Multiplicative vertex algebras and wall-crossing in equivariant K-theory

Henry Liu

Kavli IPMU

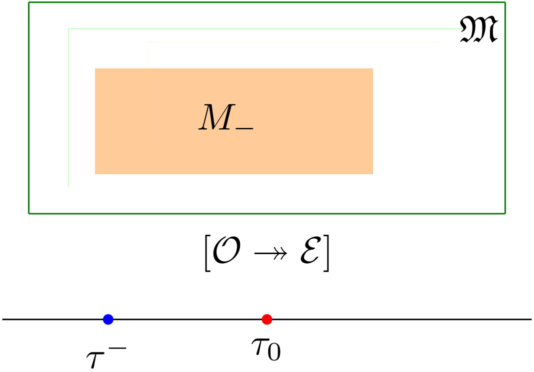
January 23, 2025

What is wall-crossing?

Study of how the geometry of (semi)stable loci in moduli spaces $\mathfrak{M}$ changes upon varying the stability condition.

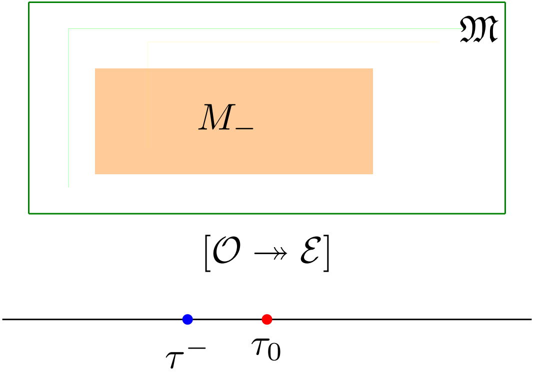
What is wall-crossing?

Study of how the geometry of (semi)stable loci in moduli spaces $\mathfrak{M}$ changes upon varying the stability condition.



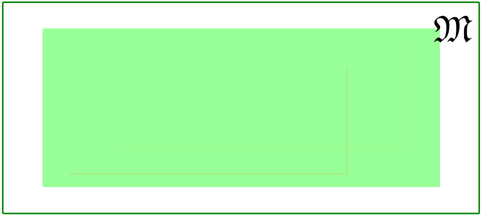
What is wall-crossing?

Study of how the geometry of (semi)stable loci in moduli spaces $\mathfrak{M}$ changes upon varying the stability condition.

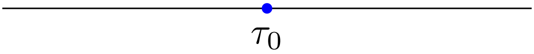


What is wall-crossing?

Study of how the geometry of (semi)stable loci in moduli spaces $\mathfrak{M}$ changes upon varying the stability condition.

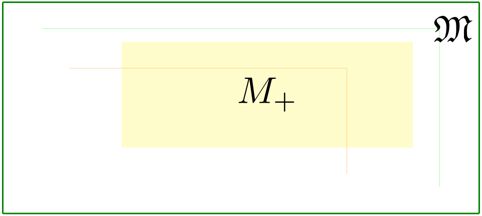


$$[\mathcal{O} \to \mathcal{E}] \oplus [0 \to \mathcal{Q}_1] \oplus \cdots \oplus [0 \to \mathcal{Q}_k]$$

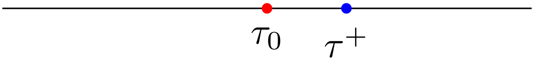


What is wall-crossing?

Study of how the geometry of (semi)stable loci in moduli spaces $\mathfrak{M}$ changes upon varying the stability condition.

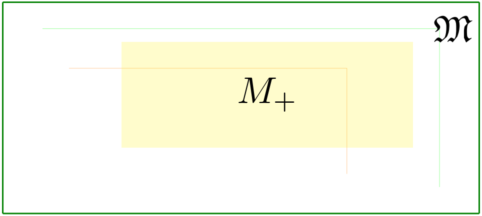


$$[0 \to \mathcal{I}_C \to \mathcal{O} \to \mathcal{E} \to \mathcal{Q} \to 0]$$

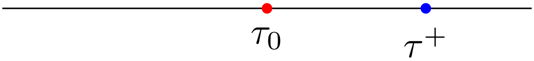


What is wall-crossing?

Study of how the geometry of (semi)stable loci in moduli spaces $\mathfrak{M}$ changes upon varying the stability condition.

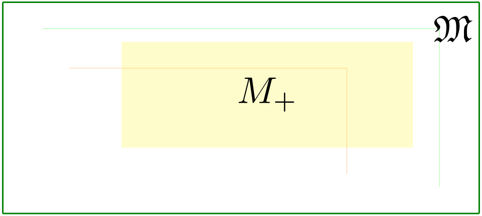


$$[0 \to \mathcal{I}_C \to \mathcal{O} \to \mathcal{E} \to \mathcal{Q} \to 0]$$

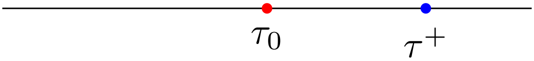


What is wall-crossing?

Study of how enumerative invariants of (semi)stable loci in moduli spaces $\mathfrak{M}$ changes upon varying the stability condition.



$$[0 \to \mathcal{I}_C \to \mathcal{O} \to \mathcal{E} \to \mathcal{Q} \to 0]$$



How do we relate M_- and M_+?

Use a master space $\mathbb{M}$. [Thaddeus

How do we relate M_- and M_+?

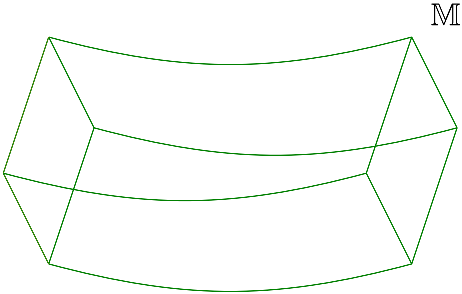
Use a master space $\mathbb{M}$. [Thaddeus, Mochizuki, Nakajima–Yoshioka, Kiem–Li, Joyce

How do we relate M_- and M_+?

Use a master space $\mathbb{M}$. [Thaddeus, Mochizuki, Nakajima–Yoshioka, Kiem–Li, Joyce, Gonzalez–Woodward, Zhou, Nesterov, ...]

How do we relate M_- and M_+?

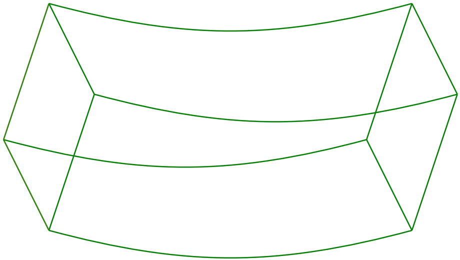
Use a master space $\mathbb{M}$. [Thaddeus, Mochizuki, Nakajima–Yoshioka, Kiem–Li, Joyce, Gonzalez–Woodward, Zhou, Nesterov, ...]



How do we relate M_- and M_+?

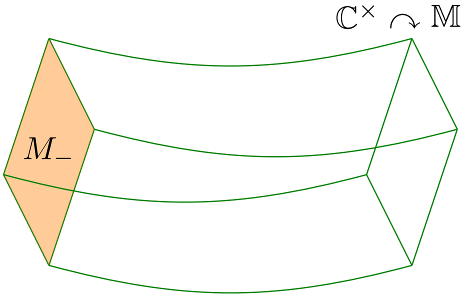
Use a master space $\mathbb{M}$. [Thaddeus, Mochizuki, Nakajima–Yoshioka, Kiem–Li, Joyce, Gonzalez–Woodward, Zhou, Nesterov, ...]

$$\mathbb{C}^\times \curvearrowright \mathbb{M}$$



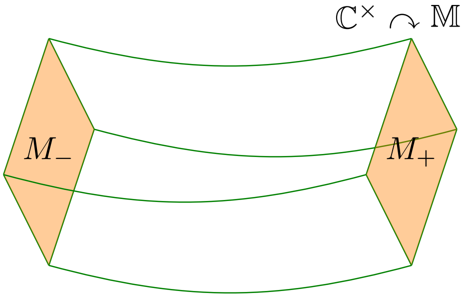
How do we relate M_- and M_+?

Use a master space $\mathbb{M}$. [Thaddeus, Mochizuki, Nakajima–Yoshioka, Kiem–Li, Joyce, Gonzalez–Woodward, Zhou, Nesterov, ...]



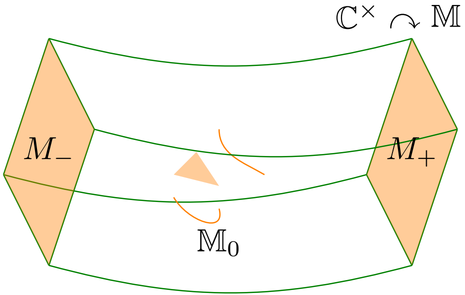
How do we relate M_- and M_+?

Use a master space $\mathbb{M}$. [Thaddeus, Mochizuki, Nakajima–Yoshioka, Kiem–Li, Joyce, Gonzalez–Woodward, Zhou, Nesterov, ...]



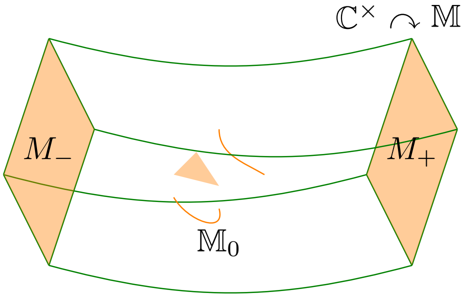
How do we relate M_- and M_+?

Use a master space $\mathbb{M}$. [Thaddeus, Mochizuki, Nakajima–Yoshioka, Kiem–Li, Joyce, Gonzalez–Woodward, Zhou, Nesterov, ...]



How do we relate M_- and M_+?

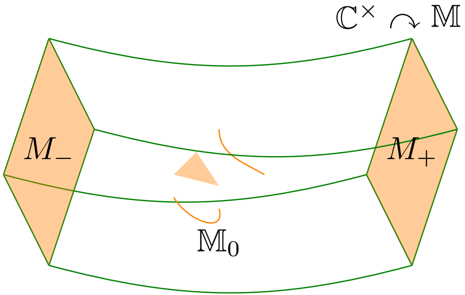
Use a master space $\mathbb{M}$. [Thaddeus, Mochizuki, Nakajima–Yoshioka, Kiem–Li, Joyce, Gonzalez–Woodward, Zhou, Nesterov, ...]



Important: $\mathbb{M}$ must be smooth and compact,
$M_\pm = \{s^\pm = 0\} \subset \mathbb{M}$ must be divisors.

How do we relate M_- and M_+?

Use a master space $\mathbb{M}$. [Thaddeus, Mochizuki, Nakajima–Yoshioka, Kiem–Li, Joyce, Gonzalez–Woodward, Zhou, Nesterov, ...]



Important: $\mathbb{M}$ must be (virtually) smooth and (equivariantly) compact,
$M_\pm = \{s^\pm = 0\} \subset \mathbb{M}$ must be divisors.

How do we relate M_- and M_+?

Example (Thaddeus '94)

For GIT quotients $M_\pm := X /\!\!/_{\mathcal{L}_\pm} G \subset [X/G]$,

How do we relate M_- and M_+?

Example (Thaddeus '94)

For GIT quotients $M_\pm := X \mathbin{/\!\!/}_{\mathcal{L}_\pm} G \subset [X/G]$, the master space is

$$\mathbb{M} := \mathbb{P}(\mathcal{L}_- \oplus \mathcal{L}_+) \mathbin{/\!\!/}_{\mathcal{O}(1)} G,$$

How do we relate M_- and M_+?

Example (Thaddeus '94)

For GIT quotients $M_\pm := X /\!\!/_{\mathcal{L}_\pm} G \subset [X/G]$, the master space is

$$\mathbb{M} := \mathbb{P}(\mathcal{L}_- \oplus \mathcal{L}_+) /\!\!/_{\mathcal{O}(1)} G,$$

with $\mathbb{C}^\times$ acting by scaling the $\mathcal{L}_-$ factor.

How do we relate M_- and M_+?

Example (Thaddeus '94)

For GIT quotients $M_\pm := X /\!\!/_{\mathcal{L}_\pm} G \subset [X/G]$, the master space is

$$\mathbb{M} := \mathbb{P}(\mathcal{L}_- \oplus \mathcal{L}_+) /\!\!/_{\mathcal{O}(1)} G,$$

with $\mathbb{C}^\times$ acting by scaling the $\mathcal{L}_-$ factor.

Exercise

Check that $\mathbb{M}^{\mathbb{C}^\times} = M_- \sqcup M_+ \sqcup \mathbb{M}_0$ *and describe* $\mathbb{M}_0$.

How do we relate M_- and M_+?

Example (Thaddeus '94)

For GIT quotients $M_\pm := X /\!\!/_{\mathcal{L}_\pm} G \subset [X/G]$, the master space is

$$\mathbb{M} := \mathbb{P}(\mathcal{L}_- \oplus \mathcal{L}_+) /\!\!/_{\mathcal{O}(1)} G,$$

with $\mathbb{C}^\times$ acting by scaling the $\mathcal{L}_-$ factor.

Exercise

Check that $\mathbb{M}^{\mathbb{C}^\times} = M_- \sqcup M_+ \sqcup \mathbb{M}_0$ *and describe* $\mathbb{M}_0$.

Exercise

Under what conditions is $\mathbb{M}$ *a smooth scheme?*

How do we relate M_- and M_+?

The master space $\mathbb{M}$ relates enumerative invariants.

How do we relate M_- and M_+?

The master space $\mathbb{M}$ relates enumerative invariants.

1. Apply $\mathbb{C}^\times$-equivariant localization:

$$\int_{\mathbb{M}} \xi = \int_{M_-} \frac{\xi|_{M_-}}{e(\mathcal{N}_{M_-/\mathbb{M}})} + \int_{M_+} \frac{\xi|_{M_+}}{e(\mathcal{N}_{M_+/\mathbb{M}})} + \int_{\mathbb{M}_0} \frac{\xi|_{\mathbb{M}_0}}{e(\mathcal{N}_{\mathbb{M}_0/\mathbb{M}})}$$

How do we relate M_- and M_+?

The master space $\mathbb{M}$ relates enumerative invariants.

1. Apply $\mathbb{C}^\times$-equivariant localization:

$$\int_{\mathbb{M}} \xi = \int_{M_-} \frac{\xi|_{M_-}}{k_- u + \alpha_-} + \int_{M_+} \frac{\xi|_{M_+}}{k_+ u + \alpha_+} + \int_{\mathbb{M}_0} \frac{\xi|_{\mathbb{M}_0}}{e(\mathcal{N}_{\mathbb{M}_0/\mathbb{M}})}$$

where u is the weight of the $\mathbb{C}^\times$, and $k_\pm \in \mathbb{Z}_{\neq 0}$.

How do we relate M_- and M_+?

The master space $\mathbb{M}$ relates enumerative invariants.

1. Apply $\mathbb{C}^\times$-equivariant localization:

$$\int_{\mathbb{M}} \xi = \int_{M_-} \frac{\xi|_{M_-}}{k_- u + \alpha_-} + \int_{M_+} \frac{\xi|_{M_+}}{k_+ u + \alpha_+} + \int_{\mathbb{M}_0} \frac{\xi|_{\mathbb{M}_0}}{e(\mathcal{N}_{\mathbb{M}_0/\mathbb{M}})}$$

   where u is the weight of the $\mathbb{C}^\times$, and $k_\pm \in \mathbb{Z}_{\neq 0}$.

2. Apply a residue map $f \mapsto \mathrm{Res}_u(f\, du)$ (i.e. u^{-1} coefficient):

How do we relate M_- and M_+?

The master space $\mathbb{M}$ relates enumerative invariants.

1. Apply $\mathbb{C}^\times$-equivariant localization:

$$\int_{\mathbb{M}} \xi = \int_{M_-} \frac{\xi|_{M_-}}{k_- u + \alpha_-} + \int_{M_+} \frac{\xi|_{M_+}}{k_+ u + \alpha_+} + \int_{\mathbb{M}_0} \frac{\xi|_{\mathbb{M}_0}}{e(\mathcal{N}_{\mathbb{M}_0/\mathbb{M}})}$$

   where u is the weight of the $\mathbb{C}^\times$, and $k_\pm \in \mathbb{Z}_{\neq 0}$.

2. Apply a residue map $f \mapsto \operatorname{Res}_u(f\,du)$ (i.e. u^{-1} coefficient):

$$0 = \frac{1}{k_-}\int_{M_-} \xi|_{M_-} + \frac{1}{k_+}\int_{M_+} \xi|_{M_+} + \operatorname{Res}_u\left(\int_{\mathbb{M}_0} \frac{\xi|_{\mathbb{M}_0}}{e(\mathcal{N}_{\mathbb{M}_0/\mathbb{M}})}\right) du.$$

How do we relate M_- and M_+?

The master space $\mathbb{M}$ relates enumerative invariants.

1. Apply $\mathbb{C}^\times$-equivariant localization:

$$\int_{\mathbb{M}} \xi = \int_{M_-} \frac{\xi|_{M_-}}{k_- u + \alpha_-} + \int_{M_+} \frac{\xi|_{M_+}}{k_+ u + \alpha_+} + \int_{\mathbb{M}_0} \frac{\xi|_{\mathbb{M}_0}}{e(\mathcal{N}_{\mathbb{M}_0/\mathbb{M}})}$$

   where u is the weight of the $\mathbb{C}^\times$, and $k_\pm \in \mathbb{Z}_{\neq 0}$.

2. Apply a residue map $f \mapsto \mathrm{Res}_u(f\,du)$ (i.e. u^{-1} coefficient):

$$0 = \frac{1}{k_-} \int_{M_-} \xi|_{M_-} + \frac{1}{k_+} \int_{M_+} \xi|_{M_+} + \mathrm{Res}_u \left(\int_{\mathbb{M}_0} \frac{\xi|_{\mathbb{M}_0}}{e(\mathcal{N}_{\mathbb{M}_0/\mathbb{M}})} \right) du.$$

Relates desired invariants via a complicated quantity.

How do we relate M_- and M_+?

This strategy may be generalized to virtual enumerative invariants, to include extra group actions, etc.

How do we relate M_- and M_+?

This strategy may be generalized to virtual enumerative invariants, to include extra group actions, etc.

The challenge is to gain reasonable control over the term

$$\mathrm{Res}_u \left(\int_{\mathbb{M}_0} \frac{\xi|_{\mathbb{M}_0}}{e(\mathcal{N}_{\mathbb{M}_0/\mathbb{M}})} \right) du.$$

How do we relate M_- and M_+?

Among the easiest non-trivial examples: this term vanishes.

How do we relate M_- and M_+?

Among the easiest non-trivial examples: this term vanishes.

Example

Recall the Jacobi theta function

$$\vartheta(x;\tau) := (1 - z^{-1}) \prod_{n>0} (1 - q^n z)(1 - q^n z^{-1})$$

where $z := \exp(2\pi i x)$ and $q := \exp(\pi i \tau)$. Then for any $k \in \mathbb{Z}$,

$$\left(\mathrm{Res}_{u+x_1=0} + \cdots + \mathrm{Res}_{u+x_N=0}\right)\left(\prod_{i=1}^{N} \frac{\vartheta(u+y+x_i;\tau)}{\vartheta(u+x_i;\tau)}\right)\Bigg|_{y=\frac{k}{N}} = 0$$

How do we relate M_- and M_+?

Among the easiest non-trivial examples: this term vanishes.

Theorem (L. '24)

Let $\mathcal{N}_{\mathbb{M}_0/\mathbb{M}} = \bigoplus_{n\in\mathbb{Z}} \mathcal{N}^{(n)}_{\mathbb{M}_0/\mathbb{M}}$ *be the* $\mathbb{C}^\times$*-weight decomposition. If*

$$\sum_{n\in\mathbb{Z}} n \cdot \operatorname{rank} \mathcal{N}^{(n)}_{\mathbb{M}_0/\mathbb{M}} \equiv 0 \bmod N$$

for some integer $N > 1$*, then*

$$(\textit{Elliptic genus})_{y,q}(M_-)\Big|_{y=\frac{k}{N}} = (\textit{Elliptic genus})_{y,q}(M_+)\Big|_{y=\frac{k}{N}}.$$

Joyce's vertex algebra

Joyce applies master space wall-crossing in great generality...

Joyce's vertex algebra

Joyce applies master space wall-crossing in great generality...

Assumption 5.1. (General set-up.) Let $\mathcal{A}$ be a noetherian $\mathbb{C}$-linear abelian category, and $\mathcal{B} \subseteq \mathcal{A}$ be an exact subcategory. Assume:

(a) $\mathcal{B}$ is *closed under isomorphisms in* $\mathcal{A}$ (i.e. if $E \cong F \in \mathcal{A}$ with $E \in \mathcal{B}$ then $F \in \mathcal{B}$) and *closed under direct summands in* $\mathcal{A}$ (i.e. if $E, F \in \mathcal{A}$ with $E \oplus F \in \mathcal{B}$ then $E, F \in \mathcal{B}$).

(b) The inclusion $i : \mathcal{B} \hookrightarrow \mathcal{A}$ induces a morphism $i_* : K_0(\mathcal{B}) \to K_0(\mathcal{A})$. We should be given a surjective quotient $K_0(\mathcal{A}) \twoheadrightarrow K(\mathcal{A})$, which we use for defining (weak) stability conditions on $\mathcal{A}$ as in §3.1. Write $K(\mathcal{B})$ for the image of $i_*(K_0(\mathcal{B}))$ in $K(\mathcal{A})$, so we have a commutative diagram of abelian groups:

$$\begin{array}{ccc} K_0(\mathcal{B}) & \xrightarrow{\;i_*\;} & K_0(\mathcal{A}) \\ \downarrow & & \downarrow \\ K(\mathcal{B}) & \hookrightarrow & K(\mathcal{A}). \end{array}$$

We use the quotient $K_0(\mathcal{B}) \twoheadrightarrow K(\mathcal{B})$ in the vertex and Lie algebra theory for $\mathcal{B}$ in §4.2–§4.3. Note in particular that $C(\mathcal{B}) \subseteq C(\mathcal{A}) \subset K(\mathcal{A})$, where $C(\mathcal{A}) = \{[\![E]\!] : 0 \neq E \in \mathcal{A}\}$ and $C(\mathcal{B}) = \{[\![E]\!] : 0 \neq E \in \mathcal{B}\}$.

(c) Assumption 4.4 holds for $\mathcal{B}$, with $K_0(\mathcal{B}) \twoheadrightarrow K(\mathcal{B})$ and $C(\mathcal{B})$ as above. We will freely use the notation $\mathcal{M}, \mathcal{M}^{\rm pl}, \mathcal{M}_\alpha, \mathcal{M}_\alpha^{\rm pl}, \mathcal{E}^\bullet, \chi, \ldots$ of Assumption 4.4 and Definition 4.7, and the Lie algebra $\check{H}_{\rm even}(\mathcal{M}^{\rm pl})$ of Theorem 4.8.

(d) We are given a subset $C(\mathcal{B})_{\rm pe} \subseteq C(\mathcal{B})$ of *permissible classes*.

(e) For each $\alpha \in C(\mathcal{B})_{\rm pe}$ we are given the following data:

(i) Open $\mathbb{C}$-substacks $\dot{\mathcal{M}}_\alpha^{\rm pl} \subseteq \mathcal{M}_\alpha^{\rm pl}$ and $\dot{\mathcal{M}}_\alpha = (\Pi_\alpha^{\rm pl})^{-1}(\dot{\mathcal{M}}_\alpha^{\rm pl}) \subseteq \mathcal{M}_\alpha$, where we write $\dot{\Pi}_\alpha^{\rm pl} = \Pi_\alpha^{\rm pl}|_{\dot{\mathcal{M}}_\alpha} : \dot{\mathcal{M}}_\alpha \to \dot{\mathcal{M}}_\alpha^{\rm pl}$. Then $\dot{\Pi}_\alpha^{\rm pl} : \dot{\mathcal{M}}_\alpha \to \dot{\mathcal{M}}_\alpha^{\rm pl}$ is a principal $[*/\mathbb{G}_m]$-bundle, and so is smooth of relative dimension -1.

(ii) Perfect obstruction theories $\phi_\alpha : \mathcal{F}_\alpha^\bullet \to \mathbb{L}_{\dot{\mathcal{M}}_\alpha}$ and $\psi_\alpha : \mathcal{G}_\alpha^\bullet \to \mathbb{L}_{\dot{\mathcal{M}}_\alpha^{\rm pl}}$ on the Artin $\mathbb{C}$-stacks $\dot{\mathcal{M}}_\alpha, \dot{\mathcal{M}}_\alpha^{\rm pl}$, as in Definition 2.9, where $\phi_\alpha : \mathcal{F}_\alpha^\bullet \to \mathbb{L}_{\dot{\mathcal{M}}_\alpha}$ should be obtained by pulling back $\psi_\alpha : \mathcal{G}_\alpha^\bullet \to \mathbb{L}_{\dot{\mathcal{M}}_\alpha^{\rm pl}}$ along the smooth morphism $\dot{\Pi}_\alpha^{\rm pl} : \dot{\mathcal{M}}_\alpha \to \dot{\mathcal{M}}_\alpha^{\rm pl}$, as in Definition 2.13.

(iii) A finite-dimensional $\mathbb{C}$-vector space U_α with $\dim U_\alpha = o_\alpha \geqslant 0$, and a distinguished triangle in $\mathrm{Perf}(\dot{\mathcal{M}}_\alpha)$:

$$U_\alpha \otimes_{\mathbb{C}} \mathcal{O}_{\dot{\mathcal{M}}_\alpha}[1] \xrightarrow{u_\alpha} \Delta_{\dot{\mathcal{M}}_\alpha}^*(\mathcal{E}_{\alpha,\alpha}^\bullet)[-1] \xrightarrow{v_\alpha} \mathcal{F}_\alpha^\bullet \xrightarrow{w_\alpha} U_\alpha \otimes_{\mathbb{C}} \mathcal{O}_{\dot{\mathcal{M}}_\alpha}[2], \tag{5.1}$$

where $\Delta_{\dot{\mathcal{M}}_\alpha} : \mathcal{M}_\alpha \to \mathcal{M}_\alpha \times \mathcal{M}_\alpha \subseteq \mathcal{M}_\alpha \times \mathcal{M}_\alpha$ is the diagonal morphism and $\mathcal{E}_{\alpha,\alpha}^\bullet \to \mathcal{M}_\alpha \times \mathcal{M}_\alpha$ is as in Assumption 4.4(f). It follows that $\operatorname{rank} \mathcal{F}_\alpha^\bullet = o_\alpha - \chi(\alpha,\alpha)$, so $\operatorname{rank} \mathcal{G}_\alpha^\bullet = o_\alpha + 1 - \chi(\alpha,\alpha)$ by **(i)**,**(ii)**.

(f) Suppose $\alpha, \beta \in C(\mathcal{B})_{\rm pe}$ with $\alpha + \beta \in C(\mathcal{B})_{\rm pe}$. Define an open substack $\ddot{\mathcal{M}}_{\alpha,\beta} = (\dot{\mathcal{M}}_\alpha \times \dot{\mathcal{M}}_\beta) \cap \Phi_{\alpha,\beta}^{-1}(\dot{\mathcal{M}}_{\alpha,\beta}) \subseteq \mathcal{M}_\alpha \times \mathcal{M}_\beta$, for $\Phi_{\alpha,\beta}$ as in Assumption 4.4(b),(d), and write $\ddot{\Phi}_{\alpha,\beta} = \Phi_{\alpha,\beta}|_{\ddot{\mathcal{M}}_{\alpha,\beta}} : \ddot{\mathcal{M}}_{\alpha,\beta} \to \dot{\mathcal{M}}_{\alpha+\beta}$. Then the following should commute in $\mathrm{Perf}(\ddot{\mathcal{M}}_{\alpha,\beta})$:

$$\begin{array}{ccc} (\Delta_{\dot{\mathcal{M}}_{\alpha+\beta}} \circ \ddot{\Phi}_{\alpha,\beta})^* (\mathcal{E}_{\alpha+\beta,\alpha+\beta}^\bullet)[-1] & \xrightarrow{\ddot{\Phi}_{\alpha,\beta}^*(\phi_{\alpha+\beta} \circ v_{\alpha+\beta})} & \ddot{\Phi}_{\alpha,\beta}^*(\mathbb{L}_{\dot{\mathcal{M}}_{\alpha+\beta}}) \\ \cong \downarrow \text{from (4.3)–(4.4)} & & \\ (\Delta_{\dot{\mathcal{M}}_\alpha} \circ \Pi_{\dot{\mathcal{M}}_\alpha})^*(\mathcal{E}_{\alpha,\alpha}^\bullet)[-1] \oplus (\Delta_{\dot{\mathcal{M}}_\beta} \circ \Pi_{\dot{\mathcal{M}}_\beta})^*(\mathcal{E}_{\beta,\beta}^\bullet)[-1] & & \downarrow \mathbb{L}_{\ddot{\Phi}_{\alpha,\beta}} \\ \oplus\, \mathcal{E}_{\alpha,\beta}^\bullet|_{\ddot{\mathcal{M}}_{\alpha,\beta}}[-1] \oplus \sigma_{\alpha,\beta}^*(\mathcal{E}_{\beta,\alpha}^\bullet)|_{\ddot{\mathcal{M}}_{\alpha,\beta}}[-1] & & \\ \downarrow \text{project to first two factors} & & \\ (\Delta_{\dot{\mathcal{M}}_\alpha} \circ \Pi_{\dot{\mathcal{M}}_\alpha})^*(\mathcal{E}_{\alpha,\alpha}^\bullet)[-1] & \xrightarrow{\Pi_{\dot{\mathcal{M}}_\alpha}^*(\phi_\alpha \circ v_\alpha) \oplus \Pi_{\dot{\mathcal{M}}_\beta}^*(\phi_\beta \circ v_\beta)} & \mathbb{L}_{\ddot{\mathcal{M}}_{\alpha,\beta}} \cong \\ \oplus\, (\Delta_{\dot{\mathcal{M}}_\beta} \circ \Pi_{\dot{\mathcal{M}}_\beta})^*(\mathcal{E}_{\beta,\beta}^\bullet)[-1] & & \Pi_{\dot{\mathcal{M}}_\alpha}^*(\mathbb{L}_{\dot{\mathcal{M}}_\alpha}) \oplus \Pi_{\dot{\mathcal{M}}_\beta}^*(\mathbb{L}_{\dot{\mathcal{M}}_\beta}). \end{array} \tag{5.2}$$

We also require that $\dim U_\alpha + \dim U_\beta \geqslant \dim U_{\alpha+\beta}$, that is, $o_\alpha + o_\beta \geqslant o_{\alpha+\beta}$.

(g) We are given a set $\{(\mathcal{B}_k, F_k, \lambda_k) : k \in K\}$, where for each $k \in K$:

(i) $\mathcal{B}_k \subseteq \mathcal{B} \subseteq \mathcal{A}$ is a full exact subcategory, closed under isomorphisms and direct summands in $\mathcal{A}$ as in **(a)**, such that $E \in \mathcal{B}_k$ is an open condition on $\mathbb{C}$-points $[E]$ in $\mathcal{M}, \mathcal{M}^{\rm pl}$. Thus we have open $\mathbb{C}$-substacks $\mathcal{M}_k \subseteq \mathcal{M}$, $\mathcal{M}_k^{\rm pl} \subseteq \mathcal{M}^{\rm pl}$ which are the moduli stacks of objects in $\mathcal{B}_k$. We write $\mathcal{M}_{k,\alpha} = \mathcal{M}_\alpha \cap \mathcal{M}_k$, $\mathcal{M}_{k,\alpha}^{\rm pl} = \mathcal{M}_\alpha^{\rm pl} \cap \mathcal{M}_k^{\rm pl}$ for $\alpha \in K(\mathcal{B}_k)$. We write $C(\mathcal{B}_k) = \{[\![E]\!] : 0 \neq E \in \mathcal{B}_k\} \subseteq C(\mathcal{B})$.

(ii) $F_k : \mathcal{A} \to \mathrm{Vect}_{\mathbb{C}}$ is a $\mathbb{C}$-linear functor. Its restriction $F_k|_{\mathcal{B}_k} : \mathcal{B}_k \to \mathrm{Vect}_{\mathbb{C}}$ is an *exact functor* (i.e. it preserves short exact sequences), which extends to moduli functors, so it induces morphisms of moduli stacks

$$f_k : \mathcal{M}_k \longrightarrow \mathcal{M}_{\mathrm{Vect}_{\mathbb{C}}} = \textstyle\coprod_{d \geqslant 0} [*/\mathrm{GL}(d,\mathbb{C})],$$
$$f_k^{\rm pl} : \mathcal{M}_k^{\rm pl} \longrightarrow \mathcal{M}_{\mathrm{Vect}_{\mathbb{C}}}^{\rm pl} = \textstyle\coprod_{d \geqslant 0} [*/\mathrm{PGL}(d,\mathbb{C})].$$

There is a tautological vector bundle $\mathcal{V}_{\mathrm{Vect}_{\mathbb{C}}} \to \mathcal{M}_{\mathrm{Vect}_{\mathbb{C}}}$. We write $\mathcal{V}_k = f_k^*(\mathcal{V}_{\mathrm{Vect}_{\mathbb{C}}})$. Then $\mathcal{V}_k \to \mathcal{M}_k$ is a vector bundle with canonical isomorphisms $\mathcal{V}_k|_{[E]} \cong F_k(E)$ for all $E \in \mathcal{B}_k$. We write $\mathcal{V}_{k,\alpha} = \mathcal{V}_k|_{\mathcal{M}_{k,\alpha}}$.

Since F_k takes direct sums to direct sums, and is $\mathbb{C}$-linear, in a similar way to (4.3)–(4.6), for all $\alpha, \beta \in K(\mathcal{B}_k)$ we have isomorphisms

$$\Phi_{\alpha,\beta}^*(\mathcal{V}_{k,\alpha+\beta}) \cong \mathcal{V}_{k,\alpha} \oplus \mathcal{V}_{k,\beta}, \tag{5.3}$$
$$\Psi_\alpha^*(\mathcal{V}_{k,\alpha}) \cong L_{[*/\mathbb{G}_m]} \otimes \mathcal{V}_{k,\alpha}. \tag{5.4}$$

Joyce's vertex algebra

Joyce applies master space wall-crossing in great generality...

Assumption 5.1. (General set-up.) Let $\mathcal{A}$ be a noetherian $\mathbb{C}$-linear abelian category, and $\mathcal{B} \subseteq \mathcal{A}$ be an exact subcategory. Assume:

(a) $\mathcal{B}$ is *closed under isomorphisms in* $\mathcal{A}$ (i.e. if $E \cong F \in \mathcal{A}$ with $E \in \mathcal{B}$ then $F \in \mathcal{B}$) and *closed under direct summands in* $\mathcal{A}$ (i.e. if $E, F \in \mathcal{A}$ with $E \oplus F \in \mathcal{B}$ then $E, F \in \mathcal{B}$).

(b) The inclusion $i : \mathcal{B} \hookrightarrow \mathcal{A}$ induces a morphism $i_* : K_0(\mathcal{B}) \to K_0(\mathcal{A})$. We should be given a surjective quotient $K_0(\mathcal{A}) \twoheadrightarrow K(\mathcal{A})$, which we use for defining (weak) stability conditions on $\mathcal{A}$ as in §3.1. Write $K(\mathcal{B})$ for the image of $i_*(K_0(\mathcal{B}))$ in $K(\mathcal{A})$, so we have a commutative diagram of abelian groups:

$$\begin{array}{ccc} K_0(\mathcal{B}) & \xrightarrow{\;i_*\;} & K_0(\mathcal{A}) \\ \downarrow & & \downarrow \\ K(\mathcal{B}) & \hookrightarrow & K(\mathcal{A}). \end{array}$$

We use the quotient $K_0(\mathcal{B}) \twoheadrightarrow K(\mathcal{B})$ in the vertex and Lie algebra theory for $\mathcal{B}$ in §4.2–§4.3. Note in particular that $C(\mathcal{B}) \subseteq C(\mathcal{A}) \subset K(\mathcal{A})$, where $C(\mathcal{A}) = \{[\![E]\!] : 0 \neq E \in \mathcal{A}\}$ and $C(\mathcal{B}) = \{[\![E]\!] : 0 \neq E \in \mathcal{B}\}$.

(c) Assumption 4.4 holds for $\mathcal{B}$, with $K_0(\mathcal{B}) \twoheadrightarrow K(\mathcal{B})$ and $C(\mathcal{B})$ as above. We will freely use the notation $\mathcal{M}, \mathcal{M}^{\mathrm{pl}}, \mathcal{M}_\alpha, \mathcal{M}_\alpha^{\mathrm{pl}}, \mathcal{E}^\bullet, \chi, \ldots$ of Assumption 4.4 and Definition 4.7, and the Lie algebra $\check{H}_{\mathrm{even}}(\mathcal{M}^{\mathrm{pl}})$ of Theorem 4.8.

(d) We are given a subset $C(\mathcal{B})_{\mathrm{pe}} \subseteq C(\mathcal{B})$ of *permissible classes.*

(e) For each $\alpha \in C(\mathcal{B})_{\mathrm{pe}}$ we are given the following data:

(i) Open $\mathbb{C}$-substacks $\dot{\mathcal{M}}_\alpha^{\mathrm{pl}} \subseteq \mathcal{M}_\alpha^{\mathrm{pl}}$ and $\dot{\mathcal{M}}_\alpha = (\Pi_\alpha^{\mathrm{pl}})^{-1}(\dot{\mathcal{M}}_\alpha^{\mathrm{pl}}) \subseteq \mathcal{M}_\alpha$, where we write $\dot{\Pi}_\alpha^{\mathrm{pl}} = \Pi_\alpha^{\mathrm{pl}}|_{\dot{\mathcal{M}}_\alpha} : \dot{\mathcal{M}}_\alpha \to \dot{\mathcal{M}}_\alpha^{\mathrm{pl}}$. Then $\dot{\Pi}_\alpha^{\mathrm{pl}} : \dot{\mathcal{M}}_\alpha \to \dot{\mathcal{M}}_\alpha^{\mathrm{pl}}$ is a principal $[*/\mathbb{G}_m]$-bundle, and so is smooth of relative dimension -1.

(ii) Perfect obstruction theories $\phi_\alpha : \mathcal{F}_\alpha^\bullet \to \mathbb{L}_{\dot{\mathcal{M}}_\alpha}$ and $\psi_\alpha : \mathcal{G}_\alpha^\bullet \to \mathbb{L}_{\dot{\mathcal{M}}_\alpha^{\mathrm{pl}}}$ on the Artin $\mathbb{C}$-stacks $\dot{\mathcal{M}}_\alpha, \dot{\mathcal{M}}_\alpha^{\mathrm{pl}}$, as in Definition 2.9, where $\phi_\alpha : \mathcal{F}_\alpha^\bullet \to \mathbb{L}_{\dot{\mathcal{M}}_\alpha}$ should be obtained by pulling back $\psi_\alpha : \mathcal{G}_\alpha^\bullet \to \mathbb{L}_{\dot{\mathcal{M}}_\alpha^{\mathrm{pl}}}$ along the smooth morphism $\dot{\Pi}_\alpha^{\mathrm{pl}} : \dot{\mathcal{M}}_\alpha \to \dot{\mathcal{M}}_\alpha^{\mathrm{pl}}$, as in Definition 2.13.

(iii) A finite-dimensional $\mathbb{C}$-vector space U_α with $\dim U_\alpha = o_\alpha \geqslant 0$, and a distinguished triangle in $\mathrm{Perf}(\dot{\mathcal{M}}_\alpha)$:

$$U_\alpha \otimes_{\mathbb{C}} \mathcal{O}_{\dot{\mathcal{M}}_\alpha}[1] \xrightarrow{u_\alpha} \Delta_{\dot{\mathcal{M}}_\alpha}^*(\mathcal{E}_{\alpha,\alpha}^\bullet)[-1] \xrightarrow{v_\alpha} \mathcal{F}_\alpha^\bullet \xrightarrow{w_\alpha} U_\alpha \otimes_{\mathbb{C}} \mathcal{O}_{\dot{\mathcal{M}}_\alpha}[2], \quad (5.1)$$

where $\Delta_{\dot{\mathcal{M}}_\alpha} : \mathcal{M}_\alpha \to \mathcal{M}_\alpha \times \mathcal{M}_\alpha \subseteq \mathcal{M}_\alpha \times \mathcal{M}_\alpha$ is the diagonal morphism and $\mathcal{E}_{\alpha,\alpha}^\bullet \to \mathcal{M}_\alpha \times \mathcal{M}_\alpha$ is as in Assumption 4.4(f). It follows that $\operatorname{rank} \mathcal{F}_\alpha^\bullet = o_\alpha - \chi(\alpha,\alpha)$, so $\operatorname{rank} \mathcal{G}_\alpha^\bullet = o_\alpha + 1 - \chi(\alpha,\alpha)$ by **(i)**,**(ii)**.

(f) Suppose $\alpha, \beta \in C(\mathcal{B})_{\mathrm{pe}}$ with $\alpha + \beta \in C(\mathcal{B})_{\mathrm{pe}}$. Define an open substack $\ddot{\mathcal{M}}_{\alpha,\beta} = (\dot{\mathcal{M}}_\alpha \times \dot{\mathcal{M}}_\beta) \cap \Phi_{\alpha,\beta}^{-1}(\dot{\mathcal{M}}_{\alpha,\beta}) \subseteq \mathcal{M}_\alpha \times \mathcal{M}_\beta$, for $\Phi_{\alpha,\beta}$ as in Assumption 4.4(b),(d), and write $\ddot{\Phi}_{\alpha,\beta} = \Phi_{\alpha,\beta}|_{\ddot{\mathcal{M}}_{\alpha,\beta}} : \ddot{\mathcal{M}}_{\alpha,\beta} \to \dot{\mathcal{M}}_{\alpha+\beta}$. Then the following should commute in $\mathrm{Perf}(\ddot{\mathcal{M}}_{\alpha,\beta})$:

$$\begin{array}{ccc}
(\Delta_{\dot{\mathcal{M}}_{\alpha+\beta}} \circ \ddot{\Phi}_{\alpha,\beta})^* (\mathcal{E}_{\alpha+\beta,\alpha+\beta}^\bullet)[-1] & \xrightarrow{\ddot{\Phi}_{\alpha,\beta}^*(\phi_{\alpha+\beta} \circ v_{\alpha+\beta})} & \ddot{\Phi}_{\alpha,\beta}^*(\mathbb{L}_{\dot{\mathcal{M}}_{\alpha+\beta}}) \\
\cong \big\downarrow \text{from (4.3)–(4.4)} & & \\
(\Delta_{\dot{\mathcal{M}}_\alpha} \circ \Pi_{\dot{\mathcal{M}}_\alpha})^*(\mathcal{E}_{\alpha,\alpha}^\bullet)[-1] \oplus (\Delta_{\dot{\mathcal{M}}_\beta} \circ \Pi_{\dot{\mathcal{M}}_\beta})^*(\mathcal{E}_{\beta,\beta}^\bullet)[-1] & & \big\downarrow \mathbb{L}_{\ddot{\Phi}_{\alpha,\beta}} \\
\oplus\, \mathcal{E}_{\alpha,\beta}^\bullet|_{\ddot{\mathcal{M}}_{\alpha,\beta}}[-1] \oplus \sigma_{\alpha,\beta}^*(\mathcal{E}_{\beta,\alpha}^\bullet)|_{\ddot{\mathcal{M}}_{\alpha,\beta}}[-1] & & \\
\big\downarrow \text{project to first two factors} & & \mathbb{L}_{\ddot{\mathcal{M}}_{\alpha,\beta}} \cong \\
(\Delta_{\dot{\mathcal{M}}_\alpha} \circ \Pi_{\dot{\mathcal{M}}_\alpha})^*(\mathcal{E}_{\alpha,\alpha}^\bullet)[-1] \oplus (\Delta_{\dot{\mathcal{M}}_\beta} \circ \Pi_{\dot{\mathcal{M}}_\beta})^*(\mathcal{E}_{\beta,\beta}^\bullet)[-1] & \xrightarrow{\Pi_{\dot{\mathcal{M}}_\alpha}^*(\phi_\alpha \circ v_\alpha) \oplus \Pi_{\dot{\mathcal{M}}_\beta}^*(\phi_\beta \circ v_\beta)} & \Pi_{\dot{\mathcal{M}}_\alpha}^*(\mathbb{L}_{\dot{\mathcal{M}}_\alpha}) \oplus \Pi_{\dot{\mathcal{M}}_\beta}^*(\mathbb{L}_{\dot{\mathcal{M}}_\beta}).
\end{array} \quad (5.2)$$

We also require that $\dim U_\alpha + \dim U_\beta \geqslant \dim U_{\alpha+\beta}$, that is, $o_\alpha + o_\beta \geqslant o_{\alpha+\beta}$.

(g) We are given a set $\{(\mathcal{B}_k, F_k, \lambda_k) : k \in K\}$, where for each $k \in K$:

(i) $\mathcal{B}_k \subseteq \mathcal{B} \subseteq \mathcal{A}$ is a full exact subcategory, closed under isomorphisms and direct summands in $\mathcal{A}$ as in **(a)**, such that $E \in \mathcal{B}_k$ is an open condition on $\mathbb{C}$-points $[E]$ in $\mathcal{M}, \mathcal{M}^{\mathrm{pl}}$. Thus we have open $\mathbb{C}$-substacks $\mathcal{M}_k \subseteq \mathcal{M}$, $\mathcal{M}_k^{\mathrm{pl}} \subseteq \mathcal{M}^{\mathrm{pl}}$ which are the moduli stacks of objects in $\mathcal{B}_k$. We write $\mathcal{M}_{k,\alpha} = \mathcal{M}_\alpha \cap \mathcal{M}_k$, $\mathcal{M}_{k,\alpha}^{\mathrm{pl}} = \mathcal{M}_\alpha^{\mathrm{pl}} \cap \mathcal{M}_k^{\mathrm{pl}}$ for $\alpha \in K(\mathcal{B}_k)$. We write $C(\mathcal{B}_k) = \{[\![E]\!] : 0 \neq E \in \mathcal{B}_k\} \subseteq C(\mathcal{B})$.

(ii) $F_k : \mathcal{A} \to \mathrm{Vect}_{\mathbb{C}}$ is a $\mathbb{C}$-linear functor. Its restriction $F_k|_{\mathcal{B}_k} : \mathcal{B}_k \to \mathrm{Vect}_{\mathbb{C}}$ is an *exact functor* (i.e. it preserves short exact sequences), which extends to moduli functors, so it induces morphisms of moduli stacks

$$f_k : \mathcal{M}_k \longrightarrow \mathcal{M}_{\mathrm{Vect}_{\mathbb{C}}} = \textstyle\coprod_{d \geqslant 0} [*/\mathrm{GL}(d,\mathbb{C})],$$
$$f_k^{\mathrm{pl}} : \mathcal{M}_k^{\mathrm{pl}} \longrightarrow \mathcal{M}_{\mathrm{Vect}_{\mathbb{C}}}^{\mathrm{pl}} = \textstyle\coprod_{d \geqslant 0} [*/\mathrm{PGL}(d,\mathbb{C})].$$

There is a tautological vector bundle $\mathcal{V}_{\mathrm{Vect}_{\mathbb{C}}} \to \mathcal{M}_{\mathrm{Vect}_{\mathbb{C}}}$. We write $\mathcal{V}_k = f_k^*(\mathcal{V}_{\mathrm{Vect}_{\mathbb{C}}})$. Then $\mathcal{V}_k \to \mathcal{M}_k$ is a vector bundle with canonical isomorphisms $\mathcal{V}_k|_{[E]} \cong F_k(E)$ for all $E \in \mathcal{B}_k$. We write $\mathcal{V}_{k,\alpha} = \mathcal{V}_k|_{\mathcal{M}_{k,\alpha}}$.

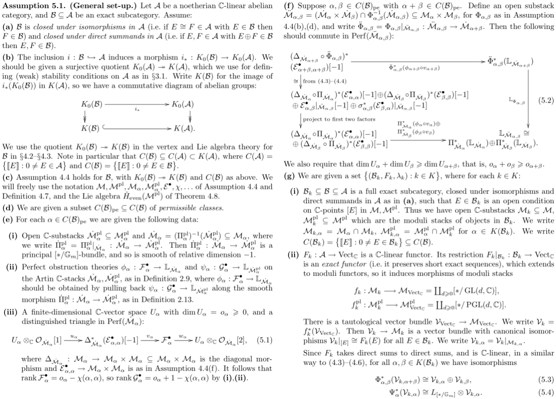
Since F_k takes direct sums to direct sums, and is $\mathbb{C}$-linear, in a similar way to (4.3)–(4.6), for all $\alpha, \beta \in K(\mathcal{B}_k)$ we have isomorphisms

$$\Phi_{\alpha,\beta}^*(\mathcal{V}_{k,\alpha+\beta}) \cong \mathcal{V}_{k,\alpha} \oplus \mathcal{V}_{k,\beta}, \quad (5.3)$$
$$\Psi_\alpha^*(\mathcal{V}_{k,\alpha}) \cong L_{[*/\mathbb{G}_m]} \otimes \mathcal{V}_{k,\alpha}. \quad (5.4)$$

... (+6 more pages).

Joyce's vertex algebra

Joyce applies master space wall-crossing in great generality...



Assumption 5.1. (General set-up.) Let $\mathcal{A}$ be a noetherian $\mathbb{C}$-linear abelian category, and $\mathcal{B} \subseteq \mathcal{A}$ be an exact subcategory. Assume:

(a) $\mathcal{B}$ is *closed under isomorphisms in* $\mathcal{A}$ (i.e. if $E \cong F \in \mathcal{A}$ with $E \in \mathcal{B}$ then $F \in \mathcal{B}$) and *closed under direct summands in* $\mathcal{A}$ (i.e. if $E, F \in \mathcal{A}$ with $E \oplus F \in \mathcal{B}$ then $E, F \in \mathcal{B}$).

(b) The inclusion $i : \mathcal{B} \hookrightarrow \mathcal{A}$ induces a morphism $i_* : K_0(\mathcal{B}) \to K_0(\mathcal{A})$. We should be given a surjective quotient $K_0(\mathcal{A}) \twoheadrightarrow K(\mathcal{A})$, which we use for defining (weak) stability conditions on $\mathcal{A}$ as in §3.1. Write $K(\mathcal{B})$ for the image of $i_*(K_0(\mathcal{B}))$ in $K(\mathcal{A})$, so we have a commutative diagram of abelian groups:

$$\begin{array}{ccc} K_0(\mathcal{B}) & \xrightarrow{i_*} & K_0(\mathcal{A}) \\ \downarrow & & \downarrow \\ K(\mathcal{B}) & \hookrightarrow & K(\mathcal{A}). \end{array}$$

We use the quotient $K_0(\mathcal{B}) \twoheadrightarrow K(\mathcal{B})$ in the vertex and Lie algebra theory for $\mathcal{B}$ in §4.2–§4.3. Note in particular that $C(\mathcal{B}) \subseteq C(\mathcal{A}) \subset K(\mathcal{A})$, where $C(\mathcal{A}) = \{[\![E]\!] : 0 \neq E \in \mathcal{A}\}$ and $C(\mathcal{B}) = \{[\![E]\!] : 0 \neq E \in \mathcal{B}\}$.

(c) Assumption 4.4 holds for $\mathcal{B}$, with $K_0(\mathcal{B}) \twoheadrightarrow K(\mathcal{B})$ and $C(\mathcal{B})$ as above. We will freely use the notation $\mathcal{M}, \mathcal{M}^{\rm pl}, \mathcal{M}_\alpha, \mathcal{M}^{\rm pl}_\alpha, \mathcal{E}^\bullet, \chi, \ldots$ of Assumption 4.4 and Definition 4.7, and the Lie algebra $\check{H}_{\rm even}(\mathcal{M}^{\rm pl})$ of Theorem 4.8.

(d) We are given a subset $C(\mathcal{B})_{\rm pe} \subseteq C(\mathcal{B})$ of *permissible classes.*

(e) For each $\alpha \in C(\mathcal{B})_{\rm pe}$ we are given the following data:

(i) Open $\mathbb{C}$-substacks $\dot{\mathcal{M}}^{\rm pl}_\alpha \subseteq \mathcal{M}^{\rm pl}_\alpha$ and $\dot{\mathcal{M}}_\alpha = (\Pi^{\rm pl}_\alpha)^{-1}(\dot{\mathcal{M}}^{\rm pl}_\alpha) \subseteq \mathcal{M}_\alpha$, where we write $\dot{\Pi}^{\rm pl}_\alpha = \Pi^{\rm pl}_\alpha|_{\dot{\mathcal{M}}_\alpha} : \dot{\mathcal{M}}_\alpha \to \dot{\mathcal{M}}^{\rm pl}_\alpha$. Then $\dot{\Pi}^{\rm pl}_\alpha : \dot{\mathcal{M}}_\alpha \to \dot{\mathcal{M}}^{\rm pl}_\alpha$ is a principal $[*/\mathbb{G}_m]$-bundle, and so is smooth of relative dimension -1.

(ii) Perfect obstruction theories $\phi_\alpha : \mathcal{F}^\bullet_\alpha \to \mathbb{L}_{\dot{\mathcal{M}}_\alpha}$ and $\psi_\alpha : \mathcal{G}^\bullet_\alpha \to \mathbb{L}_{\dot{\mathcal{M}}^{\rm pl}_\alpha}$ on the Artin $\mathbb{C}$-stacks $\dot{\mathcal{M}}_\alpha, \dot{\mathcal{M}}^{\rm pl}_\alpha$, as in Definition 2.9, where $\phi_\alpha : \mathcal{F}^\bullet_\alpha \to \mathbb{L}_{\dot{\mathcal{M}}_\alpha}$ should be obtained by pulling back $\psi_\alpha : \mathcal{G}^\bullet_\alpha \to \mathbb{L}_{\dot{\mathcal{M}}^{\rm pl}_\alpha}$ along the smooth morphism $\dot{\Pi}^{\rm pl}_\alpha : \dot{\mathcal{M}}_\alpha \to \dot{\mathcal{M}}^{\rm pl}_\alpha$, as in Definition 2.13.

(iii) A finite-dimensional $\mathbb{C}$-vector space U_α with $\dim U_\alpha = o_\alpha \geqslant 0$, and a distinguished triangle in $\mathrm{Perf}(\dot{\mathcal{M}}_\alpha)$:

$$U_\alpha \otimes_{\mathbb{C}} \mathcal{O}_{\dot{\mathcal{M}}_\alpha}[1] \xrightarrow{u_\alpha} \Delta^*_{\dot{\mathcal{M}}_\alpha}(\mathcal{E}^\bullet_{\alpha,\alpha})[-1] \xrightarrow{v_\alpha} \mathcal{F}^\bullet_\alpha \xrightarrow{w_\alpha} U_\alpha \otimes_{\mathbb{C}} \mathcal{O}_{\dot{\mathcal{M}}_\alpha}[2], \quad (5.1)$$

where $\Delta_{\dot{\mathcal{M}}_\alpha} : \mathcal{M}_\alpha \to \mathcal{M}_\alpha \times \mathcal{M}_\alpha \subseteq \mathcal{M}_\alpha \times \mathcal{M}_\alpha$ is the diagonal morphism and $\mathcal{E}^\bullet_{\alpha,\alpha} \to \mathcal{M}_\alpha \times \mathcal{M}_\alpha$ is as in Assumption 4.4(f). It follows that $\operatorname{rank} \mathcal{F}^\bullet_\alpha = o_\alpha - \chi(\alpha,\alpha)$, so $\operatorname{rank} \mathcal{G}^\bullet_\alpha = o_\alpha + 1 - \chi(\alpha,\alpha)$ by **(i)**,**(ii)**.

(f) Suppose $\alpha, \beta \in C(\mathcal{B})_{\rm pe}$ with $\alpha + \beta \in C(\mathcal{B})_{\rm pe}$. Define an open substack $\ddot{\mathcal{M}}_{\alpha,\beta} = (\dot{\mathcal{M}}_\alpha \times \dot{\mathcal{M}}_\beta) \cap \Phi^{-1}_{\alpha,\beta}(\dot{\mathcal{M}}_{\alpha,\beta}) \subseteq \mathcal{M}_\alpha \times \mathcal{M}_\beta$, for $\Phi_{\alpha,\beta}$ as in Assumption 4.4(b),(d), and write $\ddot{\Phi}_{\alpha,\beta} = \Phi_{\alpha,\beta}|_{\ddot{\mathcal{M}}_{\alpha,\beta}} : \ddot{\mathcal{M}}_{\alpha,\beta} \to \dot{\mathcal{M}}_{\alpha+\beta}$. Then the following should commute in $\mathrm{Perf}(\ddot{\mathcal{M}}_{\alpha,\beta})$:

$$\begin{array}{ccc} (\Delta_{\dot{\mathcal{M}}_{\alpha+\beta}} \circ \ddot{\Phi}_{\alpha,\beta})^*(\mathcal{E}^\bullet_{\alpha+\beta,\alpha+\beta})[-1] & \xrightarrow{\ddot{\Phi}^*_{\alpha,\beta}(\phi_{\alpha+\beta} \circ v_{\alpha+\beta})} & \ddot{\Phi}^*_{\alpha,\beta}(\mathbb{L}_{\dot{\mathcal{M}}_{\alpha+\beta}}) \\ \cong \downarrow \text{from (4.3)–(4.4)} & & \\ (\Delta_{\dot{\mathcal{M}}_\alpha} \circ \Pi_{\dot{\mathcal{M}}_\alpha})^*(\mathcal{E}^\bullet_{\alpha,\alpha})[-1] \oplus (\Delta_{\dot{\mathcal{M}}_\beta} \circ \Pi_{\dot{\mathcal{M}}_\beta})^*(\mathcal{E}^\bullet_{\beta,\beta})[-1] & & \downarrow \mathbb{L}_{\ddot{\Phi}_{\alpha,\beta}} \\ \oplus\, \mathcal{E}^\bullet_{\alpha,\beta}|_{\ddot{\mathcal{M}}_{\alpha,\beta}}[-1] \oplus \sigma^*_{\alpha,\beta}(\mathcal{E}^\bullet_{\beta,\alpha})|_{\ddot{\mathcal{M}}_{\alpha,\beta}}[-1] & & \\ \downarrow \text{project to first two factors} & & \\ (\Delta_{\dot{\mathcal{M}}_\alpha} \circ \Pi_{\dot{\mathcal{M}}_\alpha})^*(\mathcal{E}^\bullet_{\alpha,\alpha})[-1] & \xrightarrow{\Pi^*_{\dot{\mathcal{M}}_\alpha}(\phi_\alpha \circ v_\alpha) \oplus \Pi^*_{\dot{\mathcal{M}}_\beta}(\phi_\beta \circ v_\beta)} & \mathbb{L}_{\ddot{\mathcal{M}}_{\alpha,\beta}} \cong \\ \oplus\, (\Delta_{\dot{\mathcal{M}}_\beta} \circ \Pi_{\dot{\mathcal{M}}_\beta})^*(\mathcal{E}^\bullet_{\beta,\beta})[-1] & & \Pi^*_{\dot{\mathcal{M}}_\alpha}(\mathbb{L}_{\dot{\mathcal{M}}_\alpha}) \oplus \Pi^*_{\dot{\mathcal{M}}_\beta}(\mathbb{L}_{\dot{\mathcal{M}}_\beta}). \end{array} \quad (5.2)$$

We also require that $\dim U_\alpha + \dim U_\beta \geqslant \dim U_{\alpha+\beta}$, that is, $o_\alpha + o_\beta \geqslant o_{\alpha+\beta}$.

(g) We are given a set $\{(\mathcal{B}_k, F_k, \lambda_k) : k \in K\}$, where for each $k \in K$:

(i) $\mathcal{B}_k \subseteq \mathcal{B} \subseteq \mathcal{A}$ is a full exact subcategory, closed under isomorphisms and direct summands in $\mathcal{A}$ as in **(a)**, such that $E \in \mathcal{B}_k$ is an open condition on $\mathbb{C}$-points $[E]$ in $\mathcal{M}, \mathcal{M}^{\rm pl}$. Thus we have open $\mathbb{C}$-substacks $\mathcal{M}_k \subseteq \mathcal{M}$, $\mathcal{M}^{\rm pl}_k \subseteq \mathcal{M}^{\rm pl}$ which are the moduli stacks of objects in $\mathcal{B}_k$. We write $\mathcal{M}_{k,\alpha} = \mathcal{M}_\alpha \cap \mathcal{M}_k$, $\mathcal{M}^{\rm pl}_{k,\alpha} = \mathcal{M}^{\rm pl}_\alpha \cap \mathcal{M}^{\rm pl}_k$ for $\alpha \in K(\mathcal{B}_k)$. We write $C(\mathcal{B}_k) = \{[\![E]\!] : 0 \neq E \in \mathcal{B}_k\} \subseteq C(\mathcal{B})$.

(ii) $F_k : \mathcal{A} \to \mathrm{Vect}_{\mathbb{C}}$ is a $\mathbb{C}$-linear functor. Its restriction $F_k|_{\mathcal{B}_k} : \mathcal{B}_k \to \mathrm{Vect}_{\mathbb{C}}$ is an *exact functor* (i.e. it preserves short exact sequences), which extends to moduli functors, so it induces morphisms of moduli stacks

$$f_k : \mathcal{M}_k \longrightarrow \mathcal{M}_{\mathrm{Vect}_{\mathbb{C}}} = \textstyle\coprod_{d \geqslant 0}[*/\mathrm{GL}(d,\mathbb{C})],$$
$$f^{\rm pl}_k : \mathcal{M}^{\rm pl}_k \longrightarrow \mathcal{M}^{\rm pl}_{\mathrm{Vect}_{\mathbb{C}}} = \textstyle\coprod_{d \geqslant 0}[*/\mathrm{PGL}(d,\mathbb{C})].$$

There is a tautological vector bundle $\mathcal{V}_{\mathrm{Vect}_{\mathbb{C}}} \to \mathcal{M}_{\mathrm{Vect}_{\mathbb{C}}}$. We write $\mathcal{V}_k = f^*_k(\mathcal{V}_{\mathrm{Vect}_{\mathbb{C}}})$. Then $\mathcal{V}_k \to \mathcal{M}_k$ is a vector bundle with canonical isomorphisms $\mathcal{V}_k|_{[E]} \cong F_k(E)$ for all $E \in \mathcal{B}_k$. We write $\mathcal{V}_{k,\alpha} = \mathcal{V}_k|_{\mathcal{M}_{k,\alpha}}$.

Since F_k takes direct sums to direct sums, and is $\mathbb{C}$-linear, in a similar way to (4.3)–(4.6), for all $\alpha, \beta \in K(\mathcal{B}_k)$ we have isomorphisms

$$\Phi^*_{\alpha,\beta}(\mathcal{V}_{k,\alpha+\beta}) \cong \mathcal{V}_{k,\alpha} \oplus \mathcal{V}_{k,\beta}, \quad (5.3)$$
$$\Psi^*_\alpha(\mathcal{V}_{k,\alpha}) \cong L_{[*/\mathbb{G}_m]} \otimes \mathcal{V}_{k,\alpha}. \quad (5.4)$$

... (+6 more pages). We will work in an easy case for simplicity.

Joyce's vertex algebra

Let X be a smooth projective variety, and $\mathfrak{M}_\alpha$ be a moduli of coherent sheaves $\mathcal{E}$ on X of class α.

Joyce's vertex algebra

Let X be a smooth projective variety, and $\mathfrak{M}_\alpha$ be a moduli of coherent sheaves $\mathcal{E}$ on X of class α. Let $\mathfrak{M}_\alpha^{\mathrm{sst}}(\tau)$ denote the locus of τ-semistable objects.

Joyce's vertex algebra

Let X be a smooth projective variety, and $\mathfrak{M}_\alpha$ be a moduli of coherent sheaves $\mathcal{E}$ on X of class α. Let $\mathfrak{M}_\alpha^{\mathrm{sst}}(\tau)$ denote the locus of τ-semistable objects.

Fact (about (Mochizuki-style) master spaces in such a setting)

Assume that the wall is simple, i.e.

- *strictly τ_0-semistable objects split into ≤ 2 τ_0-stable pieces.*

Joyce's vertex algebra

Let X be a smooth projective variety, and $\mathfrak{M}_\alpha$ be a moduli of coherent sheaves $\mathcal{E}$ on X of class α. Let $\mathfrak{M}_\alpha^{\mathrm{sst}}(\tau)$ denote the locus of τ-semistable objects.

Fact (about (Mochizuki-style) master spaces in such a setting)

Assume that the wall is simple, i.e.

- *strictly τ_0-semistable objects split into ≤ 2 τ_0-stable pieces.*

Then a master space $\mathbb{M}$ exists and

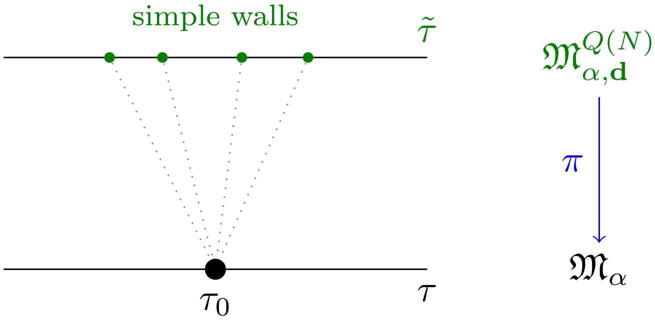
$$\begin{aligned}\mathbb{M}_0 &= \{\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2 : \tau_0(\mathcal{E}_1) = \tau_0(\mathcal{E}_2)\} \\ &= \bigsqcup_{\substack{\alpha=\alpha_1+\alpha_2 \\ \tau_0(\alpha_1)=\tau_0(\alpha_2)}} \mathfrak{M}_{\alpha_1}^{\mathrm{sst}}(\tau_0) \times \mathfrak{M}_{\alpha_2}^{\mathrm{sst}}(\tau_0)\end{aligned}$$

Joyce's vertex algebra

(We can work around the simple wall condition [Mochizuki].)

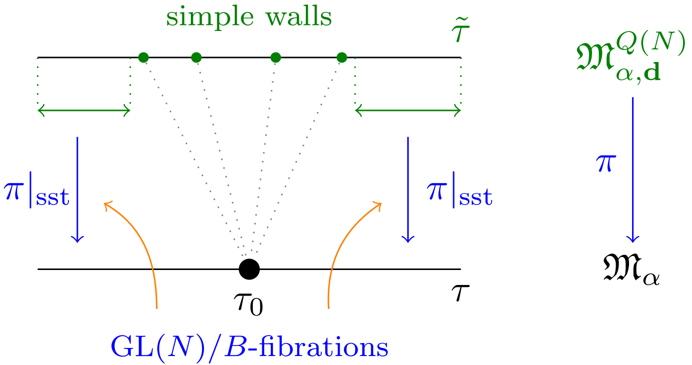
Joyce's vertex algebra

(We can work around the simple wall condition [Mochizuki].)



Joyce's vertex algebra

(We can work around the simple wall condition [Mochizuki].)



Joyce's vertex algebra

Let $\mathbb{M}_0^{\alpha_1,\alpha_2} := \mathfrak{M}_{\alpha_1}^{\mathrm{sst}}(\tau_0) \times \mathfrak{M}_{\alpha_2}^{\mathrm{sst}}(\tau_0)$ for short.

Theorem (Joyce, '21)

Let $Y(-,u)\colon H_*(\mathfrak{M}_{\alpha_1}) \otimes H_*(\mathfrak{M}_{\alpha_2}) \to H_*(\mathfrak{M}_\alpha)((u))$ be the natural operation such that

$$\int_{\mathbb{M}_0^{\alpha_1,\alpha_2}} \frac{\xi|_{\mathbb{M}_0^{\alpha_1,\alpha_2}}}{e(\mathcal{N}_{\mathbb{M}_0^{\alpha_1,\alpha_2}/\mathbb{M}})} = \Big(Y([\mathfrak{M}_{\alpha_1}^{\mathrm{sst}}(\tau_0)], u)[\mathfrak{M}_{\alpha_2}^{\mathrm{sst}}(\tau_0)] \Big) \cap \xi.$$

Then $H_*(\mathfrak{M})$ has a vertex algebra structure with state-field correspondence $Y(-,u)$.

An explicit formula for $Y(-, u)$

Let

$$\oplus\colon \mathfrak{M}_\alpha \times \mathfrak{M}_\beta \to \mathfrak{M}_{\alpha+\beta}$$
$$\Psi\colon [\mathrm{pt}/\mathbb{C}^\times] \times \mathfrak{M}_\alpha \to \mathfrak{M}_\alpha$$

be the direct sum and the scaling of automorphisms on objects.

An explicit formula for $Y(-, u)$

Let

$$\oplus\colon \mathfrak{M}_\alpha \times \mathfrak{M}_\beta \to \mathfrak{M}_{\alpha+\beta}$$
$$\Psi\colon [\mathrm{pt}/\mathbb{C}^\times] \times \mathfrak{M}_\alpha \to \mathfrak{M}_\alpha$$

be the direct sum and the scaling of automorphisms on objects. Then

$$(Y(a, u)b) \cap \xi := (a \boxtimes b) \cap \left(\frac{1}{e(\mathcal{N}_\oplus)} \cup (\Psi \times \mathrm{id})^* \oplus^* \xi \right).$$

An explicit formula for $Y(-, u)$

Let

$$\oplus\colon \mathfrak{M}_\alpha \times \mathfrak{M}_\beta \to \mathfrak{M}_{\alpha+\beta}$$
$$\Psi\colon [\mathrm{pt}/\mathbb{C}^\times] \times \mathfrak{M}_\alpha \to \mathfrak{M}_\alpha$$

be the direct sum and the scaling of automorphisms on objects. Then

$$(Y(a,u)b) \cap \xi := (a \boxtimes b) \cap \left(\frac{1}{e(\mathcal{N}_\oplus)} \cup (\Psi \times \mathrm{id})^* \oplus^* \xi \right).$$

Dually, in homology,

$$Y(a,u)b := \oplus_*(\Psi \times \mathrm{id})_* \left((a \boxtimes b) \cap \frac{1}{e(\mathcal{N}_\oplus)} \right).$$

An explicit formula for $Y(-, u)$

Let

$$\oplus \colon \mathfrak{M}_\alpha \times \mathfrak{M}_\beta \to \mathfrak{M}_{\alpha+\beta}$$
$$\Psi \colon [\mathrm{pt}/\mathbb{C}^\times] \times \mathfrak{M}_\alpha \to \mathfrak{M}_\alpha$$

be the direct sum and the scaling of automorphisms on objects. Then

$$(Y(a,u)b) \cap \xi := (a \boxtimes b) \cap \left(\frac{1}{e(\mathcal{N}_\oplus)} \cup (\Psi \times \mathrm{id})^* \oplus^* \xi \right).$$

Dually, in homology,

$$Y(a,u)b := \oplus_* (\Psi \times \mathrm{id})_* \left((a \boxtimes b) \cap \frac{1}{e(\mathcal{N}_\oplus)} \right).$$

(where $\Psi_* \colon H_*([\mathrm{pt}/\mathbb{C}^\times]) \otimes H_*(\mathfrak{M}_\alpha) \to H_*(\mathfrak{M}_\alpha)$)

An explicit formula for $Y(-, u)$

Let

$$\oplus \colon \mathfrak{M}_\alpha \times \mathfrak{M}_\beta \to \mathfrak{M}_{\alpha+\beta}$$
$$\Psi \colon [\mathrm{pt}/\mathbb{C}^\times] \times \mathfrak{M}_\alpha \to \mathfrak{M}_\alpha$$

be the direct sum and the scaling of automorphisms on objects. Then

$$(Y(a,u)b) \cap \xi := (a \boxtimes b) \cap \left(\frac{1}{e(\mathcal{N}_\oplus)} \cup (\Psi \times \mathrm{id})^* \oplus^* \xi \right).$$

Dually, in homology,

$$Y(a,u)b := \oplus_* (\Psi \times \mathrm{id})_* \left((a \boxtimes b) \cap \frac{1}{e(\mathcal{N}_\oplus)} \right).$$

(where $\Psi_* \colon H_*(\mathfrak{M}_\alpha) \to H_*(\mathfrak{M}_\alpha) \widehat{\otimes} H^*([\mathrm{pt}/\mathbb{C}^\times]))$

An explicit formula for $Y(-,u)$

Let

$$\oplus\colon \mathfrak{M}_\alpha \times \mathfrak{M}_\beta \to \mathfrak{M}_{\alpha+\beta}$$
$$\Psi\colon [\mathrm{pt}/\mathbb{C}^\times] \times \mathfrak{M}_\alpha \to \mathfrak{M}_\alpha$$

be the direct sum and the scaling of automorphisms on objects. Then

$$(Y(a,u)b) \cap \xi := (a \boxtimes b) \cap \left(\frac{1}{e(\mathcal{N}_\oplus)} \cup (\Psi \times \mathrm{id})^* \oplus^* \xi \right).$$

Dually, in homology,

$$Y(a,u)b := \oplus_* (\Psi \times \mathrm{id})_* \left((a \boxtimes b) \cap \frac{1}{e(\mathcal{N}_\oplus)} \right).$$

(where $\Psi_*\colon H_*(\mathfrak{M}_\alpha) \to H_*(\mathfrak{M}_\alpha)[[u]]$)

An explicit formula for $Y(-, u)$

Exercise (Proof of Joyce's theorem)

Verify, using this explicit formula, that $Y(-, u)$ satisfies the vertex algebra axioms.

An explicit formula for $Y(-, u)$

Exercise (Proof of Joyce's theorem)

Verify, using this explicit formula, that $Y(-, u)$ satisfies the vertex algebra axioms.

This is purely formal and uses only that the normal bundle $\mathcal{N}_{\oplus}$ is a bilinear function of $\mathbb{C}^{\times}$-weight 1 objects.

An explicit formula for $Y(-, u)$

Example

Let $\mathcal{E}$ be the universal sheaf on $\mathfrak{M} \times X$. Then

$$\tau(\gamma) := \mathrm{pr}_{1*}(\mathrm{ch}(\mathcal{E}) \cup \mathrm{pr}_2^*(\gamma)) \in H^*(\mathfrak{M})$$

are descendent classes.

An explicit formula for $Y(-, u)$

Example

Let $\mathcal{E}$ be the universal sheaf on $\mathfrak{M} \times X$. Then

$$\tau(\gamma) := \mathrm{pr}_{1*}(\mathrm{ch}(\mathcal{E}) \cup \mathrm{pr}_2^*(\gamma)) \in H^*(\mathfrak{M})$$

are descendent classes. We have

$$\begin{aligned} \oplus^* \mathcal{E} &= \mathrm{pr}_{12}^* \mathcal{E} \oplus \mathrm{pr}_{13}^* \mathcal{E} \\ \Psi^* \mathcal{E} &= \mathcal{L} \boxtimes \mathcal{E} \end{aligned}$$

An explicit formula for $Y(-,u)$

Example

Let $\mathcal{E}$ be the universal sheaf on $\mathfrak{M} \times X$. Then

$$\tau(\gamma) := \mathrm{pr}_{1*}(\mathrm{ch}(\mathcal{E}) \cup \mathrm{pr}_2^*(\gamma)) \in H^*(\mathfrak{M})$$

are descendent classes. We have

$$\begin{aligned} \oplus^* \tau(\gamma) &= \tau(\gamma) \boxtimes 1 + 1 \boxtimes \tau(\gamma) \\ \Psi^* \tau(\gamma) &= e^u \tau(\gamma). \end{aligned}$$

An explicit formula for $Y(-, u)$

Theorem (J. Gross, '19)

Suppose X is of class D; roughly:

- $\mathfrak{M} = \bigsqcup_{\alpha \in K_0^{\mathrm{Top}}(X)} \mathfrak{M}_\alpha$ *is a decomposition into connected components;*
- $H^*(\mathfrak{M}_\alpha)$ *is a free $\mathbb{Q}$-algebra generated by descendents.*

An explicit formula for $Y(-, u)$

Theorem (J. Gross, '19)

Suppose X is of class D; roughly:

- $\mathfrak{M} = \bigsqcup_{\alpha \in K_0^{\mathrm{Top}}(X)} \mathfrak{M}_\alpha$ *is a decomposition into connected components;*
- $H^*(\mathfrak{M}_\alpha)$ *is a free $\mathbb{Q}$-algebra generated by descendents.*

Then Joyce's vertex algebra on $H_*(\mathfrak{M})$ is a (super-)lattice vertex algebra on the (super-)lattice $(K_0^{\mathrm{Top}}(X) \oplus K_1^{\mathrm{Top}}(X), \chi)$.

An explicit formula for $Y(-, u)$

Theorem (J. Gross, '19)

Suppose X is of class D; roughly:

- $\mathfrak{M} = \bigsqcup_{\alpha \in K_0^{\mathrm{Top}}(X)} \mathfrak{M}_\alpha$ *is a decomposition into connected components;*
- $H^*(\mathfrak{M}_\alpha)$ *is a free $\mathbb{Q}$-algebra generated by descendents.*

Then Joyce's vertex algebra on $H_*(\mathfrak{M})$ is a (super-)lattice vertex algebra on the (super-)lattice $(K_0^{\mathrm{Top}}(X) \oplus K_1^{\mathrm{Top}}(X), \chi)$.

Proof.

By reconstruction theorem and explicit calculation. □

Joyce's vertex algebra

Problem

Where is the conformal field theory? What is the physical meaning of all this?

Lifting from cohomology to K-theory

The "cohomology to K-theory" dictionary:

Lifting from cohomology to K-theory

The "cohomology to K-theory" dictionary:

$$
\begin{array}{c|}
H_*^{\mathrm{BM}}(\mathfrak{M}) \\
\int_M \xi \\
e(\mathcal{N}) = \prod_{\mathcal{L}} c_1(\mathcal{L}) \\
H_*(\mathfrak{M})
\end{array}
$$

Lifting from cohomology to K-theory

The “cohomology to K-theory” dictionary:

$H_*^{\mathrm{BM}}(\mathfrak{M})$	$K(\mathfrak{M}) := K_0(\mathrm{Coh}(\mathfrak{M}))$
$\int_M \xi$	$\chi(M, \mathcal{F})$
$e(\mathcal{N}) = \prod_{\mathcal{L}} c_1(\mathcal{L})$	$\wedge_{-1}^{\bullet}(\mathcal{N}^{\vee}) = \prod_{\mathcal{L}}(1 - \mathcal{L}^{\vee})$
$H_*(\mathfrak{M})$	a K-homology group $K_{\circ}(\mathfrak{M})$

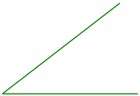
Lifting from cohomology to K-theory

The "cohomology to equivariant K-theory" dictionary:

$H_*^{\mathrm{BM}}(\mathfrak{M})$	$K_{\mathsf{T}}(\mathfrak{M}) := K_0(\mathrm{Coh}_{\mathsf{T}}(\mathfrak{M}))$
$\int_M \xi$	$\chi(M, \mathcal{F})$
$e(\mathcal{N}) = \prod_{\mathcal{L}} c_1(\mathcal{L})$	$\wedge^{\bullet}_{-1}(\mathcal{N}^{\vee}) = \prod_{\mathcal{L}}(1 - \mathcal{L}^{\vee})$
$H_*(\mathfrak{M})$	a K-homology group $K_{\circ}^{\mathsf{T}}(\mathfrak{M})$

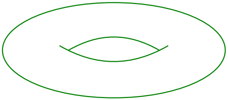
$\mathsf{T} = (\mathbb{C}^{\times})^r$ ($r \geq 0$) is a split torus.

Lifting from cohomology to K-theory



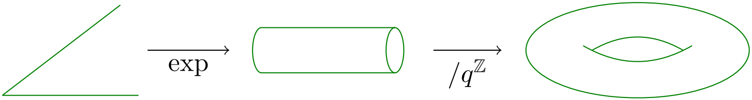
$\operatorname{Spec} H_{\mathbb{C}^\times}(\mathrm{pt})$
$= \mathbb{C}$

$\operatorname{Spec} K_{\mathbb{C}^\times}(\mathrm{pt})$
$= \mathbb{C}^\times$



$\mathrm{Ell}_{\mathbb{C}^\times}(\mathrm{pt})$
$= E$

Lifting from cohomology to K-theory

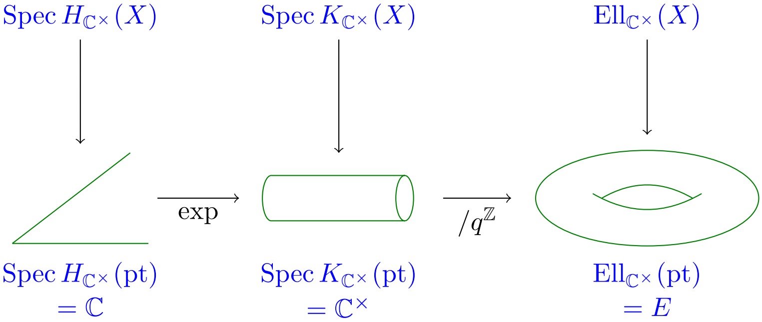


$\operatorname{Spec} H_{\mathbb{C}^\times}(\mathrm{pt}) = \mathbb{C}$

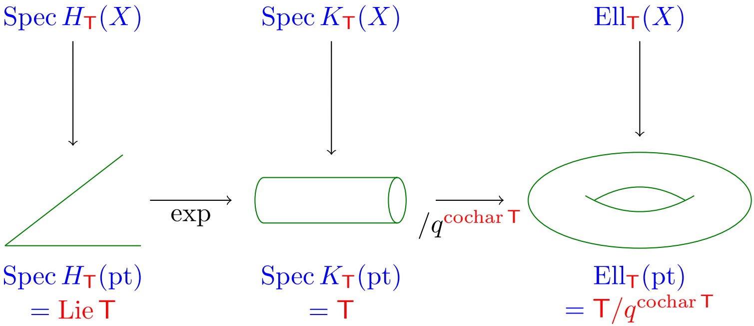
$\operatorname{Spec} K_{\mathbb{C}^\times}(\mathrm{pt}) = \mathbb{C}^\times$

$\mathrm{Ell}_{\mathbb{C}^\times}(\mathrm{pt}) = E$

Lifting from cohomology to K-theory



Lifting from cohomology to K-theory



Joyce's vertex algebra, in equivariant K-theory

Theorem (L. '22)

Let
$Y(-,z)\colon K_\circ^{\mathsf{T}}(\mathfrak{M}_{\alpha_1}) \otimes K_\circ^{\mathsf{T}}(\mathfrak{M}_{\alpha_2}) \to K_\circ^{\mathsf{T}}(\mathfrak{M}_{\alpha})[\![1-z]\!][(1-zt^{\mu})^{-1}]$
be the natural operation such that

$$\chi\left(\mathbb{M}_0^{\alpha_1,\alpha_2}, \frac{\mathcal{F}\big|_{\mathbb{M}_0^{\alpha_1,\alpha_2}}}{\wedge_{-1}^{\bullet}(\mathcal{N}^{\vee}_{\mathbb{M}_0^{\alpha_1,\alpha_2}/\mathbb{M}})}\right) = \left(Y(\mathsf{Z}_{\alpha_1}(\tau_0), z)\mathsf{Z}_{\alpha_2}(\tau_0)\right)(\mathcal{F}).$$

Then $K_\circ^{\mathsf{T}}(\mathfrak{M})$ *has a equivariant multiplicative vertex algebra structure with state-field correspondence* $Y(-,z)$.

Equivariant multiplicative vertex algebras

Definition

A vertex algebra is the data of:

- a $\mathbb{Z}$-module V and a vacuum vector $\mathbf{1} \in V$;
- a translation operator $D(u)\colon V \to V[\![u]\!]$;
- a state-field correspondence

$$Y(-, u)\colon V \otimes V \to V((u)).$$

This data must satisfy the axioms:

1. (vacuum) $Y(\mathbf{1}, u) = \mathrm{id}$ and $Y(a, u)\mathbf{1} = D(u)a$;
2. (skew symmetry) $Y(a, u)b = D(u)Y(b, u)a$;
3. (weak associativity) $Y(Y(a, u)b, v) \equiv Y(a, u + v)Y(b, v)$

Equivariant multiplicative vertex algebras

Definition

A vertex algebra is the data of:

- a $\mathbb{Z}$-module V and a vacuum vector $\mathbf{1} \in V$;
- a translation operator $D(u)\colon V \to V[\![u]\!]$;
- a state-field correspondence

$$Y(-,u)\colon V \otimes V \to V((u)).$$

This data must satisfy the axioms:

1. (vacuum) $Y(\mathbf{1},u) = \mathrm{id}$ and $Y(a,u)\mathbf{1} = D(u)a$;
2. (skew symmetry) $Y(a,u)b = D(u)Y(b,u)a$;
3. (weak associativity) $Y(Y(a,u)b,v) \equiv Y(a,u+v)Y(b,v)$ where $\equiv$ means matrix elements of both sides are the expansion of the same element in

$$\mathbb{Z}[\![u,v]\!]\left[u^{-1},v^{-1},(u-v)^{-1}\right].$$

Equivariant multiplicative vertex algebras

Definition

A vertex algebra is the data of:

- a $\mathbb{Z}$-module V and a vacuum vector $\mathbf{1} \in V$;
- a translation operator $D(u)\colon V \to V[\![u]\!]$;
- a state-field correspondence

$$Y(-,u)\colon V \otimes V \to V[\![u]\!][u^{-1}].$$

This data must satisfy the axioms:

1. (vacuum) $Y(\mathbf{1},u) = \mathrm{id}$ and $Y(a,u)\mathbf{1} = D(u)a$;
2. (skew symmetry) $Y(a,u)b = D(u)Y(b,u)a$;
3. (weak associativity) $Y(Y(a,u)b,v) \equiv Y(a,u+v)Y(b,v)$ where $\equiv$ means matrix elements of both sides are the expansion of the same element in

$$\mathbb{Z}[\![u,v]\!]\left[u^{-1}, v^{-1}, (u-v)^{-1}\right].$$

Equivariant multiplicative vertex algebras

Definition (L. '22)

A T-equivariant multiplicative vertex algebra is the data of:

- a $\mathbb{Z}[t^\mu]$-module V and a vacuum vector $\mathbf{1} \in V$;
- a translation operator $D(z)\colon V \to V[\![1-z]\!]$;
- a state-field correspondence

$$Y(-,z)\colon V \otimes V \to V[\![1-z]\!]\left[(1-z^i t^\mu)^{-1}\right].$$

This data must satisfy the axioms:

1. (vacuum) $Y(\mathbf{1}, z) = \mathrm{id}$ and $Y(a,z)\mathbf{1} = D(z)a$;
2. (skew symmetry) $Y(a,z)b = D(z)Y(b,z)a$;
3. (weak associativity) $Y(Y(a,z)b, w) \equiv Y(a, zw)Y(b,w)$ where $\equiv$ means matrix elements of both sides are the multiplicative expansion of the same element in

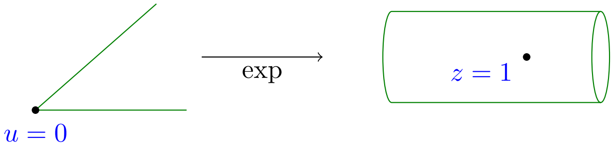
$$\mathbb{Z}[\![1-z, 1-w]\!]\left[(1-z^i t^\mu)^{-1}, (1-w^j t^\nu)^{-1}, (1-z^i w^j t^\rho)^{-1}\right].$$

Equivariant multiplicative vertex algebras

In other words, singularities exist at:

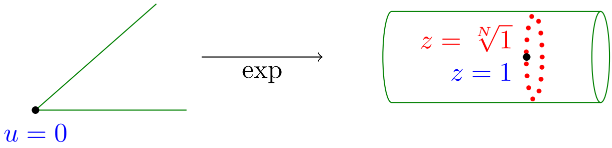
Equivariant multiplicative vertex algebras

In other words, singularities exist at:



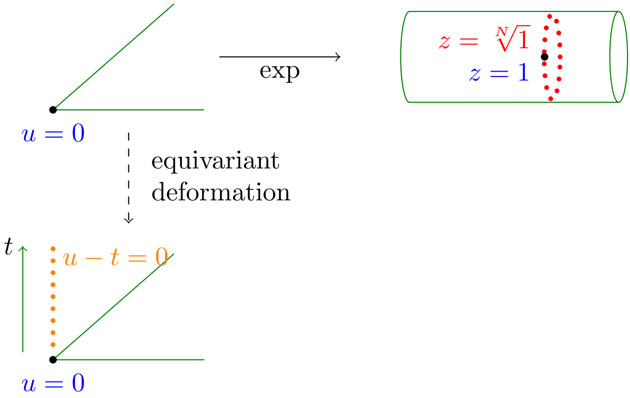
Equivariant multiplicative vertex algebras

In other words, singularities exist at:



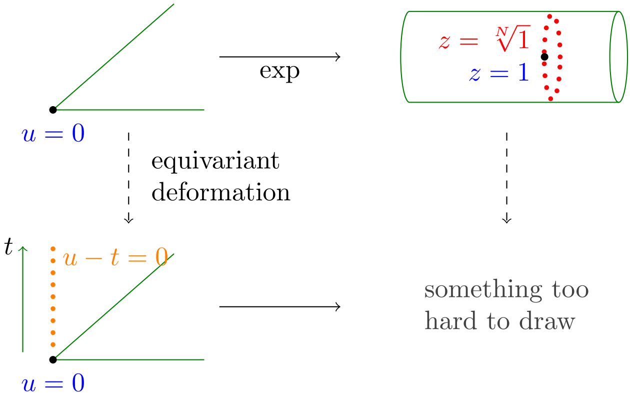
Equivariant multiplicative vertex algebras

In other words, singularities exist at:



Equivariant multiplicative vertex algebras

In other words, singularities exist at:



Equivariant multiplicative vertex algebras

Many familiar structures/results lift, e.g.:

Equivariant multiplicative vertex algebras

Many familiar structures/results lift, e.g.:

- (locality) $Y(a, z)Y(b, w) \equiv Y(b, w)Y(a, z)$;

Equivariant multiplicative vertex algebras

Many familiar structures/results lift, e.g.:

- (locality) $Y(a,z)Y(b,w) \equiv Y(b,w)Y(a,z)$;
- (OPE)

$$:Y(a,z)Y(b,w):\; := \begin{array}{l} \text{singular terms } \frac{f(w)}{(1-z^i w^j t^\mu)^n} \\ \text{in } Y(Y(a,zw)b,w) \end{array}$$

Equivariant multiplicative vertex algebras

But there is still much work that can be done!

Equivariant multiplicative vertex algebras

But there is still much work that can be done!

Problem

Define a multiplicative ("quantized"?) Zhu algebra and associated variety.

Equivariant multiplicative vertex algebras

But there is still much work that can be done!

Problem

Define a multiplicative ("quantized"?) Zhu algebra and associated variety.

Problem

Prove a reconstruction theorem. What is a (equivariant) multiplicative lattice vertex algebra?

Equivariant multiplicative vertex algebras

But there is still much work that can be done!

Problem

Define a multiplicative ("quantized"?) Zhu algebra and associated variety.

Problem

Prove a reconstruction theorem. What is a (equivariant) multiplicative lattice vertex algebra?

Problem (Possibly harder...)

Does it make sense to introduce a conformal element? What are equivariant multiplicative/K-theoretic Virasoro constraints?

K-homology groups

What is $K_\circ^{\mathsf{T}}(\mathfrak{M})$?

K-homology groups

What is $K_\circ^{\mathsf{T}}(\mathfrak{M})$? For everything to work, must:

- admit pushforwards along arbitrary (T-equivariant) morphisms, and external tensor product;

K-homology groups

What is $K_\circ^{\mathsf{T}}(\mathfrak{M})$? For everything to work, must:

- admit pushforwards along arbitrary (T-equivariant) morphisms, and external tensor product;
- satisfy $K_\circ([\mathrm{pt}/\mathbb{C}^\times]) = \mathbb{Z}[\phi]$

K-homology groups

What is $K_\circ^{\mathsf{T}}(\mathfrak{M})$? For everything to work, must:

- admit pushforwards along arbitrary (T-equivariant) morphisms, and external tensor product;
- satisfy $K_\circ([\mathrm{pt}/\mathbb{C}^\times]) = \mathbb{Z}[\phi]$ (not a typo);

K-homology groups

What is $K_\circ^{\mathsf{T}}(\mathfrak{M})$? For everything to work, must:

- admit pushforwards along arbitrary (T-equivariant) morphisms, and external tensor product;
- satisfy $K_\circ([\mathrm{pt}/\mathbb{C}^\times]) = \mathbb{Z}[\phi]$ (not a typo);
- allow for equivariant localization;

K-homology groups

What is $K_\circ^{\mathsf{T}}(\mathfrak{M})$? For everything to work, must:

- admit pushforwards along arbitrary (T-equivariant) morphisms, and external tensor product;
- satisfy $K_\circ([\mathrm{pt}/\mathbb{C}^\times]) = \mathbb{Z}[\phi]$ (not a typo);
- allow for equivariant localization;
- (!) have a (non-equivariant) evaluation pairing $K_\circ(\mathfrak{M}) \otimes K^\circ(\mathfrak{M}) \to K^\circ(\mathrm{pt})$ such that

$$\phi \otimes (1 - \mathcal{L})^{\otimes N} \mapsto 0 \quad \forall N \gg 0.$$

Here $K^\circ(\mathfrak{M}) := K_0(D_{\mathsf{Perf}}(\mathfrak{M}))$.

K-homology groups

Definition (L. '22)

Roughly,

$$K_\circ^{\mathsf{T}}(\mathfrak{M}) \subset \mathrm{Hom}_{K_{\mathsf{T}}(\mathrm{pt})}(K^\circ_{\mathsf{T}}(\mathfrak{M}), K_{\mathsf{T}}(\mathrm{pt}))$$

is the largest $K_{\mathsf{T}}(\mathrm{pt})$-submodule that satisfies all these properties.

K-homology groups

Definition (L. '22)

Roughly,

$$K_\circ^{\mathsf{T}}(\mathfrak{M}) \subset \mathrm{Hom}_{K_{\mathsf{T}}(\mathrm{pt})}(K^\circ_{\mathsf{T}}(\mathfrak{M}), K_{\mathsf{T}}(\mathrm{pt}))$$

is the largest $K_{\mathsf{T}}(\mathrm{pt})$-submodule that satisfies all these properties. So it contains "universal" enumerative invariants

$$\mathsf{Z}_\alpha(\tau) := \left[\mathcal{E} \mapsto \chi(\mathfrak{M}_\alpha^{\mathrm{sst}}(\tau), \mathcal{E})\right]$$

whenever $\mathfrak{M}_\alpha^{\mathrm{sst}}(\tau)$ has no strictly τ-semistable objects.

K-homology groups

Definition (L. '22)

Roughly,

$$K_\circ^{\mathsf{T}}(\mathfrak{M}) \subset \mathrm{Hom}_{K_{\mathsf{T}}(\mathrm{pt})}(K^\circ_{\mathsf{T}}(\mathfrak{M}), K_{\mathsf{T}}(\mathrm{pt}))$$

is the largest $K_{\mathsf{T}}(\mathrm{pt})$-submodule that satisfies all these properties.
So it contains "universal" enumerative invariants

$$\mathsf{Z}_\alpha(\tau) := \left[\mathcal{E} \mapsto \chi(\mathfrak{M}_\alpha^{\mathrm{sst}}(\tau), \mathcal{E} \otimes \mathcal{O}^{\mathrm{vir}})\right]$$

whenever $\mathfrak{M}_\alpha^{\mathrm{sst}}(\tau)$ has no strictly τ-semistable objects.

K-homology groups

Definition (L. '22)

Roughly,

$$K_\circ^{\mathsf{T}}(\mathfrak{M}) \subset \mathrm{Hom}_{K_{\mathsf{T}}(\mathrm{pt})}(K^\circ_{\mathsf{T}}(\mathfrak{M}), K_{\mathsf{T}}(\mathrm{pt}))$$

is the largest $K_{\mathsf{T}}(\mathrm{pt})$-submodule that satisfies all these properties. So it contains "universal" enumerative invariants

$$\mathsf{Z}_\alpha(\tau) := \left[\mathcal{E} \mapsto \chi(\mathfrak{M}_\alpha^{\mathrm{sst}}(\tau), \mathcal{E} \otimes \mathcal{O}^{\mathrm{vir}})\right]$$

whenever $\mathfrak{M}_\alpha^{\mathrm{sst}}(\tau)$ has no strictly τ-semistable objects.

This module is basically uncomputable.

K-homology groups

Problem

Find a smaller, more explicitly computable $K_{\mathsf{T}}(\mathrm{pt})$*-module* $K_{\circ}^{\mathsf{T}}(\mathfrak{M})$ *which still satisfies all the desired properties.*

K-homology groups

Problem

Find a smaller, more explicitly computable $K_{\mathsf{T}}(\mathrm{pt})$-module $K_{\circ}^{\mathsf{T}}(\mathfrak{M})$ which still satisfies all the desired properties.

In particular, it should still carry a equivariant multiplicative vertex algebra structure, and contain all the invariants $\mathsf{Z}_{\alpha}(\tau)$.

K-homology groups

Problem

Find a smaller, more explicitly computable $K_{\mathsf{T}}(\mathrm{pt})$*-module* $K_{\circ}^{\mathsf{T}}(\mathfrak{M})$ *which still satisfies all the desired properties.*

In particular, it should still carry a equivariant multiplicative vertex algebra structure, and contain all the invariants $\mathsf{Z}_{\alpha}(\tau)$*.*

What is the analogue of Gross's theorem?

Universal wall-crossing formulas

Returning to wall-crossing...

Proposition (Borcherds '86)

Let $(V, \mathbf{1}, D(u), Y(-,u))$ *be a vertex algebra. Then*

$$[a,b] := \operatorname{Res}_u Y(a,u)b\,du$$

is a Lie bracket on $V / \operatorname{im}(\operatorname{id} - D(u))$*.*

Universal wall-crossing formulas

Returning to wall-crossing...

Proposition (L. '22)

Let $(V, \mathbf{1}, D(z), Y(-, z))$ *be an equivariant multiplicative vertex algebra. Suppose it is reduced, i.e.* $Y(-, z)$ *takes values in*

$$V[\![1-z]\!]\left[(1 - z t^\mu)^{-1}\right] \subset V[\![1-z]\!]\left[(1 - z^i t^\mu)^{-1}\right].$$

Then

$$[a, b] \coloneqq \sum_\mu \operatorname{Res}_{zt^\mu = 1} Y(a, z) b \, \frac{dz}{z}$$

is a Lie bracket on $V / \operatorname{im}(\operatorname{id} - D(z))$.

Universal wall-crossing formulas

Theorem (Joyce '21, L. '22)

For $\dim X \le 2$ *and appropriately-defined enumerative invariants* $\mathsf{Z}_\alpha(\tau)$, *in (equivariant) (K-)homology,*

$$\mathsf{Z}_\alpha(\tau') = \sum_{\substack{n\geq 1 \\ \alpha_1+\cdots+\alpha_n=\alpha \\ \forall i:\, \tau(\alpha_i)=\tau(\alpha)}} \widetilde{U}(\alpha_1,\ldots,\alpha_n;\tau,\tau')\cdot \left[\left[\cdots\left[\mathsf{Z}_{\alpha_1}(\tau),\mathsf{Z}_{\alpha_2}(\tau)\right],\ldots\right],\mathsf{Z}_{\alpha_n}(\tau)\right]$$

where the $\widetilde{U}$ *are universal wall-crossing coefficients.*

Universal wall-crossing formulas

Theorem (Joyce '21, L. '22)

For $\dim X \leq 2$ *and appropriately-defined enumerative invariants* $\mathsf{Z}_\alpha(\tau)$*, in (equivariant) (K-)homology,*

$$\mathsf{Z}_\alpha(\tau') = \sum_{\substack{n\geq 1\\ \alpha_1+\cdots+\alpha_n=\alpha\\ \forall i:\, \tau(\alpha_i)=\tau(\alpha)}} \widetilde{U}(\alpha_1,\ldots,\alpha_n;\tau,\tau')\cdot \left[\left[\cdots\left[\mathsf{Z}_{\alpha_1}(\tau),\mathsf{Z}_{\alpha_2}(\tau)\right],\ldots\right],\mathsf{Z}_{\alpha_n}(\tau)\right]$$

where the $\widetilde{U}$ *are universal wall-crossing coefficients.*

If $\mathfrak{M}_\alpha^{\mathrm{sst}}(\tau)$ has no strictly semistable objects, then in homology,

$$\mathsf{Z}_\alpha(\tau) = [\mathfrak{M}_\alpha^{\mathrm{sst}}(\tau)].$$

Universal wall-crossing formulas

Theorem (Joyce '21, L. '22)

For $\dim X \leq 2$ *and appropriately-defined enumerative invariants* $\mathsf{Z}_\alpha(\tau)$*, in (equivariant) (K-)homology,*

$$\mathsf{Z}_\alpha(\tau') = \sum_{\substack{n \geq 1 \\ \alpha_1 + \cdots + \alpha_n = \alpha \\ \forall i \colon \tau(\alpha_i) = \tau(\alpha)}} \widetilde{U}(\alpha_1, \ldots, \alpha_n; \tau, \tau') \cdot [[\cdots[\mathsf{Z}_{\alpha_1}(\tau), \mathsf{Z}_{\alpha_2}(\tau)], \ldots], \mathsf{Z}_{\alpha_n}(\tau)]$$

where the $\widetilde{U}$ *are universal wall-crossing coefficients.*

If $\mathfrak{M}_\alpha^{\mathrm{sst}}(\tau)$ has no strictly semistable objects, then in homology,

$$\mathsf{Z}_\alpha(\tau) = [\mathfrak{M}_\alpha^{\mathrm{sst}}(\tau)]^{\mathrm{vir}}.$$

Universal wall-crossing formulas

Theorem (Kuhn–L.–Thimm, early '25)

The same universal wall-crossing formula also holds for $\dim X = 3$*, assuming the moduli stack* $\mathfrak{M}$ *has a symmetric obstruction theory*

Universal wall-crossing formulas

Theorem (Kuhn–L.–Thimm, early '25)

The same universal wall-crossing formula also holds for $\dim X = 3$*, assuming the moduli stack* $\mathfrak{M}$ *has a symmetric obstruction theory (*$\approx$ *is locally a critical locus).*

Universal wall-crossing formulas

Theorem (Kuhn–L.–Thimm, early '25)

The same universal wall-crossing formula also holds for $\dim X = 3$, *assuming the moduli stack* $\mathfrak{M}$ *has a symmetric obstruction theory (*$\approx$ *is locally a critical locus).*

Theorem (Bojko–Kuhn–L.–Thimm, later in '25)

The same also holds for $\dim X = 4$, *assuming (???).*

Universal wall-crossing formulas

Defining and computing the "coproduct"

$$[a, b] \cap \xi =: (a \boxtimes b) \cap \Delta\xi,$$

and its iterates $\Delta^{k+1} := (\Delta \otimes \mathrm{id})^k \Delta$, is therefore of utmost importance for wall-crossing applications.

Universal wall-crossing formulas

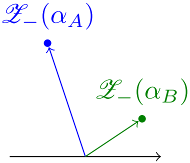
Example

Let α_A and α_B be classes such that:

Universal wall-crossing formulas

Example

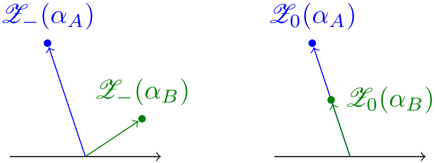
Let α_A and α_B be classes such that: ($\tau = \arg \mathscr{Z}$)



Universal wall-crossing formulas

Example

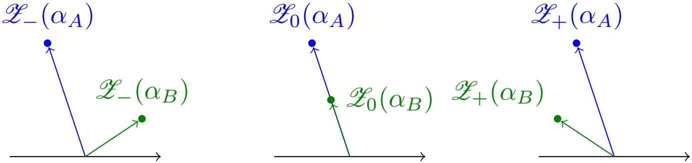
Let α_A and α_B be classes such that: ($\tau = \arg \mathscr{Z}$)



Universal wall-crossing formulas

Example

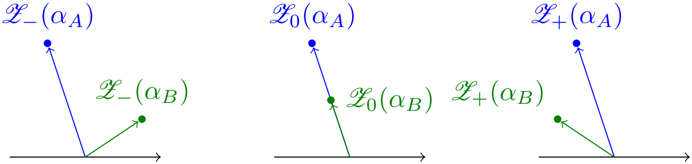
Let α_A and α_B be classes such that: ($\tau = \arg \mathscr{Z}$)



Universal wall-crossing formulas

Example

Let α_A and α_B be classes such that: ($\tau = \arg \mathscr{Z}$)



Simplest possible wall-crossing formula:

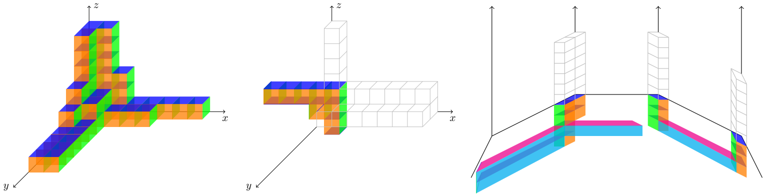
$$\mathsf{Z}_n(\tau_+) = \exp\left(\mathrm{ad}_{\mathsf{Z}_0(\tau_-)}\right) \mathsf{Z}_n(\tau_-)$$

where $\mathsf{Z}_n(\tau) \coloneqq \sum_{m \in \mathbb{Z}} Q^m \mathsf{Z}_{n\alpha_A + m\alpha_B}(\tau)$.

Universal wall-crossing formulas

Example (continued)

In fact, such wall-crossings occur in real life, e.g. in the study of vertices in Donaldson–Thomas-like theories.



Universal wall-crossing formulas

Example (continued)

Here, for any integer partitions λ, μ, ν:

$$\mathsf{Z}_{\lambda,\mu,\nu}(\tau^{\mathsf{DT}}) = \exp\left(\mathrm{ad}_{\mathsf{Z}_0(\tau^{\mathsf{PT}})}\right) \mathsf{Z}_{\lambda,\mu,\nu}(\tau^{\mathsf{PT}}).$$

Universal wall-crossing formulas

Example (continued)

Here, for any integer partitions λ, μ, ν:

$$\mathsf{Z}_{\lambda,\mu,\nu}(\tau^{\mathsf{DT}}) = \exp\left(\mathrm{ad}_{\mathsf{Z}_0(\tau^{\mathsf{PT}})}\right) \mathsf{Z}_{\lambda,\mu,\nu}(\tau^{\mathsf{PT}}).$$

Using $\Delta(1) = (\text{scalar}) \cdot (1 \boxtimes 1)$ and

$$\mathsf{Z}_{\emptyset,\emptyset,\emptyset}(\tau^{\mathsf{PT}})(\xi) = \begin{cases} 1 & \xi = 1 \\ 0 & \text{otherwise,} \end{cases}$$

Universal wall-crossing formulas

Example (continued)

Here, for any integer partitions λ, μ, ν:

$$\mathsf{Z}_{\lambda,\mu,\nu}(\tau^{\mathsf{DT}}) = \exp\left(\mathrm{ad}_{\mathsf{Z}_0(\tau^{\mathsf{PT}})}\right) \mathsf{Z}_{\lambda,\mu,\nu}(\tau^{\mathsf{PT}}).$$

Using $\Delta(1) = (\text{scalar}) \cdot (1 \boxtimes 1)$ and

$$\mathsf{Z}_{\emptyset,\emptyset,\emptyset}(\tau^{\mathsf{PT}})(\xi) = \begin{cases} 1 & \xi = 1 \\ 0 & \text{otherwise}, \end{cases}$$

one immediately obtains the
DT/PT vertex correspondence [Pandharipande–Thomas '09, Nekrasov–Okounkov '15, proved in Kuhn–L.–Thimm '23 (messy) '25 (cleaner)]

$$\begin{aligned} \mathsf{Z}_{\lambda,\mu,\nu}(\tau^{\mathsf{DT}})(1) &= \left(\exp\left(\mathrm{ad}_{\mathsf{Z}_0(\tau^{\mathsf{PT}})}\right) \mathsf{Z}_{\lambda,\mu,\nu}(\tau^{\mathsf{PT}})\right)(1) \\ &= \mathsf{Z}_{\emptyset,\emptyset,\emptyset}(\tau^{\mathsf{DT}})(1) \cdot \mathsf{Z}_{\lambda,\mu,\nu}(\tau^{\mathsf{PT}})(1) \end{aligned}$$

Universal wall-crossing formulas

Example (continued)

In fact, in this sort of setting,

$$\Delta(1) = \mathrm{rank}(\mathcal{N}_{\oplus}^{\mathrm{vir},\frac{1}{2}}) \cdot (1 \boxtimes 1)$$

so that the cohomological Lie bracket is

$$[a, b](1) = \mathrm{rank}(\mathcal{N}_{\oplus}^{\mathrm{vir},\frac{1}{2}}) a(1) b(1).$$

Universal wall-crossing formulas

Example (continued)

In fact, in this sort of setting,

$$\Delta(1) = [\,\mathrm{rank}(\mathcal{N}_{\oplus}^{\mathrm{vir},\frac{1}{2}})]_{\kappa} \cdot (1 \boxtimes 1)$$

so that the K-theoretic Lie bracket is

$$[a, b](1) = [\,\mathrm{rank}(\mathcal{N}_{\oplus}^{\mathrm{vir},\frac{1}{2}})]_{\kappa} a(1) b(1).$$

Universal wall-crossing formulas

Example (continued)

In fact, in this sort of setting,

$$\Delta(1) = [\,\mathrm{rank}(\mathcal{N}_{\oplus}^{\mathrm{vir},\frac{1}{2}})]_{\kappa} \cdot (1 \boxtimes 1)$$

so that the K-theoretic Lie bracket is

$$[a, b](1) = [\,\mathrm{rank}(\mathcal{N}_{\oplus}^{\mathrm{vir},\frac{1}{2}})]_{\kappa} a(1) b(1).$$

Here $[n]_\kappa$ is a quantum integer, quantizing n.

Vertex coalgebras and quantum groups

Theorem (L. '22)

Let $\curlywedge : V^* \to (V^* \otimes V^*)(((1-z)^{-1}))$ *be the natural operation such that*

$$Y(a,z)b \cap \xi =: (a \boxtimes b)\curlywedge \xi.$$

Then V^* *has a (multiplicative) vertex coalgebra structure with vertex coproduct* $\curlywedge$.

Vertex coalgebras and quantum groups

Theorem (L. '22)

Let $\curlywedge : V^* \to (V^* \otimes V^*)(((1-z)^{-1}))$ *be the natural operation such that*

$$Y(a,z)b \cap \xi =: (a \boxtimes b) \curlywedge \xi.$$

Then V^* *has a (multiplicative) vertex coalgebra structure with vertex coproduct* $\curlywedge$.

Theorem (Latyntsev '21 (cohomology), L. '22 ((critical) K-theory))

If V^* *has the structure of*

- *a (multiplicative) vertex coalgebra with vertex coproduct* $\curlywedge$, *and*
- *a (K-theoretic) Hall algebra with Hall product* $*$,

and both structures are defined using $\mathcal{N}_\oplus$, *then they are compatible.*

Vertex coalgebras and quantum groups

Problem

Can one use the quantum group to gain a better understanding of the vertex coproduct, beyond the compatibility theorem?

Vertex coalgebras and quantum groups

Problem

Can one use the quantum group to gain a better understanding of the vertex coproduct, beyond the compatibility theorem?

For instance, quantum group description of the "Ext operator" $e(\mathcal{N}_\oplus)$ or $\wedge^\bullet_{-1}(\mathcal{N}_\oplus^\vee)$ should not be difficult.

Finale: some ambitious problems

Problem

Does the (multipicative) vertex (co)algebra act on H^* or K of $\mathfrak{M}^{\mathrm{sst}}(\tau)$?

Finale: some ambitious problems

Problem

Does the (multipicative) vertex (co)algebra act on H^* or K of $\mathfrak{M}^{\mathrm{sst}}(\tau)$?

Possible application: enumerative invariants similar to $\mathsf{Z}_{\lambda,\mu,\nu}(\tau^{\mathsf{DT}})$ are conjectured to be modular or Jacobi forms (e.g. S-duality in Vafa–Witten theory); can vertex algebra techniques be used to attack such conjectures?

Finale: some ambitious problems

Problem

What is an (equivariant) elliptic vertex (co)algebra?

Finale: some ambitious problems

Problem

What is an (equivariant) elliptic vertex (co)algebra?

Possible application: better understanding of wall-crossing for invariants in elliptic cohomology.

Finale: some ambitious problems

Problem

What is an (equivariant) elliptic vertex (co)algebra?

Possible application: better understanding of wall-crossing for invariants in elliptic cohomology. E.g. topological conditions are known such that equivariant (virtual chiral) elliptic genus has trivial wall-crossing [L. '24].

Thank you!