatorname{Spec} R$? We will draw the norm-1 part of the torus $\operatorname{Spec} R$, and use logarithmic coordinates:




So $K_{\mathbb{C}^\times}(\mathbb{P}(V))$ is a finite union of components, and these components all intersect at $(a, s) = (0, 0)$ and $(a, s) = (1, 1)$. There is also another interesting point of order 2 in between.

The point $a = 1$ is easy to understand. It corresponds to restricting to the *trivial* subgroup. Its fiber is just $K(\mathbb{P}^2) = \mathbb{Z}[s^{\pm 1}]/\langle (1 - s)^3\rangle$, which is a point of multiplicity 3. The interesting point corresponds to the subgroup of order 2, and we have not computed the equivariant K-theory of $\mathbb{P}^2$ with respect to it yet. The fiber there consists of a double point and another point.

In general we actually have a *universal* formula for the fiber over a point $a \in \operatorname{Spec} R$: it is $K(X^a)$. Given $a \in T$, let $A$ be the smallest subgroup generated by $a$, i.e. $A := \overline{\langle a^n \rangle}$. Then

$$a \in A \setminus \bigcup_{\substack{A' \subset A \\ \text{proper subgroup}}} A',$$

and there should be an identification

$$K_A(X)\big|_{A \setminus \bigcup A'} = K(X^A) = K(X^a).$$

In other words, we want to restrict to a *complement* of subgroups. This is called **localization**.

Consider the inclusion of the fixed locus $X^A \hookrightarrow X$. Since $A$ doesn't act on $X^A$,

$$K_A(X^A) = K(X^A) \otimes R.$$

There is an exact sequence

$$\cdots \to K_A^1(X \setminus X^A) \to K_A(X^A) \to K_A(X) \to K_A(X \setminus X^A) \to 0.$$

We want to prove that $K_A^1$ and $K_A$ of $X \setminus X^A$ are supported only on $\bigcup_{A' \subset A} A'$, i.e. they are torsion. Hence if we restrict to the complement of $\bigcup A'$, the map $K_A(X^A) \to K_A(X)$ is an isomorphism. It would also imply that the map $K_A^1(X \setminus X^A) \to K_A(X^A)$ is the zero map, because $K_A(X^A)$ is a free module.

The idea of the proof is to do the comparison in stages, via a sequence

$$X^A \subset \cdots \subset X^{A'} \subset \cdots \subset X.$$

Suppose we have $X^{\mathbb{C}^\times} \subset X$, and there is nothing intermediate. This means $\mathbb{C}^\times$ acts freely, because otherwise we can take the element that acts with fixed locus and use that fixed locus. In other words, $\mathbb{C}^\times$ acts freely on $X \setminus X^{\mathbb{C}^\times}$. Let

$$Y \coloneqq \left(X \setminus X^{\mathbb{C}^\times}\right)/\mathbb{C}^\times$$

be the quotient. Then the sequence is

$$K^1(Y) \to K_{\mathbb{C}^\times}(X^{\mathbb{C}^\times}) \to K_{\mathbb{C}^\times}(X) \to K(Y) \to 0.$$

Importantly, everything in $K^1(Y)$ and $K(Y)$ sit at $1 \in \mathbb{C}^\times$, because the only data remembered by the quotient by $\mathbb{C}^\times$ is the *rank* of the sheaf (which is evaluation at 1).

In our $\mathbb{P}^2$ example earlier, we had a chain of inclusions

$$(3 \text{ pts}) \subset \mathbb{P}^1 \sqcup \text{pt} \subset \mathbb{P}^2.$$

The second inclusion has a free $\mathbb{C}^\times$-action on $\mathbb{P}^2 \setminus (\mathbb{P}^1 \sqcup \text{pt})$, so

$$\operatorname{supp} K(\mathbb{P}^2 \setminus (\mathbb{P}^1 \sqcup \text{pt})) = \{1\}.$$

The first inclusion similarly has $\operatorname{supp} = \{\pm 1\}$.

This is *not* how localization is usually used. Its primary application is in the computation of pushforwards. Consider

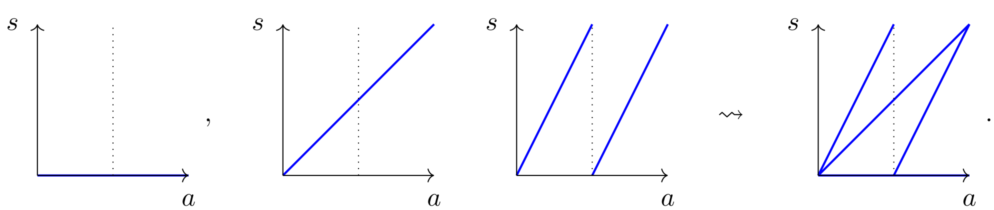
$$\begin{array}{ccc} X^A & \longrightarrow & X \\ & \searrow & \big\downarrow{\scriptstyle \text{proper}} \\ & & \text{pt} \end{array}$$

and suppose $X \to \text{pt}$ is proper. We want to compute $\chi(\mathcal{F}) \in R$ for a sheaf $\mathcal{F}$ on $X$. Since $R$ is torsion free, it is enough to know the function on any open set of $A = \operatorname{Spec} R$. If we tensor with the field of fractions, then there is an isomorphism

$$\begin{array}{ccc} K(X^A) \otimes_R \operatorname{Frac}(R) & \xrightarrow{\sim} & K_A(X) \otimes_R \operatorname{Frac}(R) \\ & \searrow & \big\downarrow \\ & & \operatorname{Frac}(R) \end{array}.$$

So to push forward from $K_A(X)$, we may as well push forward from $K_A(X^A)$. In general it is hard to identify what corresponds to $\mathcal{F} \in K_A(X)$ inside $K_A(X^A)$. But if $X$ is smooth, there is a formula.

To understand the formula, consider the map $i_*\colon K_A(X^A) \to K_A(X)$ from earlier. This map is hard to draw, because it is *not* an algebra homomorphism. However there is an opposite map $i^*\colon K_A(X) \to K_A(X^A)$ which *is* an isomorphism of algebras. This map is just the normalization map for our earlier picture over $\operatorname{Spec} R$:




So $i^*$ is easy to understand. The idea is to compute $i^* i_*$ in order to get a handle on $i_*$. Let $X^A = \bigsqcup F_i$, and let $\mathcal{E}$ be a sheaf on a component $F_i$. The components $F_i$ are smooth. To compute $i^* i_* \mathcal{E}$, we use the Koszul complex, and get

$$i^* i_* \mathcal{E} = \mathcal{E} \otimes \left(\mathcal{O}_{F_i} - N^\vee_{X/F_i} + \wedge^2 N^\vee_{X/F_i} - \cdots\right).$$

## 6 (Feb 10) Localization (cont'd)

Let $X$ have an action by $A$, where $A$ is a semisimple algebraic abelian group (sometimes called a quasi-torus). These are all products of finite abelian groups and tori. Any $A$-orbit is of the form $A/A'$ where $A'$ is a (discrete) subgroup. There are infinitely many such $A/A'$, but on $X$ there is a finite stratification by free orbits of things like $A/A'$. One can imagine that $X$ sits inside some $\mathbb{P}(V)$, in which case this statement is clear. Each stratum is of the form

$$\mathring{X}^{A'} := X^{A'} \setminus \bigsqcup_{A'' \supset A'} X^{A''}.$$

If we take $K_A$ of each stratum, there is clearly a map

$$K_A(\mathring{X}^{A'}) \to K_{A/A'}(\mathring{X}^{A'}),$$

and we view $K_A(\mathring{X}^{A'})$ as pulled back from $K_{A/A'}$. But since $A/A'$ acts *freely* on $\mathring{X}^{A'}$,

$$K_{A/A'}(\mathring{X}^{A'}) = K(\mathring{X}^{A'}/(A/A')).$$

In summary,

$$\begin{aligned} K_A(\mathring{X}^{A'}) &= R(A) \otimes R(A/A') K_{A/A'}(\mathring{X}^{A'}) \\ K_{A/A'}(\mathring{X}^{A'}) &= K(\mathring{X}^{A'}) \otimes_{\mathbb{Z}} R(A/A'). \end{aligned}$$

The K-theory $K_A(\mathring{X}^{A'})$ is therefore only supported at $1 \in A/A'$. It follows that in the sequence

$$\to K_A^i(X^A) \to K_A^i(X) \to K_A^i(X \setminus X^A) \to \cdots,$$

all the elements in $K_A^i(X \setminus X^A)$ are torsion, because $X \setminus X^A$ decomposes as a union of strata of the form $\mathring{X}^{A'}$. Note also that the first term $K_A^i(X^A)$ is free, because it is $K(X^A) \otimes_{\mathbb{Z}} R(A)$.

Suppose $X$ is smooth. Let $i^* \colon K_A(X) \to K_A(X^A)$ be the pullback map. It is an algebra homomorphism. Given a sheaf $\mathcal{F} \in K_A(X^A)$, we can compute using Koszul resolution that

$$i^* i_* \mathcal{F} = \mathcal{F} \otimes (\text{Koszul}) = \mathcal{F} \otimes \sum_i (-1)^i \wedge^i N_{X/X^A}^\vee.$$

It is important to observe now that the term $\sum_i (-1)^i \wedge^i N_{X/X^A}^\vee$ is *invertible* on the relevant locus in $A$. This is because if $a \in A$ acts with some weights $w(a)$ in $N_{X/X^A}$, then we can decompose

$$N_{X/X^A} = \bigoplus w_i(a) L_i.$$

Without loss of generality, we assume the $L_i$ are line bundles. (In K-theory it suffices to prove formulas for vector bundles that decompose as a sum of line bundles, by the usual splitting principle argument.) Then the Koszul expression becomes

$$\sum_i (-1)^i \wedge^i N_{X/X^A}^\vee = \prod (1 - w_i^{-1} L_i^{-1}) \in K_A(X^A).$$

The operator $L \otimes -$ of multiplication by a line bundle is *unipotent*, because by trivializing the line bundle we see that the operator has eigenvalue 1 on each $F^i K(X^A)/F^{i+1} K(X^A)$. Hence the operator $\prod (1 - w_i L_i^{-1})$ has eigenvalues $1 - w_i$. The only way for this to be zero is if $w_i$ is a *fixed* weight, which is impossible by definition. It follows that

$$i_* \colon K_A(X^A) \left[ \frac{1}{1-w} \right] \to K_A(X) \left[ \frac{1}{1-w} \right]$$

is an isomorphism. The inverse map is $i^*$ up to the Koszul factor, namely

$$i_*\left(\frac{i^*\mathcal{E}}{\prod(1-w_i^{-1}L_i^{-1})}\right)=\mathcal{E}$$

for $\mathcal{E}\in K_A(X)$.

In K-theory, if $V=\sum x_i$ is the decomposition of $V$ into its Chern roots, then $\wedge^k V=e_k(x_i)$ where $e_k$ is the $k$-th elementary symmetric polynomial. It is very convenient to think of $w_iL_i$ as a Chern root, since it is an equivariant line bundle. We will write it as just $w_i$ from now on.

Consider a proper map $p\colon X\to Y$. Given $\mathcal{E}$ on $X$, to compute the pushforward $p_*\mathcal{E}$ we may as well think about $\mathcal{E}$ on $X^A$, via

$$\begin{array}{ccc} X^A & \xrightarrow{\ i\ } & X \\ & \searrow & \downarrow{\scriptstyle p} \\ & & Y \end{array}\ .$$

If the pushforward $p_*$ is actually not proper but the composition $p\circ i$ is, then one might even want to *define* the pushforward $p_*$ using $p\circ i$. If $Y$ is a point, the formula becomes

$$\chi(X,\mathcal{E})=\chi\left(X^A,\frac{i^*\mathcal{E}}{\prod(1-w_i^{-1})}\right).$$

**Example 6.1.** Take $\mathcal{O}(d)$ on $\mathbb{P}(\mathbb{C}^2)$, and let

$$\begin{pmatrix} a_1 & 0\\ 0 & a_2\end{pmatrix}\in \mathrm{GL}_2$$

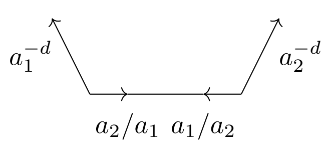
act on $\mathbb{P}(\mathbb{C}^2)$. Recall that $\mathcal{O}(1)$ has fibers *dual* to the corresponding line represented by a point of $\mathbb{P}(\mathbb{C}^2)$. Hence

$$\mathcal{O}(d)|_{[1:0]}=a_1^{-d},\quad \mathcal{O}(d)|_{[0:1]}=a_2^{-d}.$$

We discussed last time that

$$T_\ell\mathbb{P}(V)=\mathrm{Hom}(\underbrace{\ell}_{a_1},\underbrace{V/\ell}_{a_2})=a_2/a_1.$$

As a picture, we think of this setup as




.

Applying localization,

$$\chi(\mathbb{P}(\mathbb{C}^2),\mathcal{O}(d))=\frac{a_1^{-d}}{1-a_1/a_2}+\frac{a_2^{-d}}{1-a_2/a_1}=\frac{a_1^{-d-1}-a_2^{-d-1}}{a_1^{-1}-a_2^{-1}}.$$

There are two cases here. If $d\geq 0$, we get $a_1^{-d}+a_1^{-d+1}a_2^{-1}+\cdots+a_2^{-d}$. If $d<0$ we get something else. Note that we knew this is the correct answer, because for $d\geq 0$ the space of sections is spanned by degree-$d$ monomials in two variables.

An equivalent way to think of $\chi(\mathcal{E})$ is that we are computing the trace of an operator $a\in A$ acting on the cohomology of $\mathcal{E}$. We write this as

$$\mathrm{str}_{H^\bullet(\mathcal{E})}(a)=\sum(-1)^i\,\mathrm{tr}_{H^i(\mathcal{E})}(a),$$

where str denotes a *supertrace*. We'll just call it a trace. If $\dim X^A = 0$, then localization says this simplifies dramatically to

$$\operatorname{tr}_{H^\bullet(\mathcal{E})} a = \sum_{\text{fixed pts } p} \frac{\operatorname{tr}_{\mathcal{E}_p} a}{\det(1-a^{-1})}$$

where the denominator is acting on $T_pX$.

Consider an abelian variety $X = V/\Lambda$, where $V$ is a $\mathbb{C}$-vector space of dimension $n$. Let $a \in \mathrm{GL}(V) \cap \mathrm{GL}(2n,\mathbb{Z})$ preserve the lattice $\Lambda$. Take the simplest sheaf on $X$, namely $\mathcal{O}_X$. To compute $\operatorname{tr}_{H^\bullet(\mathcal{O}_X)} a$, one way is to use the Dolbeault resolution

$$\Omega^{0,0}(X) \xrightarrow{\bar\partial} \Omega^{0,1}(X) \xrightarrow{\bar\partial} \Omega^{0,2}(X) \to \cdots .$$

Then

$$H^i(\mathcal{O}_X) = H^{0,i}(X) = \wedge^i \overline{V}^\vee.$$

Conveniently, from the perspective of the element $a$, we can replace $\overline{V}^\vee$ by just $V$. It follows that

$$\operatorname{tr}_{H^\bullet(\mathcal{O}_X)} a = \det_{\mathbb{C}}(1-a),$$

where $\det_{\mathbb{C}}$ reminds us to take determinant in $\mathrm{GL}(V)$ and not $\mathrm{GL}(2n,\mathbb{Z})$. However, by localization,

$$\operatorname{tr}_{H^\bullet(\mathcal{O}_X)} a = \sum_{\text{fixed pt } p} \frac{1}{\det_{N_{X/p}}(1-a^{-1})}.$$

Since $N_{X/p} = V$ for *every* fixed point $p$,

$$\#\,(\text{fixed pts}) = \det_{\mathbb{C}}(1-a)\det_{\mathbb{C}}(1-a^{-1}) = \det_{\mathbb{R}}(1-a).$$

This is an incarnation of the Lefschetz fixed point formula, where the number of fixed points is computed as the degree of the map $(1-a)\colon X \to X$.

*Remark.* There are fixed point formulas from differential geometry as well, by Atiyah and Bott. Given a sheaf $\mathcal{E}$, the Dolbeault resolution

$$\mathcal{E} \otimes \Omega^{0,1}(X) \xrightarrow{\bar\partial} \mathcal{E} \otimes \Omega^{0,2}(X) \to \cdots$$

would yield

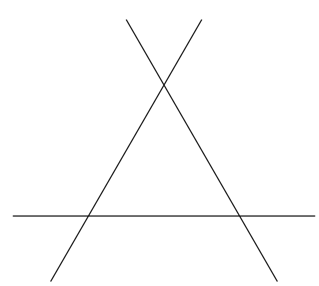
$$\operatorname{tr} - \chi(\mathcal{E}) a = \sum_p \frac{\operatorname{tr}_{\mathcal{E}_p} a \cdot \det(1-a^{-1})}{\det_{T_p}(1-a^{-1}) \cdot \det(1-a^{-1})}.$$

Here $\det_{T_p}$ is the character of an action on a formal neighborhood $\varprojlim \mathcal{O}_X/\mathfrak{m}_p^N$ of $p$. The two additional terms correspond to antiholomorphic functions and antiholomorphic forms. Upon canceling them, we get the usual algebraic localization formula.

**Example 6.2.** Let's look at $\mathbb{P}^2 = \mathbb{P}(\mathbb{C}^3)$ with action

$$\begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix}.$$

The picture corresponding to $\mathcal{O}(d)$ on $\mathbb{P}^2$ is a triangle of size $d$, on the lattice $a_1 + a_2 + a_3 = 0$.



Then the vertices of the triangle correspond to $a_1^{-d}$, $a_2^{-d}$, and $a_3^{-d}$. Take a Cech cover

$$\begin{aligned} U_i &:= \{x_i \neq 0\} \cong \mathbb{C}^2 \\ U_{ij} &:= U_i \cap U_j \cong \mathbb{C} \times \mathbb{C}^\times \\ U_{123} &:= U_1 \cap U_2 \cap U_3 \cong (\mathbb{C}^\times)^2. \end{aligned}$$

Let's do a Cech computation for $\chi(\mathcal{O}(4))$. On $U_1$, the character of $\mathcal{O}(4)$ will be of the form

$$\sum_{d_1, d_2 \geq 0} a_1^{-4} \left(\frac{a_1}{a_2}\right)^{d_1} \left(\frac{a_1}{a_3}\right)^{d_2} = \frac{a_1^{-4}}{(1 - a_1/a_2)(1 - a_1/a_3)}.$$

We recognize this as a term in the localization formula, where $a_1^{-4} = \mathcal{O}(4)|_{[1:0:0]}$ and the denominators are tangent weights. A similar computation holds for the other two fixed points. Repeating this calculation on $U_{12} = \mathbb{C} \times \mathbb{C}^\times$ and $U_{123}$, we can keep track of which areas of the polytope are included/excluded. The final result is that only the interior triangle contributes. From our localization formula, we knew this must be the result. All other regions in the polytope have *torsion* contributions, because they contain invertible functions like $x_1/x_2$ which contribute terms like

$$\sum_{k \in \mathbb{Z}} (a_1/a_2)^k.$$

These are delta functions.

Suppose $V$ is an infinite-dimensional space, and we want to compute $\operatorname{tr}_V a$. This may be too much to ask for, but maybe we can compute $\operatorname{tr}_V \int a\varphi(a)\, da$.

# 7 (Feb 12) Character formulas

Let $X := G/B$, and let $\mathcal{L}$ be a line bundle on $X$. Then

$$H^i(X, \mathcal{L}) = \begin{cases} 0 \\ \text{induced rep of } G \end{cases}$$

and therefore a computation of $\chi(X, \mathcal{L})$ will yield the Weyl character formula. The group acting on $X$ is the maximal torus $A \subset G$. Let's see this in action for $\mathbb{P}^1$.

**Example 7.1.** If $X = \mathbb{P}^1 = \mathrm{GL}(2)/B$, then $\mathcal{L} = \mathcal{O}(d)$ for some $d$. Let

$$A = \begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix}.$$

We have a freedom to choose the linearization of the line bundle $\mathcal{O}(d)$. In other words, we are allowed to twist $\mathcal{L}$ by an equivariant character. We computed last time, using the canonical linearization, that

$$\chi(X, \mathcal{O}(d)) = \frac{a_1^{-d-1} - a_2^{-d-1}}{a_1^{-1} - a_2^{-1}}.$$

If $d \geq 0$, this all comes from $H^0$; if $d < 0$, this all comes from $H^1$. Note that this answer is essentially a generating function like $\sum_{i=0}^d x^i$, but it must

- analytically continue to $d < 0$, and
- satisfy the duality $\overline{\chi}|_{(d+1) \mapsto (-d-1)} = \chi$, where $\overline{\chi}$ means the substitution $\overline{a}_i := a_i^{-1}$.

This duality operation corresponds to taking a dual representation. Explicitly, Serre duality tells us

$$H^1(\mathcal{L}^{-1} \otimes K) = H^0(\mathcal{L})^\vee.$$

So we expect the duality to be a symmetry around the *square root* $K^{1/2}$ of the canonical $K$.

In terms of localization, taking duals yields

$$\overline{\chi}(\mathcal{E}) = \sum_p \frac{\overline{\mathcal{E}}|_p}{\prod \overline{(1-w^{-1})}} = \sum_p (-1)^{\dim} \frac{\overline{\mathcal{E}}|_p \cdot \prod w^{-1}}{\prod (1-w^{-1})}.$$

This product $\prod w^{-1}$ is exactly the weight of the canonical $K_X$.

**Proposition 7.2.** *Let $X = G/B$. For every line bundle $\mathcal{L} \in \mathrm{Pic}(X)$:*

1. *$H^0(X, \mathcal{L})$ is either zero or an irrep of $G$;*
2. *all irreps of $G$ appear in this way.*

*Proof.* Let $V = H^0(X, \mathcal{L})$. Then $V$ has at most one highest weight vector $v$, namely a vector $v \in V$ such that

$$b \cdot v = \chi_v(b) v$$

for some character $\chi_v$. Let $U \subset B$ be the unipotent subgroup. In fact there is at most one $U$-invariant highest vector. Otherwise if $v_1, v_2 \in V$ are two such vectors, $v_1/v_2$ is a $U$-invariant rational function on $X$. But $U$ acts with an open orbit, so $v_1/v_2$ is actually constant.

Here is a more down-to-earth proof. We discussed that $\mathrm{Pic}_G(G/B) = \mathrm{Pic}_B(\mathrm{pt})$. Hence a section in $H^0(X, \mathcal{L})$ is a function $f(g)$ on $G$ such that $f(gb) = f(g)\chi(b)$ for some character $\chi$. Peter–Weyl says that functions live in $\bigoplus V \otimes V^*$. The character $\chi$ specifies $V^*$ uniquely.

Given an irrep $V$, note that $V = H^0(\mathbb{P}(V^\vee), \mathcal{O}(1))$. Inside $\mathbb{P}(V^\vee)$ is $G/B$. The restriction map of *linear* equations to a smaller sub-variety is injective, unless the sub-variety has a linear relation, in which case the functions must come from a *smaller* $\mathbb{P}(W^\vee)$. ☐

To understand $T_{eB}(G/B)$ for localization, write

$$\mathfrak{g} = \underbrace{\mathfrak{h} \oplus \mathfrak{n}_+}_{\mathfrak{b}} \oplus \mathfrak{n}_-.$$

Then it is clear that $T_{eB}(G/B) = \mathfrak{n}_- = \bigoplus_{\alpha<0} \alpha$ in $K_B(\mathrm{pt})$.

On $G/B$, we have $K_{G/B} = \mathcal{O}(-\sum_{\alpha>0} \alpha)$. If we let $\rho \coloneqq (1/2)\sum_{\alpha>0} \alpha$ denote the half-sum of all positive roots, $K_{G/B} = \mathcal{O}(-2\rho)$. Hence the symmetry coming from Serre duality involves a *shifted* version of the Weyl chambers, by exactly $-\rho$. If we label each (shifted) chamber by its length $i$, i.e. the minimum number of reflections across hyperplanes necessary to reach it from the *dominant* cone, it is true that only $H^i \neq 0$ for a line bundle in that chamber. Equivalently, we can restate this as follows.

**Proposition 7.3.** *There is a shifted action*

$$w \cdot \lambda \coloneqq w(\lambda + \rho) - \rho,$$

*and $H^i(X, \lambda) \neq 0$ iff $i = \ell(w)$ where $w \cdot \lambda$ lands in the dominant cone.*

*Proof.* Let $\alpha_i$ be a simple root, and $r_i$ be the corresponding reflection in $W$. There is an associated parabolic $P_i \subset G$ whose Lie algebra is $\mathfrak{b} \oplus \mathfrak{g}_{-\alpha}$. Then there is a fibration

$$\begin{array}{ccc} P/B = \mathbb{P}^1 & \hookrightarrow & G/B \\ & & \downarrow \pi \\ & & G/P \end{array}.$$

On the $\mathbb{P}^1$ fibers, the $G$ action is by $\langle \alpha_i^\vee, \lambda \rangle$. Fiber-wise, Serre duality tells us that if $\langle \alpha_i^\vee, \lambda \rangle \geq 0$,

$$\pi_*(\mathcal{L}_\lambda) = \pi_* \mathcal{L}_{r_i \cdot \lambda}[1],$$

from our discussion earlier about $\mathbb{P}^1$. Since the maximal length is exactly $\dim X$, it follows that the only non-zero $H^i$ is exactly $i = \ell(w)$. $\square$

To wrap up our general discussion of K-theory, let's discuss **rigidity**. This is when you *think* you have a group action, but in fact it doesn't act non-trivially. Here is the main example. Take a connected group $G$ and a smooth proper $X$. Then we can look at $H^p(\Omega^q X)$, which a priori is some non-trivial $G$-rep. In fact, this is the *trivial* rep, because by Hodge theory there is an embedding

$$H^p(\Omega^q X) \hookrightarrow H^{p+q}_{\mathsf{Top}}(X, \mathbb{C})$$

but the $G$-action is trivial on this.

Typically we make rigidity arguments differently. For a reductive group $G$, to prove a rep is trivial it is enough to prove it is trivial upon restriction to the maximal torus $A \subset G$. Individual cohomology groups are hard to deal with, so let's compute a generating function. Define

$$\Omega_t^\bullet(X) \coloneqq \sum_q \Omega^q t^q.$$

This is a generating function for exterior powers, so we have things like

$$\Omega_t^\bullet(X) = \Omega_t^\bullet(X_A) \otimes \wedge_t^\bullet(N_{X/X^A}^\vee)$$

because on the fixed locus, the tangent bundle $T_X$ splits in K-theory as $T_{X^A} + N_{X/X^A}$. By localization,

$$\chi(X, \Omega_t^\bullet(X)) = \chi\left(X^A, \Omega_t^\bullet \frac{1 + t w_i^{-1}}{1 - w_i^{-1}}\right),$$

where the $w_i$ are weights of the normal bundle $N_{X/X^A}$. Since we assumed $X$ is proper,

$$\chi(X, \Omega_t^\bullet(X)) \in K_G(\mathrm{pt})[t] \subset (\text{Laurent polynomials on } A)[t].$$

The statement of rigidity is that actually this expression for $\chi$ has to be constant. This is because the rational function

$$\frac{1 + t w_i^{-1}}{1 - w_i^{-1}}$$

is *bounded* at every infinity of the torus $A$. Hence the resulting $\chi$ must actually be a *constant* Laurent polynomial, i.e. a trivial rep.

In addition, rigidity implies this Laurent polynomial can be computed. We can choose any direction in $A$ to go to infinity. Let $\mathbb{C}^\times \to A$ be a generic cocharacter, so that for $z \in \mathbb{C}^\times$ each weight $w_i(z) \to 0, \infty$. Call the 0 limit *attracting* and the $\infty$ limit *repelling*. Taking the limit for $\chi$, each attracting weight $w_i$ yields a factor $(-t)$, and therefore

$$\frac{1 + t w_i^{-1}}{1 - w_i^{-1}} = (-t)^{\#(\text{attracting weights})}.$$

Actually the quantity $\chi(X, \Omega_y^\bullet(X))$ is usually called $\chi_y$**-genus**. We have just proved that

$$\chi_y(X) = \sum_{\substack{\text{components } F \text{ of} \\ \text{fixed locus}}} \chi_y(F) \cdot y^{\#(\text{attracting weights})},$$

where to compute attracting weights we take a generic cocharacter $\mathbb{C}^\times \to A$.

**Example 7.4.** For $\mathbb{P}^1$, we have $H^0(\mathcal{O}) = \mathbb{C}$ and $H^1(\Omega^1) = \mathbb{C}$, so

$$\chi_y(\mathbb{P}^1) = 1 + y.$$

Indeed, given a torus action on $\mathbb{P}^1$, everything is attracted to *one* of the two fixed points, giving $1 + 1 \cdot y$.

Note that here it is crucial that $X$ is compact. For something like $\mathbb{C}^n$, there is only $H^0$ but this is a non-trivial infinite-dimensional rep. In fact, there is a motivic decomposition

$$X = \sum X^A \times \mathbb{A}^{\#(\text{attracting weights})}$$

in the "ring of varieties", and $\chi_y$ genus is the homomorphism $\mathbb{A} \mapsto y$.

**Exercise 7.1.** Suppose $X$ is such that $\mathcal{L} = K_X^s$ where $0 < s < 1$ is some fractional power. For example, if $X = \mathbb{P}^n$, then $K_X = \mathcal{O}(-n-1)$ and $\mathcal{O}(-p) = K_X^{p/(n+1)}$ where $0 < p < n+1$. Then show that

$$\chi(\mathcal{L}) = 0.$$

# 8 (Feb 17) Correspondences acting on K-theory

*(Notes by Davis Lazowski)*

We've talked in general about K-theory. For us, K-theory will be an arena in which representation theory will play out.

If we want a map $K(Y) \to K(X)$, you write the analogous thing to a matrix: some element $\mathcal{E} \in K(Y \times X)$. You then get an operator by pushforward and pullback. I.e. if $\mathcal{F} \in K(X)$, we send

$$\mathcal{F} \to p_{Y,\star}(p_X^\star \mathcal{F} \otimes \mathcal{E})$$

Note, for pushforward to make sense, we need the map to be proper, for pullback to make sense, we need the map to be flat.

Now suppose we are working equivariantly. Let $T$ a torus of rank $r$. Then everything we write down is over the ring

$$K_T(\text{pt}) = \mathbb{Z}[t_1^\pm, \dots, t_r^\pm]$$

In representation theory, we typically work over a field. For the representation theory over a *general* ring is often a nightmare. Nonetheless, the fact that we're over $\mathbb{Z}[\{t_i^\pm\}]$ is somehow an added feature. Becuse this ring is very nice. If we tensor with a field, $K_T(\text{pt}) \otimes_{\mathbb{Z}} \mathbb{Q}$, we understand its fibres very well.

Some of the big successes of this business are the work of Ginzburg, in his book, and of Kazhdan–Lusztig. They took the Hecke algebra, which has a parameter $q$. If $q$ is generic (not a root of unity), things are very different from at a root of unity. If you interpret $q$ as a coordinate on a copy of $\mathbb{G}_m$ acting, the reason for this difference is very transparent.

*Remark.* We could also work over a base scheme $S$. Then we would have a correspondence

$$\begin{array}{ccc} & K_T(Y \times X \times S) & \\ \swarrow & & \searrow \\ K_T(Y \times S) & & K_T(X \times S) \end{array}$$

We have discussed at length the failure of the Kunneth formula in K-theory, but nonetheless, $K_T(Y \times S)$ is not *that* different from $K_T(Y) \otimes K_T(S)$, and may agree for many varieties. We may view this as a family of sheaves fibred over $S$. We can take a further correspondence,

$$\begin{array}{ccc} & \Delta_Y \boxtimes \beta \in K_T(Y \times Y \times S) & \\ \swarrow & & \searrow \\ K_T(Y) & & K_T(Y \times S) \end{array}$$

If the Kunneth formula fails, in some sense, that's ok : it just means you need to add an extra copy of $S$ every time we compose. So indices just pile up.

**Example 8.1.** Let $S$ a surface, $\mathrm{Hilb}_n(S)$ the variety parametrising $n$-tuples of points on $S$. The Nakajima correspondence then says, for example, that if I had a configuration of five points on $S$, corresponding to an ideal $I_n$, I could consider adding one extra point on $S$ to get $I_{n+1}$. I could consider adding a point, which would correspond to a short exact sequence

$$0 \to I_{n+1} \to I_n \to \mathcal{O}_p \to 0$$

In fact, $\{0 \to I_{n+1} \to I_n \to \mathcal{O}_p\} \subset \mathrm{Hilb}_n(S) \times \mathrm{Hilb}_{n+1}(S) \times S$ is *smooth*.

If $S$ is a general surface, e.g. $K3$, there is no way we can decompose the ideal. So this correspondence is really parameterized by $S$.

The operator induced by the correspondence that *adds* a point is usually called $\alpha_{-1}$. That which removes a point is $\alpha_1$. Then

$$[\alpha_1, \alpha_{-1}] \sim \Delta_{\mathrm{Hilb}} \times \Delta_S$$

(i.e., if one adds a point somewhere and removes a spatially distant point, these operations commute.) We write this as a map

$$\mathrm{Hilb}_n \to \mathrm{Hilb}_n \times S \times S$$

where this map lives in $K$-theory, but we have dropped the $K_T(\bullet)$ from our notation.

**Example 8.2.** We will start with the simplest possible Nakajima quiver variety. Pick some $n \in \mathbb{Z}_{\geq 0}$, then let

$$X = TG(n) = \sqcup_{k=0}^{n} T^\star \operatorname{Gr}(k, n).$$

$GL(n) \times GL(1)$ acts on $TG(n)$, where

- $GL(1)$ scales fibres;
- $GL(n)$ acts on the base.

Lying inside here, we write the torus as

$$T = \{ \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \times \hbar^{-1} \} = A \times \mathbb{C}^\times,$$

where the matrix is in some basis $\{b_i\}$, and look at $K_T(X)$.

- On one hand, we've proved for any $B$, $K(T^\star B) = K(B)$.
- On the other hand, we've proved that $K(\operatorname{Gr}(k, n))$ is generated by $\Lambda^i Taut$, $Taut$ denoting a tautological bundle.

So what does equivariant $K$-theory look like as a scheme

$$\operatorname{Spec}(K_T(T^\star \operatorname{Gr}(k, n))) \to \operatorname{Spec}(K_T(\mathrm{pt}))?$$

We write $e_i(x_1, \ldots, x_k) = \Lambda^i(Taut)$, where the $x_i$ are coordinates on the base $(\mathbb{C}^\times)^k / S(k)$. The fixed locus, w.r.t to $A$, is spanned by coordinate vector spaces, i.e. $\mathrm{span}\{b_{i_1}, \ldots, b_{i_k}\}$.

So the fixed locus is the union of $\binom{n}{k}$ parts. Therefore,

$$\text{rank } K(X) = \sum_{k=0}^{n} \binom{n}{k} = 2^n$$

we would like this to be a representation of something that deforms $\mathbb{C}^2(a_1) \otimes \mathbb{C}^2(a_2) \otimes \ldots \mathbb{C}^2(a_n)$ where each one of these is going to be a representation of $\mathfrak{gl}(2, \mathbb{C}[u])$.

If one then looks at a point in the torus, say $a_i$, I may evaluate this Lie algebra at that point. So this representation is for general $a_1, \ldots, a_n$, a very big algebra, and for specific choices of $a_1, \ldots, a_n$ it's an evaluation of this.

*Remark.* $\mathbb{C}^2(a_i)$ *should* be written as $\mathbb{C}^2[a_i^{\pm}]$, the round brackets are purely convention.

What this representation will be will end up being called an **R-matrix**.

**Example 8.3.** Here is a coordinate point of view. Take $\mathfrak{gl}(2, \mathbb{C}[u^{\pm}])$. These are generated by matrices like

$$\begin{pmatrix} u^n & 0 \\ 0 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

The first kind of matrices roughly come from $\Lambda^k Taut$. The $\binom{n}{k}$ above correspond to summing coefficient in the $|\uparrow\rangle, |\downarrow\rangle$ basis.

There is a Nakajima-style correspondence here on $\mathrm{Gr}(k, n) \times \mathrm{Gr}(k+1, n)$, which by taking conormals gives a correspondence on $T^\star(\mathrm{Gr}(k, n) \times \mathrm{Gr}(k+1, n))$.

But this is not how you should be thinking about it. Just like you don't think of an algebraic variety in terms of coordinates, you think about it in terms of the coherent sheaves on it, you don't think of an algebra in terms of generators-and-relations, but rather in terms of its category of modules.

*Remark.* Note that if $G$ is a group, $\mathbb{C}G$ an algebra with an extra structure of a coproduct and antipode making it a **Hopf algebra**, note that these have the same category of modules, But the extra structure on $\mathbb{C}G$ gives $G - mod$ the structure of a tensor category.

Observe that totally generally, if $\mathcal{A}$ is an algebra,

$$\mathcal{A} = \text{all operators in M that commute with Hom in Mod-}\mathcal{A}$$

So inside $\mathcal{A} = \mathbb{C}G$, we can find $G$ inside as the subset solving

$$\Delta g = g \otimes g$$

where $\Delta$ denotes the coproduct.

*Remark.* The same story holds when $G$ a group is replaced by $\mathfrak{g}$ a Lie algebra. Then the analogue to $\mathbb{C}G$ is the universal enveloping algebra $U(\mathfrak{g})$, and the coproduct equation identifying $\mathfrak{g}$ inside is now

$$\Delta \xi = \xi \otimes 1 + 1 \otimes \xi$$

**Definition 8.4.** A **quantum group** is a deformation of $U\mathfrak{g}$ or $C\mathbb{G}$ in the class of Hopf algebras for which the coproduct $\Delta$ is no longer commutative.

One may think this makes the story more complicated, but in fact in many ways it is easier. Previously, we had

$$(12)\Delta = \Delta$$

but this no longer holds. The point is, how can this not be the same? It can not be the same if now

$$\mathbb{C}^2(a_1) \otimes \mathbb{C}^2(a_2) \neq \mathbb{C}^2(a_2) \otimes \mathbb{C}^2(a_1)$$

How can it not be the same? Well, it means that sometimes these representations might be reducible. In fact, they're still the 'same', just there is a nontrivial isomorphism, $R(a_1/a_2)^\star$, from one side to the other. $R(a_1/a_2)$ is a rational function of its parameter. The points where the representations are reducible corresponds to the points where the $R$-matrix has zero determinant. We will discuss this in more detail in the next lecture.

*Remark.* For notational convention, it may often be easier to use the same space but the opposite coproduct.

For $\mathbb{C}^2(u)\otimes$ anything = V has some $R$-matrix,

$$R(u):\mathbb{C}^2(u)\otimes V\to\mathbb{C}^2(u)\otimes V$$

where $u$ is a rational function of $u$ with values in $End(\mathbb{C}^2\otimes V)$. To have one function in this space is the same as to have four operators on $V$, by taking matrix elements.

The strategy of how we are going to make this $R$-matrix is as follows. $TG(1)$ consists of two points.

Take $K(TG(1)\times TG(n))$, construct two different maps (of $U_\hbar\widehat{\mathfrak{gl}}_2$) in $K(TG(n+1))$, such that they are both isomorpisms after localisation in u. I.e. we want to send a tuple $(u)\times\hbar^{-1}$, $\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n\end{pmatrix}\times\hbar^{-1}$ into a matrix $\begin{pmatrix} u & & & \\ & a_1 & & \\ & & \ddots & \\ & & & a_n\end{pmatrix}$. The Kunneth formula holds for the Grassmannian, so

$$K(TG(1)\times TG(n))\simeq K(TG(1))\otimes K(TG(n)).$$

So take the matrix elements there. This is the direction in which our discussion will proceed next time.

# 9 (Feb 19) Reconstruction for quantum groups

*(Notes by Davis Lazowski)*

**Setup**. Today, we will look at $\mathcal{A}=U_\hbar(\widehat{\mathfrak{g}})$. This will be a Hopf algebra deformation of $U(\mathbb{C}^\star\to\mathfrak{g})$. The $\mathbb{C}^\times$ automorphisms will be preserved in $\mathcal{A}$. It will act on $K(X_n)=\sqcup_{k=0}^n T^\star\operatorname{Gr}(k,n))$.

$\mathcal{A}-mod$ will be a tensor category with an action of $\mathbb{C}^\star$, $M\to M(u)$, generically braided so that there is a rational matrix of $u$ inducing an isomorphism

$$R(u_1/u_2):M_1(u_1)\otimes M_2(u_2)\to M_1(u_1)\otimes_{op}M_2(u_2)$$

Further, $R$ will satisfy a certain consistency equation, the **Yang-Baxter equation**,

$$R_{12}(u_1/u_2)R_{13}(u_1/u_3)R_{23}(u_2/u_3)=R_{23}(u_2/u_3)R_{13}(u_1/u_3)R_{13}(u_1/u_3)$$

**Fact.** *For $X_n=TG(n)$, it will be that case that $K(X_n)\otimes\operatorname{Frac}(K(pt))\simeq K(X_1)^{\otimes n}$. Further, $K(X_1)=pt\sqcup pt$. Therefore it suffices to know*

$$R_{K(X_1),K(X_1)}(u)$$

*Remark.* Suppose we have a solution $R_{M_i,M_j}(u)$ of the Yang-Baxter equation for some collection $\{M_i\}_{i\in I}$ of projective modules over your ground ring (for us, basically vector spaces).

Then this makes $M_{i_1}(a_1)\otimes\cdots\otimes M_{i_k}(a_k)$ a module over a certain Hopf algebra with automorphism $\mathbb{C}^\times$, where the automorphism sends $a_i\to qa_i$, constructed as follows.

We take the further requirement

$$R_{21}(u^{-1})R_{12}(u)=1$$

- **Step 1**. Define

$$R_{M_{i_1}(a_1)\otimes\cdots\otimes M_{i_k}(a_k),M_{j_1}(b_1)\otimes\cdots\otimes M_{j_\ell}(b_\ell)}=R_{i_1j_\ell}(a_{i_1}b_\ell)\dots R_{M_{i_k},M_{j_1}}(a_k/b_1)$$

  Then we may check this satisfies the Yang-Baxter equation. The span of all matrix elements of our $R$-matrices will be the putative algebra $\mathcal{A}$.

- **Step 3**. So we can think about $R_{W,M}$ , where we call $W$ the **auxiliary** space and $M$ the **physical** space. This acts on $W \otimes M$. What we do is take an *arbitrary* matrix element on $W$.

  Tensor product in auxiliary space $W$ corresponds to multiplication of operators; the corresponding relation from YBE on the tensor product of two auxiliary spaces and a physical space is known as RTT=TTR. Meanwhile, tensor product in physical space corresponds to *coproduct*.

  For picking a matrix element in $W$, the Yang-Baxter equation for $W \otimes M_1 \otimes M_2$ implies

  $$\Delta\text{matrix element of R} = \text{same matrix element of } R_{01}R_{02}$$

  where the label 0 is in $W$, $1, 2$ in physical space.

- **Step 3**. Now we want to study commutation relations in $\mathcal{A}$: we want to understand how $W \otimes W' \otimes M, M \otimes W \otimes W'$ compare. I.e. we would like to compare $R_{0'1}R_{01}$ and $R_{01}R_{0'1}$.

  We know by Yang-Baxter

  $$R_{0'0}R_{0'1}R_{01} = R_{01}R_{0'1}R_{0'0}$$

- I.e. the algebra is the algebra of all matrix elements in auxiliary space of $R$-matrices $R_{W,M}$. It is an algebra because we may tensor auxiliary spaces, with coproduct coming from tensoring representations of $M$.

  There are relations: for instance, given by Yang-Baxter. But there could be further relations, i.e. a map $W \to W'$ which universally commutes with $R$-matrices. These correspond one-to-one to morphisms in the tensor category, hence one-to-one to relations in $\mathcal{A}$.

**Example 9.1.** This is very familiar from the case of an algebra which is commutative but not cocommutative, e.g. the example of functions on a group. E.g. $G \subset GL(N)$,

We have a map $\mathbb{C}[End(V)] \to \mathbb{C}[G]$, which is a surjection. The relations are $Sym^\bullet End(V)$, which goes like

$$\otimes_{k=0}^{\infty}(End\, V)^{\otimes k} \to S^\bullet End(V) \to \mathbb{C}[End(V)] \to \mathbb{C}[G]$$

The map $\otimes_k (End\, V)^{\otimes k} \to S^\bullet End(V)$ is $x_{ij} \otimes x_{k\ell} x_{k\ell} \otimes x_{ij}$, the analogue of the Yang-Baxter equation. Any $G$-invariant map $S^\bullet End(V) \to \mathbb{C}[End(V)]$ leads to another relation.

**Moral**. Any time you are faced by something which is like a quantum group, do not try to think about relations in the algebra. Better is to think about morphisms in the tensor category one will generate, for one has a chance to get a handle on these e.g. geometrically.

**Our case**. We are trying to use this general setup to compute an algebra $\mathfrak{gl}(2, \mathbb{C}[u^{\pm 1}])$. There is a map

$$\mathcal{U}(\mathfrak{gl}(2, \mathbb{C}[u^{\pm}])) \to \otimes \mathcal{U}(\mathfrak{gl}(2))$$

by evaluating at some sets of points. So the representation theory is sort of a copy of $\mathfrak{gl}_2$ for every point, and they don't talk to each other. But as we deform, that will change.

**Hope**. We would like

$$K_{GL(n)}(TG(n)) \otimes \text{Frac}(K_{\text{GL}(n)}(\text{pt})) \simeq \otimes_{i=1}^{n} \mathbb{C}^2(a_i)$$

where in fact we take a torus $\text{diag}(a_1, \ldots, a_n) \subset \text{GL}(n)$.

**Fact.** *There will be two maps $\Delta, \Delta_{op} : K_{eq}(TG(n_1)) \otimes K_{eq}(TG(n_2)) \rightrightarrows K_{eq}(TG(n))$. which will be isomorphisms after we invert some certain elements. Further, composing one with the other, we will get the $R$-matrix. It will be given by a particular correspondence between these two Grassmannians.*

For Grassmannians, the decomposition theorem holds, so this really will be two correspondences

$$TG(n_1) \times TG(n_2) \rightrightarrows TG(n_1 + n_2)$$

This will be a vey special case of a more general idea. Observe that if I have a torus $\text{diag}(u, \dots, u, 1, \dots, 1)$ with $u$ in the first $n_1$ entries and 1 in the last $n_2$ entries, $TG(n_1) \times TG(n_2) \subset TG(n_1 + n_2)$ as the fixed locus of this torus.

There is an interesting correspondence associated to such a situation, and much more generally. For if I have a general torus $A$ acting on some $X$, with fixed locus $F = X^A$, there are two natural other subvarieties associated to this set.

**Example 9.2.** We will do the special case of $T^\star \mathbb{P}^1$. There is a torus action $\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}$ with two fixed points, 0 and $\infty$.

We can study the attracting and repelling loci of the fixed point set on $T^\star \mathbb{P}^1$. The two correspondences we are interested in will live by taking an isomorphism of the fixed loci in $K$-theory with the attracting or repelling loci, then pushfoward to $K_{eq}(X)$ (which will be an isomorphism after localisation). The really interesting thing here will be the isomorphism with the atracting loci.

The geometric question this story raises:

$$0 \to K(X_{attr,\infty}) \to K(X_{attr,\infty} \sqcup X_{attr,0}) \to K(X_{attr,0}) \to 0$$

How may we relate a sheaf on the union of the attracting loci onto the whole space?

# 10 (Feb 24) Nakajima quiver varieties

*(Notes by Davis Lazowski)*

*Andrei is away this week, so Henry Liu is lecturing instead this week.*

This week I'll introduce you guys to Nakajima quiver varieties. They will be the prototypical varieties we discuss for the rest of the course.

But before we talk about them specifically, I think it's best to start with a larger picture, and view them of as a special kind of *(conical) symplectic resolution.*

We want to view them as such because symplectic resolutions are already a very nice class of algebraic variety, with many nice properties, inherited by Nakajima quiver varieties.

**Definition 10.1.** Let $(X, \omega)$ a smooth algebraic symplectic variety, $X_0 = Spec(\Gamma(X_0))$ is affinisation. If $X \to X_0$ is a resolution of singularities (i.e. proper, birational,surjective), we call $(X, \omega)$ a **symplectic resolution**.

Algebraic symplectic means that the symplectic form is an algebraic, i.e. holomorphic, 2-form.

**Example 10.2.** $T^\star G/B$, $Hilb(\mathbb{C}^2)$ are smooth symplectic spaces. ($Hilb(\mathbb{C}^2)$ inherits a symplectic form from $\mathbb{C}^2$).

- $Spec(\Gamma(T^\star G/B))$ is the **nilcone**, $\mathcal{N} = \{x \in \mathfrak{g} | ad(x) \text{ nilpotent } \}$.
- $Hilb(\mathbb{C}^2)$ is the moduli space of $n$ points on $\mathbb{C}^2$, possibly overlapping. Its affinisation is $Sym(\mathbb{C}^2)$ (which is n-tuples of points on $\mathbb{C}^2$.) Moving to the affinisation, we lose the data of how points hit each other.

  For instance, $(x^2, y), (x, y^2)$ are different in the Hilbert scheme, but both the same multiple of zero in $Sym$.

Examples of symplectic resolutions we have are locally one of these two. So it's not a theorem but generally, locally, it suffices to study these two.

*Remark.* These two examples are more special than general symplectic resolutions: there is a $\mathbb{C}^\times$ action on the base.

**Definition 10.3.** A **conical symplectic resolution** is a symplectic resolution with a $\mathbb{C}^\times$ action on $X_0$ which scales $X_0$ down to a point.

In the opposite category, this means $\mathbb{C}[X_0]_0 = \mathbb{C}$. In particular, therefore the symplectic form is scaled by a positive weight.

So we may look at $[\omega] \in H^2(X, \mathbb{C})$. We learned a few classes ago that $H^2(X, \mathbb{C})$ is the trivial representation – i.e. weight zero. Hence,

**Claim.** *If $(X, \omega)$ is a conical symplectic resolution, $\omega$ is exact.*

Lots of other consequences follow from considering the $\mathbb{C}^\times$ action just like this.

*Remark.* For symplectic resolutions, not necessarily algebraic, $X \to X_0$ is stratified nicely, in some sense.

It's nice because e.g.

$$\begin{array}{ccc} X_i & \xrightarrow{\subset} & X \\ \downarrow & & \downarrow \\ (X_0)_i & \xrightarrow{\subset} & X_0 \end{array}$$

If $(X_0)_i$ is a downstairs stratum, then the symplectic form $\omega_{|X_i}$ on $X_i$ is pulled back from the symplectic form $(\omega_0)_{|(X_0)_i}$ downstairs. This automatically implies every symplectic resolution is semismall.

*Definition* 10.4. A resolution $X \to Y$ is **semismall** if $codim_Y Y_i = 2\ell$, then $codim_X X_i \geq \ell$.

We care about the semismall property because of the **BBDG decomposition theorem**, which implies that we can understand the cohomology of $X$ in terms of the cohomology of the strata of $Y_i$.

The semismall property follows from this property on symplectic forms because the symplectic form upstairs is nondegenerate. So look at the fibre over $x_0 \in (X_0)_i$. We know $dim(X_i) + dim(X_i)_{x_0} \leq dim(X)$. Hence because $dim(X) = dim(X_0)$ and $dim(X_i)_{x_0} = dim(X_i) - dim(X_0)_i$,

$$\begin{aligned} 2dim(X) &\geq 2dim(X_i) + dim(X) - dim(X_0)_i \\ &\implies 2codim_X X_i \geq codim_{X_0}(X_0)_i \end{aligned}$$

So now we know symplectic resolutions are important: how many we manufacture them? We do it via *algebraic symplectic reduction.* Suppose $G$ a Lie group, reductive, acts on $M$. Then

$$M \,/\!\!/\!\!/\!\!/_{\theta,\zeta} G = \mu^{-1}(\zeta) /\!\!/_\theta G = \mu^{-1}(\zeta)^{\theta-\text{semistable}}/G$$

I am assuming most of you have seen symplectic reduction, to at least some extent, so I will focuse on the GIT side of things. But first, let's go over symplectic reduction quickly.

**Symplectic reduction**.

**Definition 10.5.** If $G$ acts on $(M, \omega)$, we say the action is **Hamiltonian** if the map

$$\mathfrak{g} \to Vect(M, \omega)$$

is such that $\iota_V \omega$ is exact. Very roughly, this says that you have 'enough conserved quantities', i.e. Noether's theorem applies.

In this case, we may extend the sequence

$$\mathfrak{g} \to Vect(M, \omega) \to C^\infty(M)$$

Then the moment map is the associated map $\mu : M \to \mathfrak{g}^\star$.

*Remark.* If you haven't done it, it is a good exercise to compute the moment map of the $S^1$ action on $S^2$.

*Remark.* From this description of the moment map, note that we can't just plug in any $\zeta$ in for algebraic symplectic reduction: you also need $\zeta$ to be a coadjoint fixed point, i.e. fixed under the action of $G$ on $\mathfrak{g}^\star$.

**Geometric invariant theory**.

**Example 10.6.** Let $\mathbb{G}_m$ act on $k^n$ diagonally, by $diag(t, t, \ldots, t)$.

$\mathbb{G}_m$-invariant functions on $k[x_1, \ldots, x_n] = k$ itself, just constant functions. Taking *Spec*, we just get a point.

The problem is: we don't have enough functions. Viewing $k[x_1, \ldots, x_n]$ as $H^0(X, \mathcal{O}_X)$, we will find more functions by choosing a different line bundle.

To do GIT, you

- Pick $\mathcal{L} \in Pic_G(X)$, ample;
- Let $\mathbb{C}[X]^{G,\mathcal{L}} := \oplus_{n\geq 0} H^0_G(X, \mathcal{L}^{\otimes n})$.
- Take Proj of this to get a geometric space.

**Definition 10.7.** The **GIT quotient**, $X /\!\!/_{\mathcal{L}} G = Proj \mathbb{C}[X]^{G,\mathcal{L}}$.

*Remark.* No one ever computes a GIT quotient this way. Note that $X /\!\!/_{\mathcal{L}} G$ has affine charts, $X_f/sslashG$, where the functions really are invariant functions: it's parameterising orbits of your action.

So we let $X^{\mathcal{L}-semistable} = \{x \in X |$ some $f \in \mathbb{C}[X]^{G,\mathcal{L}}$ has $f(x) \neq 0$ $\}$.

Here, you're thorwing away fixed points where all equivariant functions are zero which would otherwise destroy your quotient.

*Remark.* For us $X$ will almost always be affine. Then a line bundle is the same thing as a character $\mathcal{L} \simeq \mathcal{O} \in Hom(G, \mathbb{C}^\times)$.

**Example 10.8.** Let $X = \mathbb{C}^n$ , acted on by $\mathbb{C}^\times$ (the $k = \mathbb{C}$ case of the example above). Choose a character $\theta > 0$.

Then $0 \in \mathbb{C}^n$ is *not* $\theta$-semistable, because $f(t \cdot 0) = t^n f(0)$, by equivariance. $t$ is arbitrary, so take the limit as $t \to 0$.

What made this work was that 0 was a $t$-fixed point, and the weight of $f$ under $t$ is in some sense a positive weight. Keep those two observations in mind, we will come back to them later.

I claim every other point is semistable and leave it to you to check. Then

$$\mathbb{C}^n /\!\!/_\theta \mathbb{C}^\times = (\mathbb{C}^n \setminus 0)/\mathbb{C}^\times = \mathbb{P}^{n-1}$$

What would happen if you took $\theta < 0$, rather than $\theta > 0$? There are no functions with negative weights, so that space is just empty.

So,

- If $\theta > 0$, we get $\mathbb{P}^{n-1}$;
- If $\theta = 0$, we get a point;
- If $\theta < 0$, we get the empty set.

*Remark.* Observe that

$$f(x) = \theta(\gamma(t))^n f(\gamma(t)x)$$

Just like in the case above, we should try to take a limit, and compare $f(x)$ with $\lim_{t\to 0} \theta(\gamma(t))^n f(\gamma(t)x)$.

If $lim_{t\to 0}\gamma(t)x$ exists as $t \to 0$ and $\theta(\gamma(t))^n$ is of positive weight, *then*, by the same argument as in our earlier special case, $x$ cannot be $\theta$-semistable.

The **Hilbert-Mumford criterion** says that this is an if and only if.

However, this is not enough to get a good quotient, where good is a technical term. For even some semistable points may have very large (possibly infinite) stabilisers. This are, from the point of view of algebraic geometry, very bad.

**Definition 10.9.** In the situation above, a point $x \in X$ is **$\theta$-stable** if

1. The stabiliser $G_x$ is finite;

2. The orbit $Gx$ is closed.

**Fact.** *There is an analogue of the Hilbert-Mumford criterion for stability. Rather than requiring a weight $t > 0$, we require $t \geq 0$. This will give you $\theta$-stable points. In particular, being $\theta$-stable is an open condition.*

*Remark.* Because of this, it's not hard to believe the following picture: look at the space of all stability conditions. Inside will be a certain hyperplane arrangement. Within each chamber, the $\theta$-stable locus will be unchanged. AS you take a limit and hit one of the walls of the chamber, some of these stable points may become semistable. As you go past a chamber wall, these points may becaome unstable and other points which were unstable may not be stable.

You're still not quite done: even after such a GIT quotient, there may be singularities coming from finite stabilisers. You need some additional structure to guarantee smoothness.

*Remark.* We have talked a lot about what happens when you vary $\theta$. But we could also vary the moment map parameter $\zeta$. You will get a similar hyperplane picture. The point is, the more walls you're on, the more resolved you are. If you're on the intersection of all walls (e.g. 0), you are maximally resolved.

Now, time to define Nakajima quiver varieties. These things have great properties.

**Definition 10.10.**

- Let $\vec{Q}$ a quiver ( a directed graph with possible self-loops at vertices and possibly more than one edge between any two nodes ).

- A **representation of** $\vec{Q}$ is the assignment of a vector space to each vertex, and a map between the corresponding vector spaces for each edge.

- Let $\vec{v}$ a fixed **dimension vector**, with a dimension at each vertex, constraining the dimension of the vector space at that vertex, $Rep_{\vec{Q}}\vec{v}$.

- Add (drawn square) **framing vertices**, one for each vertex in $\vec{Q}$ and an arrow from the framing vertex to its associated original vertex. Add a dimension vector to constrain the dimensions at framing vertices. Denote this $Rep_{\overline{Q}}(\vec{v}, \vec{w})$.

- Now $GL(V_i)$ acts on $T^\star Rep_{\overline{Q}}(\vec{v}, \vec{w})$.

- Take the space $T^\star Rep_{\vec{Q}}(\vec{v}\vec{w}) \,////\, \prod_i GL(V_i)$.

Note, we do not quotient by the action on framing vertices.

*Remark.* Why quivers? We'll talk about this next time. But for various reasons, this quotient will not have finite stabilisers. Hence this will turn out to be a smooth space. So these will alwys be symplectic resolutions, with an incredibly nice group action by the framing vertices.

I will leave you with a small exercise:

**Exercise**. Let $\vec{A}_1$ the quiver with one node and no edges. What is the associated Nakajima quiver variety?

You should find that it is $T^\star Gr(v, w)$, for generic stability condition and $\zeta = 0$.

# 11 (Feb 26) Examples of Nakajima quiver varieties

*(Notes by Davis Lazowski)*

Last time, we wrote down the definition of a Nakajima quiver variety.

**Example 11.1.** The simplest possible such example corresponds to the quiver with no edges and one node, $\vec{A}_1 = \bullet$.

1. Add a framing vertex, $\bullet \leftarrow \boxdot$.
2. Associate to $\bullet$ a dimension vector (scalar) $k$, and to $\boxdot$ a dimension vector (scalar) $n$.
3. Then

$$Rep_{T^\star \overline{Q}} = Hom(W, V) \oplus Hom(V, W)$$

we need to compute the moment map for the $GL(V)$ action on this quiver. Call the coordinate on $Rep(W, V) = q$, on $Rep(V, W) = p$. We need to differentiate the action,

$$\xi(q, p) = (\xi q, -p\xi)$$

Contract this with the symplectic form $dq \wedge dp$,

$$\iota_\xi \omega = \xi q dp + p \xi dq = d(p \xi q)$$

The natural dual map $\mathfrak{gl} \to \mathfrak{gl}_V$ sends $A \to tr(A\bullet)$. so So, the moment map

$$\mu : M \to \mathfrak{gl}_V^\star$$
$$(p, q) \to (\xi \to tr(p\xi q))$$

4. So,

$$\mu^{-1}(0) = \{(i, j) \in Hom(W, V) \oplus Hom(V, W) | ij = 0\}$$

5. Choose stability parameter $\theta > 0$. I claim now stability requires that $j$ be injective. For suppose $j$ were not injective. Then were $j(e_1) = 0$, take the one-parameter subgroup scaling along that direction, $\gamma(t) = diag(t, 1, 1, \ldots, 1)$, which doesn't change $j$. So

$$\lim_{t \to 0} j\gamma(t)^{-1}$$

exists.

6. Hence $j \in Gr(k, n)$. $i$ is perpendicular to $j$, so $i$ is a cotangent vector at $j$. So this is $T^\star Gr(k, n)$.

*Remark.* What happens if you pick $\theta < 0$? Then we find that $i$ is surjective, by a dual argument to the one above for $j$. So $ker(i) \in Gr(n-k, n)$. The moment map is unchanged, so $j$ is still perpendicular to $i$. So now you get $T^\star Gr(n-k.n)$, still the same space.

*Remark.* In general, for any Nakajima quiver variety, any generic stability parameter $\theta$ and $-\theta$ will give you the same space.

For a general quiver, you'll mod out by $\prod_{i \in I} GL(V_i)$. So we pick a stability parameter living in $\mathbb{R}^I$. In here there will be some hyperplane arrange to avoid if we want all semistable points to be stable. What is this hyperplane arrangement?

The Hilbert-Mumford criterion says that the walls live where $\theta(\gamma(t)) = 0$ exactly. So they should lie where $\theta \cdot$ something is zero. Where?

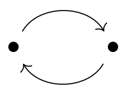
Let $A_{ij}$ the adjacency matrix of $\vec{Q}$. Write down the Kac-Moody Lie algebra associated to this $A_{ij}$ by taking $C = 2I - A$. This algebra might be horribly, but nonetheless gives a Lie algebra associated to the quiver $\vec{Q}$, $I^{\vec{Q}}$.

Associated to that matrix $C$ is a pairing on the quiver's vertices.

**Definition 11.2.** A **root** of the quiver is $\alpha \in N^I$ so that $(\alpha, \alpha) \leq 2$, where $(\bullet, \bullet)$ is the pairing associated to the Cartan matrix $C$.

**Theorem 11.3.** *(Nakajima). The hyperplanes are* $\{\theta \cdot \alpha = 0\}$ *for some root* $\alpha$.

**Example 11.4.** Let $\widehat{A}_1$ the cyclic $\vec{A}_1$-quiver,



associated to it is the affine Lie algebra $\widehat{\mathfrak{sl}_2}$. The roots will look like $\begin{pmatrix} 2\alpha + n \\ n \end{pmatrix}$. So there are tons of roots. and lots of chambers – some very interesting.

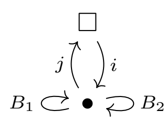
**Example 11.5.** Recall $Hilb^n(\mathbb{C}^2) = \{I \subset \mathcal{O}_{\mathbb{C}^2} | dim \mathbb{C}[x, y]/I = n\}$ where $I$ is an ideal.

How do we think about this? Well, there's a group action $(\mathbb{C}^\times)^2 = diag(x, y)$ acting on $\mathbb{C}^2$, inherited by the Hilbert scheme. On $\mathbb{C}^2$ the only fixed point is zero. So what are the fixed points on $Hilb_n(\mathbb{C}^2)$? It's all points suppoted at zero.

For instance, $(x^2, xy, y^2), (x^3, y), (x, y^3)$ are the length three fixed points at zero. Actually, every fixed point of $Hilb_n\mathbb{C}^2$ is the same as a 2D partition: for choose homogeneous generators on the $x, y$ lattice.

This is the quiver variety associated to the **Jordan quiver**,

$T^\star\overline{Q}$ looks like




To compute $\mu$, note the $\mu_{A \oplus B} = \mu_A \oplus \mu_B$. Now the moment map on $T^\star Hom(W, V)$ is just $ij$, like before, and we may compute that the moment map on $\mu$ is just a commutator, $[B_1, B_2]$.

So, $\mu = [B_1, B_2] + ij$.

Now how do we find the semistable points? Fortunately, in the case of quivers we may use a perhaps more tractable description than that of GIT stability, due to King:

**Theorem 11.6.** *For quiver representations, GIT stability is the same as 'slope' stability.*

The advantage of slope stability is that it is much easier to apply than the Hilbert-Mumford criterion.

**Definition 11.7.** If $V \in Rep_{\vec{Q}}$,

$$slope_\theta(V) = \frac{\theta \cdot dim(V)}{(1, 1, \ldots, 1) \cdot dim(V)}$$

we say $V$ is **slope semi-stable** if $slope_\theta V' \leq slope_\theta V$ for all $V' \subset V$. It is **slope stable** should the inequality be strict.

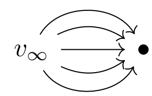
Unfortunately, we cannot apply the criterion directly to our framed quivers: to do so, we would be also modding by $GL(W)$, the global symmetry group, and we don't want to do this. But there is a very nice trick.

**Fact.** *Let* $\overline{Q}$ *a framed quiver. From it we may produce an unframed quiver* $Q^{\vec{w}}$. *To make it,*

1. *Take your framed quiver*

$$\square w_1 \to \bullet \leftarrow \square w_2$$

2. *Forget your framed vertices. Add a new vertex $v_\infty$. For each framing vertex $\square$ with dimension vector $w_1$, add $w_1$ arrows from $v_\infty$:*




3. *We may perform King's stability criterion for our new quiver $Q^{\vec{w}}$. The stability parameter adds*

$$\theta_\infty = -\theta \dim(V)$$

**Example 11.8.** Let us run this for the Jordan quiver. For framing dimension one and vertex dimension $n$. Choose $\theta = -1$. So $\theta_\infty - n$. Slope stability says that $(\theta, \theta_\infty)(dimV', dimV'_\infty) = 0$.

- If $dim(V'_\infty) = 1$, then $V' = V$. If $\dim(V'_\infty = 0)$, then there are no restrictions on $V'$.
- Hence, $V$ is stable if $V = \langle B_1, B_2 \rangle \, im(i)$.
- So, the Nakajima quiver variety is

$$\mathcal{M} = \{(B_1, B_2, i, j) | [B_1, B_2] + ij = 0 \text{and is stable}\}$$

Why is this the Hilbert scheme?

A fact is that stability implies $j = 0$. Hence $[B_1, B_2] = 0$. Therefore $V = \mathbb{C}[x, y]/I$. The $x, y$ actions on $V$ are exactly the $B_1$, $B_2$ actions. The image of the preferred vector $i$ is the unit.

That's the bijection on points. Sadly, to show an isomorphism on spaces is harder: so we'll not do so.

*Remark.* Some of you may know the ADHM construction of the Hilbert scheme. The ADHM equations say

$$[B_1, B_2] + ij = 0$$
$$[B_1, B_1^\dagger] + [B_2, B_2^\dagger] + ii^\dagger - j^\dagger j = -i\theta$$

The first is the moment map equation, the second equation is an alternate to the stability condition.

It's true this will produce for you the Hilbert scheme; why?

The answer is that in GIT there is

*Theorem* 11.9. ***(Kempf-Ness)**. $X /\!\!/_\theta G = \mu_{\mathbb{R}}^{-1}(-i\theta)/G_{\mathbb{R}}$*

The real moment map equation is the second ADHM equation. So we often call the top equation the complex moment map equation, the latter the real moment map equation.

*Remark.* If we took $\theta = 1$ instead, then $i = 0$. The bijection will show you $V = (\mathbb{C}[x, y]/I)^\star$. Here we have $(1, -n)(dim\, V', dimV'_\infty) \leq 0$, i.e. a subrep wher $V'_\infty = 0$ means $V = 0$. So the picture is a dual partition picture where 1 cogenerates, rather than generates. So you flip a generator to a cogenerator.

*Remark.* These are the two primary examples. Some other nice examples: an $\vec{A}_n$ quiver with appropriate framing will give you a flag variety.

*Remark.* What is $T\mathcal{M}$ in K-theory? Since $\mathcal{M} = [V/G]$, a tangent direction in the quotient looks as follows. Take a $G$-orbit $Gv$. A tangent vector is a vector perpendicular to the orbit. I.e. $T[V/G] = TV - TG$, approximately. The tangent bundle to a vector space is the vector space itself.

So, $TM = T^\star Q - (1 + \hbar) \oplus_i Ext(V_i, V_i)$. Further, $T^\star Q = Q + \hbar Q^\vee$.

Since $M$ is still symplectic, this tangent bundle must split as $T^{1/2} + \hbar T^{1/2,\vee}$ (true for every symplectic manifold in K-theory). We will this splitting a **polarisation**. There are many ways to make this choice, but there is a natural way to do so for Nakajima quiver varieties, corresponding to the tangent bundle of $\vec{Q}$ inside $TT^\star \vec{Q}$.

## 12 (Mar 02) Vertex models

Let's say we take equivariant K-theory

$$K_{\mathrm{eq}}(X_n) \quad X_n \coloneqq \bigsqcup_{k=0}^{n} T^*\operatorname{Gr}(k,n).$$

This is a module for the quantum group $U_q(\hat{\mathfrak{gl}}(2))$, where we think of $\mathfrak{gl}(2)$ as $2\times 2$ matrices in $\mathbb{C}[u^{\pm}]$, or, equivalently, maps $\mathbb{C}^\times \to \mathfrak{gl}(2)$. Commutators are taken "point-wise" in this latter picture. To be more precise, there is a torus $T = A \times \mathbb{C}^\times_\hbar$ where the $\mathbb{C}^\times_\hbar$ scales the cotangent direction in $T^*\operatorname{Gr}$ and $A \subset \operatorname{GL}(n)$ is the Cartan.

We have discussed this setup for a while now, with the conclusion that the key ingredient for constructing such a quantum group action is an R-matrix. Namely, the equivariant K-theory will be a deformation of the representation

$$\bigotimes_{i=0}^{n} \mathbb{C}^2(a_i),$$

where $\mathbb{C}^2(a_i)$ is the evaluation representation at $u = a_i$, and because quantum groups are not cocommutative, the non-trivial isomorphism

$$R_{\mathbb{C}^2,\mathbb{C}^2}(a_1/a_2)\colon \mathbb{C}^2(a_1)\otimes\mathbb{C}^2(a_2) \to \mathbb{C}^2(a_1)\otimes_{\mathrm{opp}}\mathbb{C}^2(a_2)$$

is the data we need. For example, when $n=2$, we get

$$X_2 = \mathrm{pt} \sqcup T^*\mathbb{P}^1 \sqcup \mathrm{pt},$$

and therefore, as a matrix,

$$R = \begin{pmatrix} 1 & & & \\ & ? & ? & \\ & ? & ? & \\ & & & 1 \end{pmatrix}.$$

The interesting $2\times 2$ block, corresponding to $T^*\mathbb{P}^1$, turns out to be

$$\begin{pmatrix} \hbar^{1/2}\frac{1-u}{\hbar-u} & u\frac{1-\hbar}{u-\hbar} \\ \frac{1-\hbar}{u-\hbar} & \hbar^{1/2}\frac{1-u}{\hbar-u} \end{pmatrix}.$$

This $2\times 2$ matrix originated far earlier than quantum groups, in other areas of mathematics. Note that if $\hbar = 1$, then the isomorphism is trivial and the matrix is the identity matrix. Note also that $\hat{\mathfrak{gl}}(2)$ is a quantum *loop* group, one can also plug in things like $u = 0$ or $u = \infty$. (All other points in $\mathbb{C}^\times$ are undistinguished, because of $\operatorname{Aut}(\mathbb{C}^\times)$.) For example, at $u = 0$ we get

$$\begin{pmatrix} \hbar^{-1/2} & 0 \\ 1-\hbar^{-1} & \hbar^{-1/2} \end{pmatrix}.$$

This is a valid R-matrix. But it is a very different animal from the R-matrix we care about. One reason is the unitarity

$$R_{21}(u^{-1})R_{12}(u) = 1,$$

which yields something like the second Reidemeister move. On the other hand, the degenerate version of the R-matrix really yields something like a braid group.

The first time this R-matrix appeared, its entries were weights in the *six-vertex model*. This is historically one of the first examples of an *exactly solvable* (Euclidean) model in statistical field theory. We begin by discretizing space, as a lattice $\mathbb{Z}^2$. Quantum field theories are theories of fluctuating "stuff", so what fluctuates on a lattice? One can try to assign degrees of freedom to the *vertices* of the lattice. This yields models like the Ising model, for ferromagnetism, where the degrees of freedom are just vectors which point either up or down at each vertex. This is modeled by a map $\sigma\colon \mathbb{Z}^2 \to S^0 = \{\pm 1\}$. Fluctuations are described probabilistically. The basic principle in statistical physics is that a system is *in thermal equilibrium* if

$$\mathrm{Prob}(\text{configuration}) \propto \exp(-E/T) \tag{1}$$

where $E$ is energy and $T$ is temperature. For the Ising model,

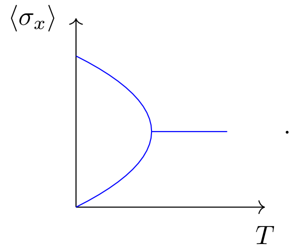
$$E := -\sum_{x,y \text{ neighbors}} \sigma(x)\sigma(y).$$

A system out of equilibrium tries to resample itself, conserving energies, so that it goes back into equilibrium. For the Ising model, spins flip based on the energy difference of being up or down. This yields a measure on the configuration space of maps $\sigma\colon \mathbb{Z}^2 \to S^0$. An equilibrium state, or *Gibbs state*, is such a measure with the property that once spins are fixed along a given contour, the spins inside are fixed by the formula (1).

**Theorem 12.1** (Onsager)**.** *The moduli of such measures is:*

- *one point, above a critical temperature $T_c$;*
- *two points, below the critical temperature $T_c$.*

The two points, at temperatures $T < T_c$, correspond to states of predominantly spins $+1$, or states of predominantly spins $-1$. (These are *magnetized* states.) Concretely, we measure this by the observable $\langle \sigma_x \rangle$. One can think of this as the graph




Onsager's main achievement was the exact diagonalization of the transfer matrix. Working on an infinite domain is difficult, so we usually take a finite volume and then take the limit. It is easiest to impose periodic boundary conditions, so that the lattice now forms a cylinder. The transfer matrix is very useful tool in this situation. Write the sum over all states as the partition function $Z$. The transfer matrix is a map

$$T\colon \mathbb{C}^{2^\ell} \to \mathbb{C}^{2^\ell}$$

where $2^\ell$ represents the states along one row. It tells us how to "transfer" spins from one row to the next. Then

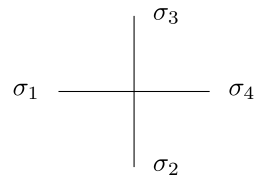
$$Z = \langle T^N v_0, v_N \rangle$$

for a cylinder of height $N$. (One can, of course, insert more operators than just $T$.) Onsager diagonalized this matrix $T$ in a way which is very dear to representation theorists: $\mathbb{C}^{2^\ell} \to \mathbb{C}^{2^\ell}$ is the spin representation, and diagonalizing $T$ means to find eigenvalues/eigenvectors in this representation.

People started looking for generalizations of this miracle. A class of models called *vertex models* were found (by Baxter?) that could also be exactly solved, in a similar but slightly different flavor. The fields are defined not on the vertices, but rather on the *edges* of the lattice:

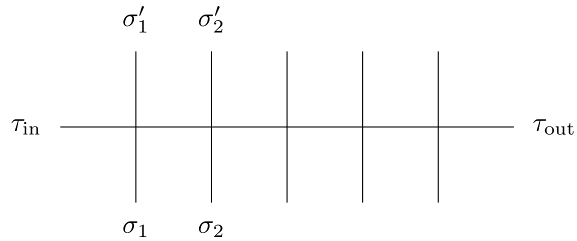
$$\sigma\colon (\text{edges in } \mathbb{Z}^2) \to (\text{target}).$$

So interactions now happen at *vertices*; corresponding to every vertex is some interaction term. If the four states, one for each edge at a vertex, are $\sigma_1, \sigma_2, \sigma_3, \sigma_4$, then one can write down the interaction term




$$= R^{\sigma_3\sigma_4}_{\sigma_1\sigma_2} = \exp(-E/T).$$

The transfer matrix in this language is then a composition of these $R$'s, in the form




If we think of $R$ as an operator $\mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2$, then the transfer matrix as drawn here is an operator

$$\mathbb{C}^2 \otimes \mathbb{C}^{2^L} \to \mathbb{C}^2 \otimes \mathbb{C}^{2^L}$$

where we view $\mathbb{C}^2$ as an auxiliary space and $\mathbb{C}^{2^L}$ as the physical space.

Baxter looked for an algebraic trick to diagonalize this transfer matrix. It turned out that the crucial equation that $T$ has to satisfy is the Yang–Baxter equation. In addition, we have the freedom to assign a variable $a_i$ to each edge in an interaction, in which case the R-matrix becomes a function of the form $R(a_1/a_2)$. Then the Yang–Baxter equation and unitarity are upgraded to involve these $a_i$, e.g.

$$R_{21}(a_2/a_1)R(a_1/a_2) = 1.$$

Then, formally, we define $T(u)$ by "closing" the auxiliary direction (whose variable is $u$) into a loop:

$$T(u) = \operatorname{tr}_{\mathbb{C}^2} R_{\mathbb{C}^2(u), \bigotimes \mathbb{C}^2(a_i)} \in \mathsf{Mat}(2^L).$$

Its matrix coefficients are functions of $u, a_1, \ldots, a_L$ where $a_i$ are the variables associated to the remaining edges.

**Proposition 12.2** (Baxter)**.** *In this situation,*

$$T(u)T(u') = T(u')T(u).$$

This is a strong constraint on eigenvalues of $T$. In fact, one can insert into the $\operatorname{tr}_{\mathbb{C}^2}$ any operator $z$ that commutes with the R-matrix, and this commutativity will *still* hold for the resulting operators $T_z(u)$.

*Proof.* Take two such transfer matrices. Create a little overlap using unitarity, and then use the Yang–Baxter equation on the resulting triangles inductively. □

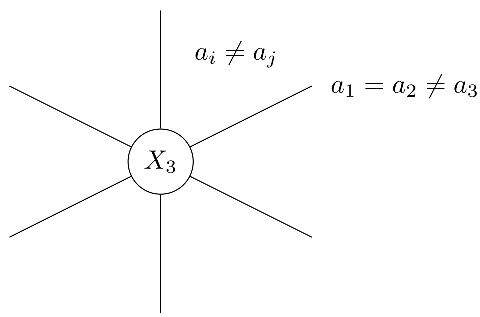
Consequently, the $T_z(u)$ form a commutative sub-algebra in $U_q(\hat{\mathfrak{gl}}(2))$, parameterized by $z \in \overline{\mathbb{C}^\times}$. But remember that $U_q(\hat{\mathfrak{gl}}(2))$ is supposed to act on $K_T(T^* \operatorname{Gr})$. The basic fact is that at $z = 0$, these are operators of tensor product by tautological bundles. For general $z \neq 0$, these are operators of *quantum* product.

# 13 (Mar 04) Stable envelopes

Let $X_n \coloneqq \bigsqcup_{k=0}^n T^* \operatorname{Gr}(k,n)$. At some point it will be important that $X_n$ is symplectic, with a group action which scales the symplectic form $\omega$. Namely, $\operatorname{Aut}(X)$ acts *non-trivially* on $\omega$. We can look at the group $\operatorname{Aut}(X,\omega)$ preserving $\omega$, and inside it will be a torus

$$A = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \subset \operatorname{GL}(n) \subset \operatorname{Aut}(X,\omega).$$

What are interesting fixed loci of $X_n$ under the $A$-action? The identity element in $A$ clearly fixes all of $X_n$. A generic element in $A$ fixes $X_1 \times X_1 \times \cdots \times X_1$, corresponding to asking which eigenvectors have common eigenvalues in $\mathbb{C}^k \subset \mathbb{C}^n$. For $X_3$, the picture to imagine is



.

We would like a correspondence which takes us between these interesting fixed loci, e.g. between $X_1 \times X_1 \times X_1$ and $X_3$ vs $X_2 \times X_1$. Note that the one-parameter subgroup

$$\operatorname{diag}(\underbrace{a, a, \dots, a}_{n_1}, \underbrace{1, 1, \dots, 1}_{n_2})$$

has fixed locus exactly $X_{n_1} \times X_{n_2}$. At $a = 0$, the fixed locus expands to $X_n$ where $n = n_1 + n_2$. Approaching from the two different sides $\pm$ of $a = 0$ yields two *different* morphisms

$$\operatorname{Stab}_\pm \colon K_T(X_{n_1} \times X_{n_2}) \to K_T(X_n)$$

called **stable envelopes**. The R-matrix will be the *ratio* of these two morphisms

$$R = (\operatorname{Stab}_-)^{-1} \operatorname{Stab}_+,$$

and from it we will recover the entire quantum group acting on $K_T(X_n)$. The R-matrix will depend on all equivariant parameters, but the variable $a$ acts trivially on the fixed loci, and so in particular there is a special dependence on $a$ which we notate as $R(a)$. Recovering the quantum group in this way makes all these maps $\operatorname{Stab}_\pm$ intertwiners for the quantum group action. In particular, they will look like permutations of factors in

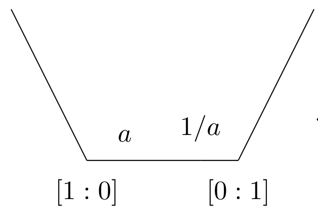
$$\mathbb{C}^2(a_1) \otimes \mathbb{C}^2(a_2) \otimes \cdots \otimes \mathbb{C}^2(a_n).$$

In this construction of the R-matrix, the Yang–Baxter equation holds automatically once we show that each "triangle" commutes in the picture above. We will prove this later. The chambers in the picture above can be explicitly described by inequalities like $a_1 \gg a_2 \gg a_3$. This means that $a_2/a_1 \to 0$ and $a_3/a_2 \to 0$ in any limit to infinity inside the chamber.

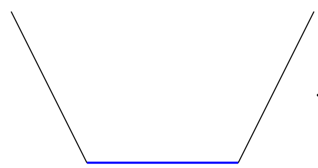
Take $X^A \subset X$. This is the locus fixed by *everything*. Its normal bundle $N_{X/X^A}$ carries an $A$-action, and the $a_i/a_j$ are weights that occur in it. There is jumping behavior (walls in the diagram) whenever $a_i/a_j = 1$ for some $i, j$, which implies that there is an *extra* fixed direction for that sub-torus. Away from the walls, elements $a$ in the interior of each chamber have $X^a = X^A$, and for every $w \in N_{X/X^A}$ either $w \to 0$ or $w \to \infty$. None can stay finite. Hence we can define the **attracting** manifold

$$\mathrm{Attr} := \{(x, f) : \lim_{a \to \infty_{\mathcal{C}}} a \cdot x = f\} \subset X \times X^A.$$

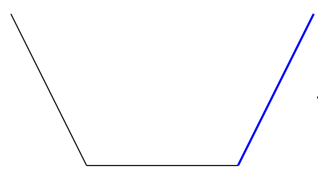
**Example 13.1.** Take $X_2 = \mathrm{pt} \sqcup T^*\mathbb{P}^1 \sqcup \mathrm{pt}$ and focus on $T^*\mathbb{P}^1$. Let $\mathrm{diag}(1, a)$ act on it, so that its toric picture is




Then the attracting manifold of $[1 : 0]$ is



and the attracting manifold of $[0 : 1]$ is



Importantly, the attracting manifold of $[1 : 0]$ is *not* a closed subset. We don't like correspondences supported on non-closed subsets, because then we pushforward and something goes bad. Of course, one can take a closure, but that yields something which is not well-behaved in families. Instead, we take the *transitive* closure: take the closure, and then take the attracting set of the closure, etc. We call this the **full attracting set**

$$\mathrm{Attr}^f \supset \mathrm{Attr}\,.$$

More precisely, $T^*\mathbb{P}^1$ has a map to the nilcone $\mathcal{N} \subset \mathfrak{sl}_2$, and the nilcone can be deformed by smoothing the conical singularity. If we look at $\mathrm{Attr}([1 : 0])$ in this family, its closure is *not* the closure fiber-wise.

Hence we would like the correspondence on $X \times X^A$ to be supported on $\mathrm{Attr}^f$. One can pick the inclusion map $X^A \hookrightarrow X$ for the correspondence, but it is not interesting because it does not take into account the chamber structure.

**Example 13.2.** Let's return to $T^*\mathbb{P}^1$. We want two different maps

$$X_1 \times X_1 \to X_2 = \mathrm{pt} \sqcup T^*\mathbb{P}^1 \sqcup \mathrm{pt}$$

whose ratio yields the non-trivial $2 \times 2$ block we wrote down last class:

$$R = \begin{pmatrix} 1 & & & \\ & ? & ? & \\ & ? & ? & \\ & & & 1 \end{pmatrix}.$$

The first attempt is to take the structure sheaf $\mathcal{O}_{\mathrm{Attr}^f}$ as the correspondence. This will unfortunately fail for a minor reason: when $\hbar = 1$, the quantum group is co-commutative, and therefore when $\hbar = 1$ the $2 \times 2$ block must be the identity map. In other words, these two different maps must be the same at $\hbar = 1$. But

$$\mathcal{O}_{\mathrm{Attr}^f_+([1:0])}|_{[1:0]} = \prod_{w \in N_{\mathrm{Attr}^f/X}} (1 - w) = 1 - a,$$

whereas in the other chamber we have

$$\mathcal{O}_{\mathrm{Attr}^f_-([1:0])}|_{[1:0]} = \prod_{w \in N_{\mathrm{Attr}^f/X}} (1 - w) = 1 - \frac{1}{\hbar a}.$$

These are close, but not the same at $\hbar = 1$.

Very generally, a symplectic manifold will have weights which come in dual pairs $w_i$ and $1/(\hbar w_i)$, and restrictions of $\mathcal{O}$ will always be of the form

$$\prod_{i=1}^n (1 - w_i^{\pm})$$

in any chamber. Changing a chamber will only change this up to a sign and a monomial. This observation necessitates that we choose a "half" of all the weights beforehand, with respect to which we do certain constructions. In K-theory, this is a **polarization**

$$T^{1/2}X \in K_T(X)$$

such that

$$TX = T^{1/2}X + \hbar^{-1}(T^{1/2}X)^\vee.$$

Note that $T^{1/2}X$ is not necessarily an actual vector bundle; it is a *virtual* bundle in general. Locally, the picture is to take $\pi \colon T^*M \to M$ and observe that

$$T(T^*M) = \pi^*TM + \hbar^{-1}(\pi^*TM)^\vee.$$

If $F \subset X^A$ is a fixed component, we would like to normalize the stable envelopes Stab such that

$$\mathrm{Stab}_F = (\text{line bundle})\mathcal{O}_{\mathrm{Attr}(F)}$$

since we have already seen it cannot be just $\mathcal{O}_{\mathrm{Attr}(F)}$ on its own. Let $w_i$ be the attracting weights at $F$; they are just Chern roots of $N_{F/X}^{>0}$. Then

$$\mathcal{O}_{\mathrm{Attr}_+(F)}|_F = \prod (1 - \hbar w_i), \quad \mathcal{O}_{\mathrm{Attr}_-(F)}|_F = \prod (1 - w_i^{-1}),$$

and so we can use the line bundle freedom to get a compromise between these two:

$$(\text{line bundle}) = (-1)^{\mathrm{rank}\, T^{1/2}_{>0}} \left( \frac{\det N_{<0}}{\det N^{1/2}} \right)^{1/2}.$$

This expression, in some sense, "symmetrizes" the products for $\mathcal{O}_{\mathrm{Attr}}|_F$:

$$(1 - \hbar w_i) \rightsquigarrow ((\hbar w_i)^{1/2} - (\hbar w_i)^{-1/2}), \quad (1 - w_i^{-1}) \rightsquigarrow (w_i^{1/2} - w_i^{-1/2}),$$

and then the specialization $\hbar = 1$ makes these two agree.

Square roots are common in K-theory, and in general some argument must be made to show they exist. In this case, note that the normal bundle can be written in two different ways:

$$\begin{aligned} N &= N_{<0} + \hbar^{-1} N_{>0}^\vee \\ &= N_{1/2} + \hbar^{-1} N_{1/2}^\vee. \end{aligned}$$

Then $\delta N := N_{<0} - N_{1/2}$ has

$$\delta N^\vee = -\hbar^{-1} \delta N.$$

It follows that

$$\det(\delta N) = \frac{\det N_{<0}}{\det N_{1/2}} = \prod \frac{1}{w_i} \cdot \frac{1}{\hbar w_i}.$$

Hence, at the price of allowing square roots of $\hbar$, the square root exists.

## 14 No more notes (classes moved online)