Wall-crossing for invariants of equivariant 3-Calabi–Yau categories

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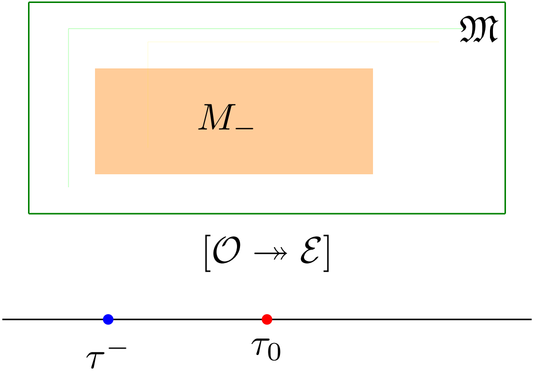

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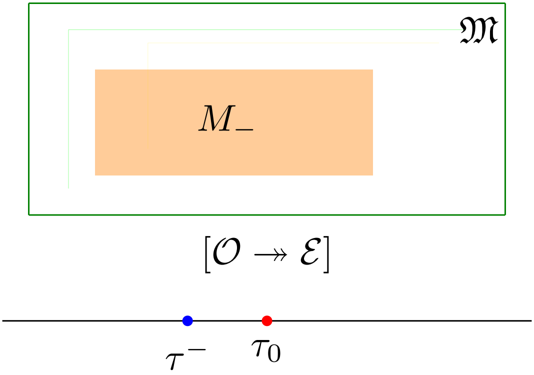

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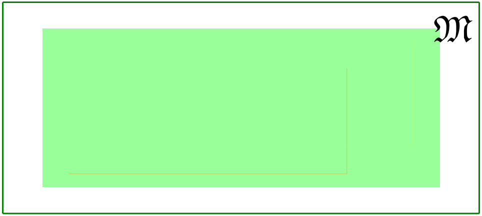
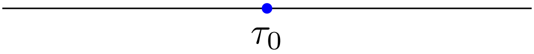


$$[\mathcal{O} \to \mathcal{E}] \oplus [0 \to \mathcal{Q}_1] \oplus \cdots \oplus [0 \to \mathcal{Q}_k]$$



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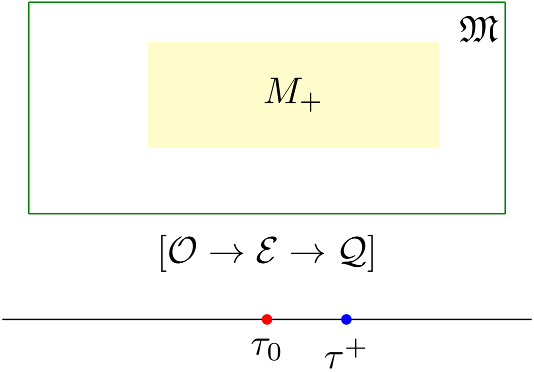

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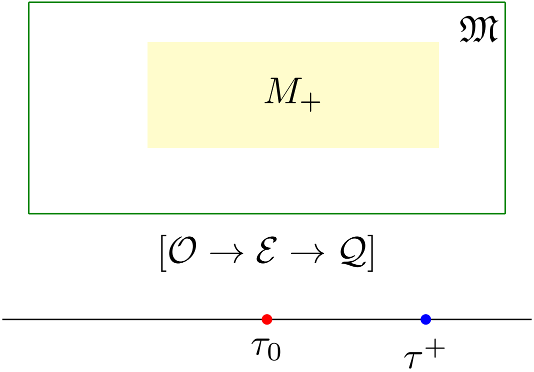

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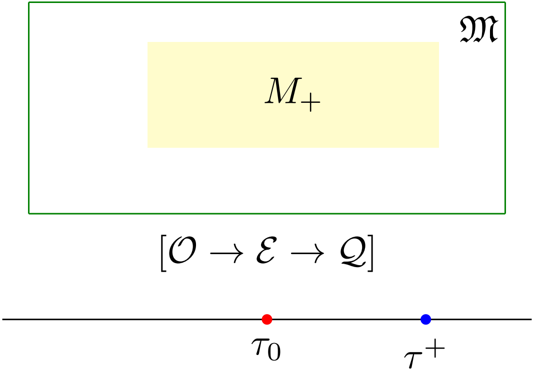

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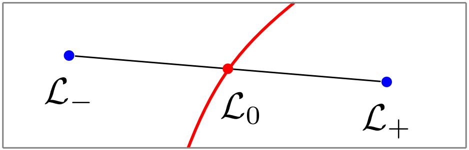
For smooth projective GIT quotients $M_\pm := X /\!\!/_{\mathcal{L}_\pm} \mathbb{C}^\times \subset [X/\mathbb{C}^\times]$,



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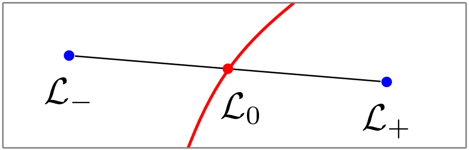

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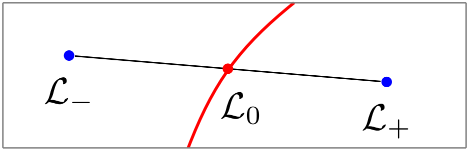

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$$\int_{M_+} \alpha - \int_{M_-} \alpha = \mathrm{Res}_{u=0} \int_{X_0} \frac{\iota^* \alpha}{\mathrm{euler}(\mathcal{N}_\iota)}, \quad \alpha \in H^*_{\mathbb{C}^\times}(X),$$

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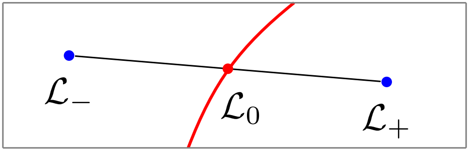

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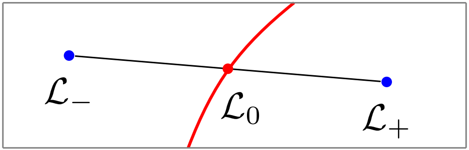

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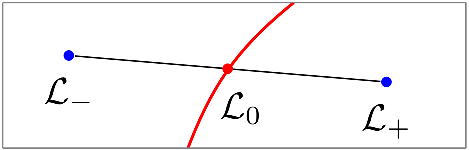

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- $\mathrm{Res}_{u=0}$ is the residue, i.e. u^{-1} coefficient.

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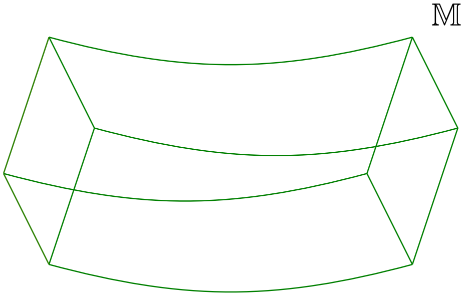

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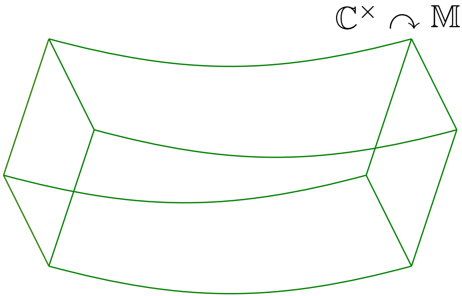

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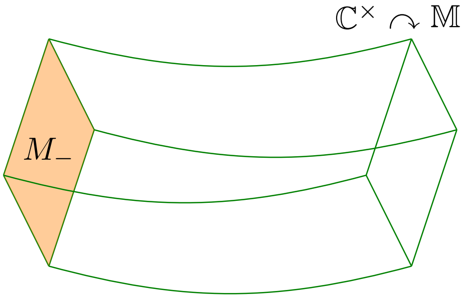

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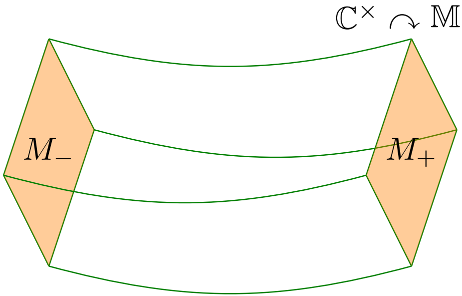

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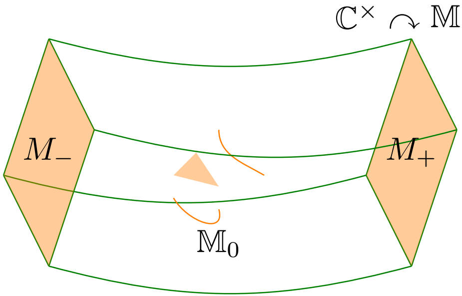

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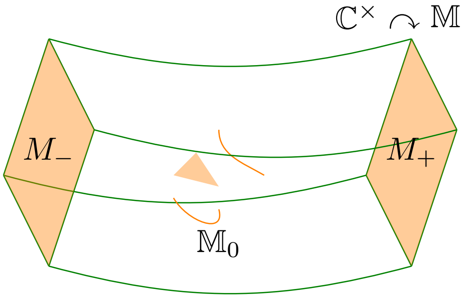

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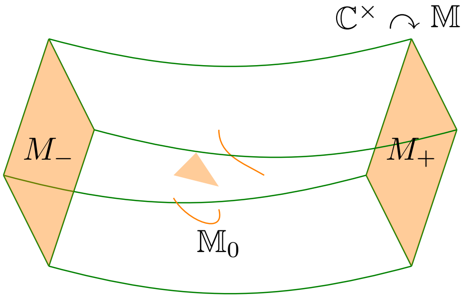

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Example (For the GIT quotients $M_\pm = X /\!\!/_{\mathcal{L}_\pm} G$)

Take $\mathbb{M} := \mathbb{P}_X(\mathcal{L}_- \oplus \mathcal{L}_+) /\!\!/_{\mathcal{O}(1)} G$, with $\mathbb{C}^\times$ scaling the $\mathbb{P}^1$ fibers.

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$$\text{(difficult!)} \rightarrow \quad \mathrm{Res}_{u=0} \int_{\mathbb{M}_0} \frac{\iota^* \alpha}{\mathrm{euler}(\mathcal{N}_\iota)}.$$

Virtual enumerative invariants

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Virtual enumerative invariants

Example ((Thomas '98), (Maulik–Okounkov–Nekrasov–Pandharipande '03), ...)

Let X be a smooth quasi-projective Calabi–Yau 3-fold, acted on by a torus T, with proper T-fixed loci.

- Abelian category: $\mathcal{C}oh(X)$.
- Stability condition: Gieseker.
- "Counting" mechanism: integration of virtual cycle (Behrend–Fantechi '96), (Nekrasov–Okounkov '15).

The result is the equivariant K-theoretic Donaldson–Thomas (DT) invariant

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Example

$X = \text{tot}(\mathcal{K}_S)$ for a surface S, and $\mathsf{T} := \mathbb{C}^\times$ scales the fibers.

The main theorem

Theorem (Kuhn–L.–Thimm (coming soon!))

Suppose that $\mathfrak{M}$ admits a κ-symmetric bilinear obstruction theory, e.g. like $\mathrm{Ext}_X(E, E)^\vee[1]$ *on* $\mathcal{Coh}(X)$.

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1. *Given any stability condition τ, there exists a unique collection $(\mathsf{z}_\alpha(\tau))_\alpha$ of operational semistable invariants such that:*
   - *if semistable=stable, $\mathsf{z}_\alpha(\tau) = \chi\left(\widehat{\mathcal{O}}^{\mathrm{vir}}_{\mathfrak{M}^{\mathrm{sst}}_\alpha(\tau)} \otimes -\right)$;*
   - *for an auxiliary operational invariant $\widetilde{\mathsf{Z}}_{\alpha,1}(\widetilde{\tau})$ counting Joyce–Song-style stable pairs,*

$$\widetilde{\mathsf{Z}}_{\alpha,1}(\widetilde{\tau}) = \sum_{\substack{n>0 \\ \alpha=\alpha_1+\cdots+\alpha_n \\ \forall i:\, \tau(\alpha_i)=\tau(\alpha)}} \frac{1}{n!}\left[\mathsf{z}_{\alpha_n}(\tau), \left[\cdots, \left[\mathsf{z}_{\alpha_2}(\tau), \left[\mathsf{z}_{\alpha_1}(\tau), \partial\right]\right]\cdots\right]\right].$$

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2. *If τ_0 is a stability condition dominating τ, there is a wall-crossing formula*

$$\mathsf{z}_\alpha(\tau) = \sum_{\substack{n>0 \\ \alpha=\alpha_1+\cdots+\alpha_n \\ \forall i\colon \tau_0(\alpha_i)=\tau_0(\alpha)}} \widetilde{U}(\alpha_1,\ldots,\alpha_n;\tau_0,\tau)\cdot [[\cdots[\mathsf{z}_{\alpha_1}(\tau_0),\mathsf{z}_{\alpha_2}(\tau_0)],\cdots],\mathsf{z}_{\alpha_n}(\tau_0)]$$

for some universal coefficients $\widetilde{U}(\alpha_1,\ldots,\alpha_n;\tau_0,\tau)\in\mathbb{Q}$.

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Roughly, means τ_0 is “on the wall” and τ is next to it.

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In this setting, the "wall-crossing term" in class α can be expressed as a bilinear operation $[-,-]$ on operational invariants in classes α_1 and α_2 (with $\alpha = \alpha_1 + \alpha_2$ and $\tau_0(\alpha_1) = \tau_0(\alpha_2)$).

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Remarkable observation (L. '22)

$[-,-]$ is a Lie bracket.

In fact, before taking K-theoretic Res*, it is the vertex product*

$$Y(-,u)\cdot -$$

for an equivariant multiplicative vertex algebra!

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Example (Rigidity)

For operational invariants ϕ_α and ψ_β in classes α and β,

$$[\phi_\alpha, \psi_\beta](\mathcal{O}) = [\chi(\alpha, \beta)]_\kappa \cdot \phi_\alpha(\mathcal{O}) \cdot \psi_\beta(\mathcal{O})$$

where $[n]_\kappa$ is the quantum integer

$$[n]_\kappa := (-1)^{n-1} \frac{\kappa^{\frac{n}{2}} - \kappa^{-\frac{n}{2}}}{\kappa^{\frac{1}{2}} - \kappa^{-\frac{1}{2}}} \in \mathbb{Z}[\kappa^{\pm\frac{1}{2}}].$$

The main theorem

This theorem is the (equivariant, K-theoretic) 3CY version of:

Enumerative invariants and wall-crossing formulae in abelian categories

Dominic Joyce

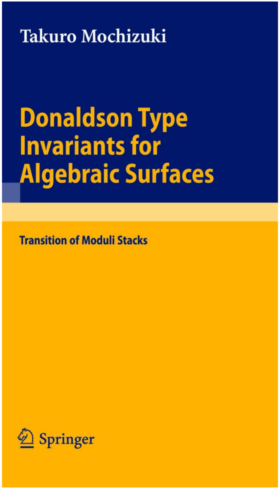
Preliminary version. Comments welcome

Abstract

An *enumerative invariant theory* in Algebraic Geometry is the study of invariants which 'count' τ-(semi)stable objects E with fixed topological invariants $[\![E]\!]=\alpha$ in some geometric problem, by means of a *virtual class* $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}$ in some homology theory, for the moduli spaces $\mathcal{M}_\alpha^{\rm st}(\tau)\subseteq\mathcal{M}_\alpha^{\rm ss}(\tau)$ of τ-(semi)stable objects. We can obtain numbers by taking integrals $\int_{[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}}\Upsilon$ for suitable universal cohomology classes Υ.

Examples include Mochizuki's invariants for coherent sheaves on surfaces [146], and Donaldson–Thomas type invariants for coherent sheaves on Calabi–Yau 3- and 4-folds and Fano 3-folds, [28, 108, 118, 152, 176].

Let $\mathcal{A}$ be a $\mathbb{C}$-linear abelian category coming from Algebraic Geometry. There are two moduli stacks of objects E in $\mathcal{A}$: the usual moduli stack $\mathcal{M}$, and the 'projective linear' moduli stack $\mathcal{M}^{\rm pl}$ modulo projective linear isomorphisms, that is, we quotient out by $\lambda\,{\rm id}_E : E\to E$ for $\lambda\in\mathbb{G}_m$. Both are Artin $\mathbb{C}$-stacks. Previous work by the author [106] gives $H_*(\mathcal{M})$ the structure of a *graded vertex algebra*, and $H_*(\mathcal{M}^{\rm pl})$ a *graded Lie algebra*, closely related to $H_*(\mathcal{M})$. Virtual



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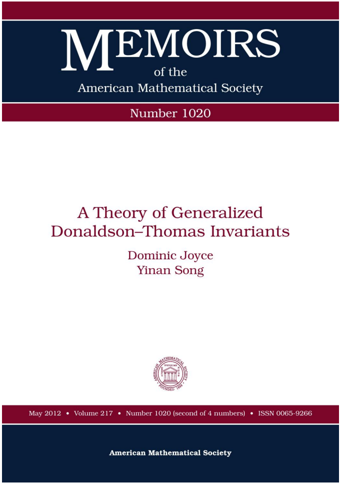
This theorem refines the wall-crossing formulas of:



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A Theory of Generalized
Donaldson–Thomas Invariants

Dominic Joyce
Yinan Song

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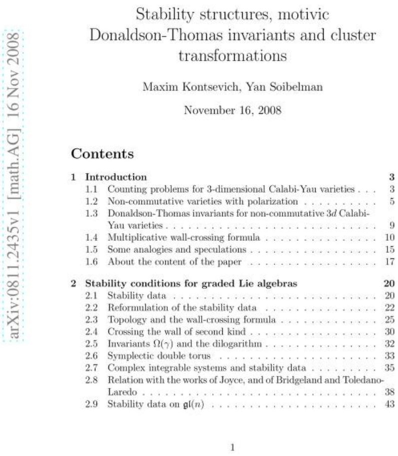

arXiv:0811.2435v1 [math.AG] 16 Nov 2008

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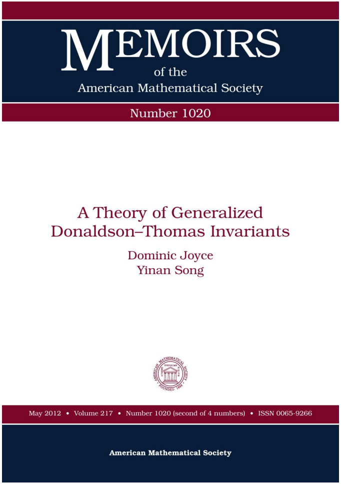

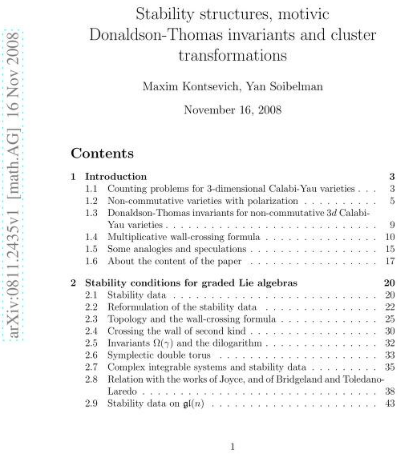

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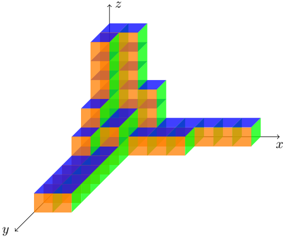
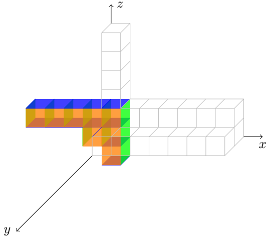


plane partitions with legs



PT configurations with legs

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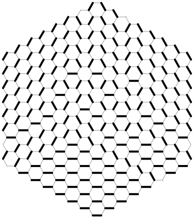

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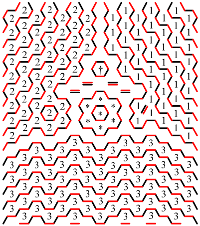

labeled double-dimer configurations

(Taken from [Jenne–Webb–Young '22])

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$$\mathcal{I}_C = [\mathcal{O}_X \twoheadrightarrow \mathcal{O}_C] \qquad [\mathcal{O}_X \to \mathcal{F}]$$

ideal sheaves PT stable pairs

($X = (\mathbb{P}^1)^3$, but objects have *prescribed restriction* λ, μ, ν to the hyperplanes at infinity.)

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Related to Gromov–Witten theory and refines the topological vertex.

[MNOP '03, MOOP '06]
[Pandharipande–Pixton '12]
many, many others...

Application: DT/PT vertex correspondence

This is an equivariant (K-theoretic) wall-crossing statement for objects on the non-compact Calabi–Yau 3-fold $\mathbb{C}^3$.

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Previous work [Toda '09, '16] [Bridgeland '10] handled only the simpler non-equivariant (cohomological) case for compact CY3's, using [Joyce–Song '08].

Application: DT/PT vertex correspondence

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In equivariant cohomology, here is $V^{PT}_{(2),(1),(1)}(1)$:

$$Q^{-3}\left(\frac{z}{x+y}\right)$$

$$+Q^{-2}\left(\frac{3\,x^3y + 5\,x^2y^2 - 2\,xy^3 + x^3z + 9\,x^2yz + 11\,xy^2z + 2\,x^2z^2 + 10\,xyz^2 + 4\,y^2z^2 + xz^3 + 4\,yz^3}{(x+2\,y)(x-y+z)xy}\right)$$

$$+Q^{-1}\Big(10\,x^7y^3 + 33\,x^6y^4 + 23\,x^5y^5 - 11\,x^4y^6 - 9\,x^3y^7 + 2\,x^2y^8 + 26\,x^7y^2z + 134\,x^6y^3z + 238\,x^5y^4z + 166\,x^4y^5z + 6\,x^3y^6z - 30\,x^2y^7z + 14\,x^7yz^2 + 154\,x^6y^2z$$

$$+Q^{0}\Big(42\,x^{10}y^3 + 169\,x^9y^4 + 185\,x^8y^5 - 16\,x^7y^6 - 100\,x^6y^7 - 7\,x^5y^8 + 17\,x^4y^9 - 2\,x^3y^{10} + 126\,x^{10}y^2z + 764\,x^9y^3z + 1798\,x^8y^4z + 1959\,x^7y^5z + 614\,x^6y^6z$$

$$+O(Q)$$

Application: DT/PT vertex correspondence

To emphasize, vertices are very explicit objects!

In non-equivariant cohomology, here is $\mathsf{V}^{\mathsf{PT}}_{(2),(1),(1)}(1)$:

$$-Q^{-3} + Q^{-2} - 4Q^{-1} + 7 + O(Q).$$

(i.e. in the Calabi–Yau limit $x + y + z \mapsto 0$)

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- Equivariant vertices, particularly in K-theory, are genuinely *much* more sophisticated.

Application: DT/PT vertex correspondence

Conjecture (Hagborg–Oblomkov '19)

There is an equality of descendent vertices

$$\mathsf{V}^{\mathsf{DT}}_{\lambda,\mu,\nu}(\widetilde{\tau}(\mathsf{p})) \equiv \mathsf{V}^{\mathsf{PT}}_{\lambda,\mu,\nu}(\widetilde{\tau}(\mathsf{p})) \cdot \mathsf{V}^{\mathsf{DT}}_{\emptyset,\emptyset,\emptyset}(\widetilde{\tau}(\mathsf{p})) \bmod x+y+z.$$

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$$\mathsf{V}^{\mathrm{DT}}_{\lambda,\mu,\nu}(\widetilde{\tau}(\mathsf{p})\widetilde{\tau}(\mathsf{p})) \equiv ?? \bmod x+y+z.$$

nobody knew!

Application: DT/PT vertex correspondence

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$$\mathsf{V}^{\mathrm{DT}}_{\lambda,\mu,\nu}(\widetilde{\tau}^{(1)}(\mathsf{p})) \equiv \mathsf{V}^{\mathrm{PT}}_{\lambda,\mu,\nu}(\widetilde{\tau}^{(1)}(\mathsf{p})) \cdot \mathsf{V}^{\mathrm{DT}}_{\emptyset,\emptyset,\emptyset}(\widetilde{\tau}^{(1)}(\mathsf{p})) \bmod x+y+z.$$

$\widetilde{\tau}^{(k)}(\mathsf{p}) = 1 - \sum_{n>0} k^n \tau_n(\mathsf{p})$ are descendent classes of $\mathsf{p} := 0 \in \mathbb{C}^3$.

$$\mathsf{V}^{\mathrm{DT}}_{\lambda,\mu,\nu}(\widetilde{\tau}^{(k_1)}(\mathsf{p})\widetilde{\tau}^{(k_2)}(\mathsf{p})) \equiv ?? \bmod x+y+z.$$

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Application: DT/PT vertex correspondence

Theorem (Kuhn–L.–Thimm (coming soon!))

There is an equality of (equivariant, K-theoretic) operational invariants

$$\mathsf{V}^{\mathsf{DT}}_{\lambda,\mu,\nu} = \exp\left(\mathrm{ad}_{\mathsf{z}}\right) \mathsf{V}^{\mathsf{PT}}_{\lambda,\mu,\nu}$$

where z *is a generating series of semistable invariants counting points.*

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In particular, z is independent of λ, μ, ν, and $\mathsf{V}^{\mathsf{PT}}_{\emptyset,\emptyset,\emptyset}$ is trivial.

Application: DT/PT vertex correspondence

Corollary (Kuhn–L.–Thimm '23)

The (K-theoretic) primary vertex correspondence holds:

$$\mathsf{V}^{\mathsf{DT}}_{\lambda,\mu,\nu}(\mathcal{O}) = \mathsf{V}^{\mathsf{PT}}_{\lambda,\mu,\nu}(\mathcal{O}) \cdot \mathsf{V}^{\mathsf{DT}}_{\emptyset,\emptyset,\emptyset}(\mathcal{O}).$$

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Proof.

Rigidity. □

Application: DT/PT vertex correspondence

Corollary

For any $N \geq 0$ and $k_1, \dots, k_N \in \mathbb{Z}$,

$$\mathsf{V}^{\mathrm{DT}}_{\lambda,\mu,\nu}\left(\sigma\{k_i\}_{i\in\underline{N}}\right) \equiv \sum_{\substack{n>0 \\ m_1,\dots,m_n>0 \\ S_1\sqcup\cdots\sqcup S_n=\underline{N} \\ \forall i:\, S_i^1\sqcup\cdots\sqcup S_i^{m_i}=S_i}} (-1)^n \cdot \mathsf{V}^{\mathrm{PT}}_{\lambda,\mu,\nu}\Big(\sigma\{\sum_{j\in S_i} k_j\}_{i=1}^n\Big) \cdot \mathsf{V}^{\mathrm{DT}}_{\emptyset,\emptyset,\emptyset}(1) \cdot \prod_{i=1}^n (m_i-1)! \prod_{j=1}^{m_i} \frac{\mathsf{V}^{\mathrm{DT}}_{\emptyset,\emptyset,\emptyset}\left(\sigma\{k_\ell\}_{\ell\in S_i^j}\right)}{-\mathsf{V}^{\mathrm{DT}}_{\emptyset,\emptyset,\emptyset}(1)} \mod x+y+z$$

where $\underline{N} := \{1,\dots,N\}$ and

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Proof.

An explicit computation of the Lie bracket + some non-trivial combinatorics. □

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$\exists$ Many other exciting and low-hanging fruits.

Thank you!