Lecture Notes on Relative GW-invariants

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0 Introduction

The purpose of these lecture notes is to outline the recent development in applying degeneration method to study moduli spaces.

Degeneration method is one of the classical tools in algebraic geometry. Many people, including the Italian school hundred years ago, used this to solve problems in algebraic geometry. The method developed in [23] provides a unified treatment to degeneration of moduli problems. This method was first utilized by Gieseker in his proof of Ramanan-Newstead conjecture [11], and by Harris-Mumford in their study of admissible covers [16]. It is also an algebro-geometric analogue of the Donaldson-Floer theory in gauge theory, and the symplectic cut formulas proved by A. Li-Ruan, Ionel-Parker and Eliashberg-Givental-Hofer [26, 18, 9].

We begin with a brief outline of the degeneration of the moduli of stable morphisms. We consider a projective family $W \to C$ of schemes of which the total space is smooth, the general fibers W_t are smooth and the central fiber W_0 is the union of two smooth varieties Y_1 and Y_2 intersecting transversally along smooth divisors $D_1 \subset Y_1$ and $D_2 \subset Y_2$. For simplicity, in this note we will only work out the moduli of stable morphisms. Let g, n be two integers and let d be a homology class. We consider the moduli space (stack) $\mathfrak{M}_{g,k}(W_t, d)$ of stable morphisms from k-pointed arithmetic genus g curves to W_t of degree d. The family

$$\mathfrak{M}_{g,k}(W^\circ/C^\circ, d) = \coprod_{t \in C^\circ} \mathfrak{M}_{g,k}(W_t, d), \quad C^\circ = C - 0 W^0 = W \times_C C^0$$

is a proper family over C°. Inserting the central fiber $\mathfrak{M}_{g,k}(W_0, d)$ into this family provides us an extension which is a degeneration of moduli spaces. To effectively study the geometry of the moduli space $\mathfrak{M}_{g,k}(W_t, d)$ via this family, we need to have the total space

$$\mathfrak{M}_{g,k}(W/C, d) = \coprod_{t \in C} \mathfrak{M}_{g,k}(W_t, d)$$

smooth and the central fiber $\mathfrak{M}_{g,k}(W_0, d)$ with normal crossing singularity, at least in the "virtual" sense. Unfortunately, this new family fails to have such property.

In this note we will demonstrate a "geometric blow up" of this family. Here we call it a "blow up" because the new degeneration is a blow up of the original family $\mathfrak{M}_{g,k}(W_C, b)$ and we characterized it as "geometric" because the new degeneration is itself a relative moduli space. The latter is important because the points of the new degenerations are moduli objects and their local and global structures can be investigated via methods of moduli studying.

To obtain the desired degeneration we need to replace the *degenerate* stable morphisms by "non-degenerate" objects. Here a map is called degenerate if it maps some irreducible components of its domain to the singular loci of W_0. The solution worked out in [23] is to replace them by regular stable morphisms to *semistable* models of W_0, which are unions of Y_1 and Y_2 with a chain of ruled varieties Δ:

$$W_0[n] = Y_1 \cup \Delta \cup \cdots \cup \Delta \cup Y_2.$$

In this note, we will sketch the main idea of this construction. We will construct the moduli of relative stable morphisms. This moduli space has its own interest, as demonstrated in many recent applications. We will investigate the central fiber of the new degenerations and show how they are related to the moduli of relative stable morphisms. The degeneration formula of Gromov-Witten invariants will be a consequence of this degeneration, after one works out the machinery of virtual moduli cycle. We will conclude this note by providing some applications of this machinery.

1 Relative morphisms

Let W be a smooth projective variety. It is known that we can construct the GW-invariants of W via the (virtual) intersection theory on the moduli of stable morphisms to W. We fix integers g, k and an algebraic class $d \in H_2(W)$. We form the moduli space

$$\mathfrak{M}_{g,k}(W,d)$$

of stable morphisms to W of algebraic class d whose domains are arithmetic genus g curves with k marked points. We know that $\mathfrak{M}_{g,k}(W,d)$ is a proper, separated DM-stack, of expected dimension

$$\operatorname{exp.dim} \mathfrak{M}_{g,k}(W,d) = (3 - \dim W)(g-1) + k + d \cdot c_1(W).$$

Using the evaluation map

$$\mathrm{ev} : \mathfrak{M}_{g,k}(W,d) \longrightarrow W^k,$$

the GW-invariants are

$$\Phi^W_{g,k,d}(\beta) = \int_{[\mathfrak{M}_{g,k}(W,d)]^{\mathrm{vir}}} \mathrm{ev}^* \beta, \quad \beta \in H^*(W^n).$$

When $\mathfrak{M}_{g,k}(W,d)$ has pure dimension and is equal to the expected dimension, $[\mathfrak{M}_{g,k}(W,d)]^{\mathrm{vir}}$ is the fundamental class of $\mathfrak{M}_{g,k}(W,d)$. In this case the above reduces to enumerations of stable maps satisfying certain incidence relations. In general, we need to use the notion of virtual cycles $[\mathfrak{M}_{g,k}(W,d)]^{\mathrm{vir}}$, constructed by Li-Tian and Behrend-Fantechi [25, 2, 3].

The easiest non-trivial example is when $W = \mathbf{P}^2$, $g = 0$ and d a multiple of a line in $\mathbf{P}^2$. Then

$$\mathfrak{M}_{0,k}(\mathbf{P}^2, d)$$

is the space of all degree d stable $f : X \to \mathbf{P}^2$ with $g(X) = 0$. We pick k so that $\dim \mathfrak{M}_{0,k}(\mathbf{P}^2, d) = 2k$. We then pick k general points $q_1, \cdots, q_k \in \mathbf{P}^2$ and form the subset

$$\{f \in \mathfrak{M}_{0,k}(\mathbf{P}^2, d) \mid f(p_i) = q_i,\ i = 1, \cdots, k\} \subset \mathfrak{M}_{0,k}(\mathbf{P}^2, d).$$

It is easy to check that this is a smooth discrete set whose degree is a GW-invariant of $\mathbf{P}^2$. For instance, when $k = 3d - 1$, $\dim \mathfrak{M}_{0,k}(\mathbf{P}^2, d) = 2k$ and their corresponding GW-invariants are $N_d = \Phi^{\mathbf{P}^2}_{0,3d-1,d}(pt^\vee)$ with

$$N_1 = 1,\ N_2 = 1,\ N_3 = 12,\ N_4 = 620,\ N_5 = 87304,\ \cdots .$$

We now work out a simple degeneration as a warm up for the general construction. We begin with a degeneration of $\mathbf{P}^2$ to a reducible surface with two smooth components. A simple example is to let the total space of the degeneration W be the blowing up of a line $L \times 0$ in the product $\mathbf{P}^2 \times \mathbf{A}^1$ with $\pi : W \to \mathbf{A}^1$ the obvious projection. Clearly, $W_t \cong \mathbf{P}^2$ for $t \neq 0 \in \mathbf{A}^1$ while W_0 is the union of $\mathbf{P}^2$ with a Hirzebruch surface $\mathbf{H}$, intersecting along the line L. We will investigate how moduli spaces $\mathfrak{M}_{0,5}(W_t, 2)$ vary when t approaches 0. To keep matters simple, we pick five sections $z_i(t)$ of $W \to \mathbf{A}^1$ and look at the subspace

$$\mathbf{M}_t \triangleq \{f \in \mathfrak{M}_{0,5}(W_t, 2) \mid f(p_i) \in z_i(t),\ i = 1, \cdots, 5\}.$$

By choosing z_i in general positions, we can assume that the scheme $\mathbf{M}_t$ consists of a single smooth point. We now choose z_i as follows.

Case 1: All $z_i(0) \in \mathbf{P}^2$. In this case, one checks that $\mathbf{M}_0$ consists of a single map

$$f : X = C_1 \cup C_2 \cup C_3 \longrightarrow \mathbf{P}^2 \cup \mathbf{H},$$

with X a chain of rational curves as indicated by the indices, $f(C_2) \subset \mathbf{P}^2$ is a quadric, $f(C_1)$ and $f(C_3) \subset \mathbf{H}$ are two fibers of $\mathbf{H}$.
Case 2: $z_1(0)$ and $z_2(0) \in \mathbf{P}^2$ and all others are in $\mathbf{H}$. In this case, one checks that $\mathbf{M}_0$ consists of a single map

$$f : X = C_1 \cup C_2 \longrightarrow \mathbf{P}^2 \cup \mathbf{H},$$

where both $C_1 \cup C_2$ are two smooth $\mathbf{P}^1$ intersecting at one point, $f(C_1) \subset \mathbf{P}^2$ is a line and $f(C_2) \subset \mathbf{H}$ is in the divisor class $L + 2F$. Here F is a fiber of $\mathbf{H}$.

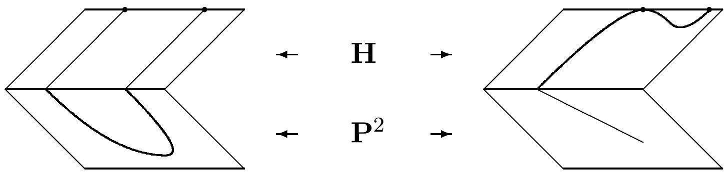

Figure 1: The left is a description of the map f in case one and the other is of the map in case two.

Note that in both cases, the domain curves X can be decomposed into two components $X = X_1 \cup X_2$ so that

1. $f(X_1) \subset \mathbf{P}^1$ and $f(X_2) \subset \mathbf{H}$;
2. There are two sets of points $q_1, \cdots, q_m \in X_1$ and $q_1', \cdots, q_m' \in X_2$ so that X is the gluing of X_1 and X_2 by identifying all pairs (q_i, q_i').

In case 1 we have $X_1 = C_2$ and $X_2 = C_1 \cup C_3$. In case 2 it is $X_1 = C_1$ while $X_2 = C_2$.

In general, when W_t specializes (degenerates) to $W_0 = Y_1 \cup Y_2$, a family of stable maps $f_t : X_t \to W_t$ will specialize to an $f_0 : X_0 \to W_0$. When f_0 is non-degenerate, it splits to a pair of maps

$$h_1 : X_1 \to Y_1 h_2 : X_2 \to Y_2$$

with smooth points (ordered artificially)

$$h_1^{-1}(D) = \{q_{1,1}, \cdots, q_{1,m}\} \in X_1 \quad \text{and} \quad h_2^{-1}(D) = \{q_{2,1}, \cdots, q_{2,m}\} \in X_2$$

so that X_0 is the gluing of X_1 and X_2 along the pairs $(q_{1,i}, q_{2,i})$ and f_0 is the gluing of h_1 and h_2. Of course for the latter to occur we must have

$$h_1(q_{1,i}) = h_2(q_{2,i})$$

for all i. Hence in the ideal situations, the counting of $f : X \to W_t$ should be equivalent to the counting of all such pairs (h_1, h_2).

This suggests that we should consider relative maps: Let $D \subset Y$ be a smooth divisor in a smooth projective variety. A relative map is an

$$f : (X, p_1, \cdots, p_n, q_1, \cdots, q_m) \longrightarrow Y$$

such that $f(q_1), \cdots, f(q_m)$ all lie in D. (Here p_i are the usual marked points.) As in the case of admissible covers, we need to specify the order of contacts of f at q_i. Namely, we must insist on

$$f^{-1}(D) = \sum \mu_i q_i$$

as Cartier divisor for a set of specified integers $\mu_1, \cdots, \mu_m$. In other words, q_i are where $f(X)$ intersects D and μ_i are the orders of tangency of $f(X)$ with D at q_i.

Before we proceed, we need to fix the topological type of the domains of the relative stable maps. The relevant data usually consist of the genus g, the number of marked points k, the number and the weights of the distinguished marked points $\mu_1, \cdots, \mu_r$. (This is the case where the domain is connected. The general case will be specified later.) We then fix a degree $d \in H_2(Y)$.

Definition 1. *A regular relative stable map $f : X \to Y^{rel} = (Y, D)$ of the given topological type consists of a collection (f, X, q_i, p_j) as follows:*
1. X is a connected nodal curve of arithmetic genus g;
2. $q_i \in X$, $i = 1, \cdots, r$ and $p_j \in X$, $j = 1, \cdots, k$ are distinct smooth points of X;
3. $f : X \to Y$ is a morphism so that as Cartier divisors $f^{-1}(D) = \sum_{i=1}^r \mu_i q_i$ and $\deg(f) = d$;
4. The morphism f considered as a morphism whose domain is X with all marked points p_i, $q_j \in X$ is a stable morphism.

It is standard to show that the moduli of all regular relative stable morphisms is a DM-stack and is separated. However this moduli space is usually not proper. Here is an example: We let (Y, D) be the pair of a line in $\mathbf{P}^2$. We pick $d = 2$, $r = 1$ and $\mu_1 = 2$. Then regular relative stable maps of the type given are quadrics in $\mathbf{P}^2$ tangent to L. It is easy to see that regular relative stable morphisms can specialize to $f: X \to \mathbf{P}^2$ with $X = \mathbf{P}^1$ and $f(\mathbf{P}^1) = L$.

One solution is to take the closure of the relative stable maps just defined, as used in [10]. This works only in limited situations. The key is that the GW-invariants are defined using the virtual moduli cycles that are defined based on having perfect obstruction theories of the moduli spaces. The compactification derived by taking closure usually does not have this property.

2 Expanded relative pairs

In this section, we will describe the expanded relative pairs that will be used to construct the desired relative stable morphisms.

We first look at a simple case where degenerate relative stable morphisms arises. Let $f_t\colon (\mathbf{P}^1, 0) \to (\mathbf{P}^2, L)$, $t \in \mathbf{A}^1 - 0$, be a family of relative stable morphisms from $\mathbf{P}^1$ to $\mathbf{P}^2$ of degree 2 with $f_t^{-1}(L) = 2[0]$. We assume f_t specializes to $f_0\colon \mathbf{P}^1 \to L \subset \mathbf{P}^2$. Of course f_0 is not a regular relative map. Our task is to find a good specialization of this family. For this purpose, we will change our perspective by viewing the family f_t as a single map

$$F(z,t) = (f(z), t) : \mathbf{P}^1 \times (\mathbf{A}^1 - 0) \longrightarrow \mathbf{P}^2 \times \mathbf{A}^1.$$

To resolve the degeneracy, we blow up $L \times 0 \subset \mathbf{P}^2 \times \mathbf{A}^1$, with $W[1]$ the resulting family. We denote the central fiber by $W[1]_0 = \Delta \cup \mathbf{P}^1$, where Δ is a ruled surface, and denote by $D[1]$ the proper transform of $L \times \mathbf{A}^1$. Then the family f_t specializes to $\bar{f}_0\colon \mathbf{P}^1 \to \Delta \subset W[1]_0$ with $\bar{f}_0^{-1}(D[1]) = 2[0]$. In other words, after we change the target from $\mathbf{P}^2$ to $\mathbf{P}^2 \cup \Delta$, we obtain a map with the same relative condition.

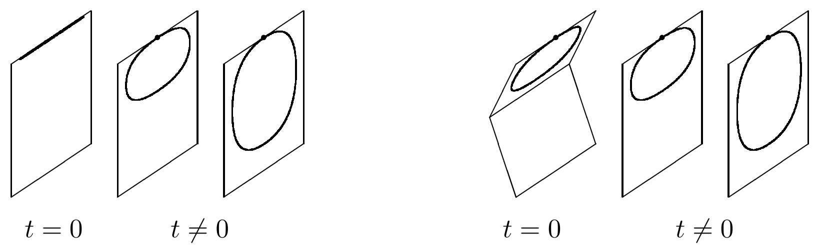

Figure 2: *On the left, the family f_t degenerates to a degenerate relative morphism. After replacing $\mathbf{P}^2$ by $\mathbf{P}^2 \cup \Delta$, the family f_t specializes to a new relative stable map on the right, which is non-degenerate. Here the top edges are the divisors D.*

In general, a "good" specialization of a family of regular relative stable morphisms to $(\mathbf{P}^1, L)$ can be a relative stable morphism to

$$\mathbf{P}^2[n]_0 = \mathbf{P}^2 \cup \Delta \cdots \Delta \cup \Delta,$$

relative to a divisor in the last copy of Δ.

Using blowing up to resolve degeneracy is a standard method in algebraic geometry. What is new is that it is possible to put maps to different target spaces together to form a moduli space that is a proper DM stack and its obstruction theory is perfect.

Let $Y^{\mathrm{rel}} = (D, Y)$ be a pair of smooth divisors in a smooth projective variety. Our first task is to put all $Y[k]_0$ together to form an (Artin) stack $\mathfrak{Y}^{\mathrm{rel}}$, which will be called the stack of all expanded relative pairs of $Y^{\mathrm{rel}} = (Y, D)$.

We let $Y[1]$ be the blowing up of $Y \times \mathbf{A}^1$ along $D \times 0$ and let $D[1]$ be the proper transform of $D \times \mathbf{A}^1$. We let $Y[1]^{\mathrm{rel}}$ be the pair $(Y[1], D[1])$, viewed as a family over $\mathbf{A}^1$. We let $Y[2]$ be the blowing up of $Y[1] \times \mathbf{A}^1$ along $D[1] \times 0$ and let $D[2]$ be the proper transform of $D[1] \times \mathbf{A}^1$. The relative pair $Y[2]^{\mathrm{rel}}$ is $(Y[2], D[2])$, viewed as a family over $\mathbf{A}^2$.

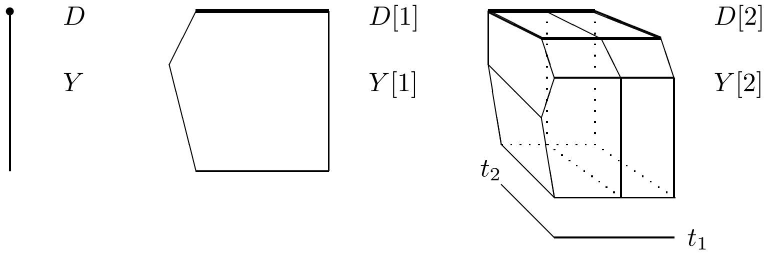

Figure 3: *The spaces Y, $Y[1]$ and $Y[2]$. The dot represents the $D \subset Y$ and the thickened lines represent $D[1] \subset Y[1]$ and $D[2] \subset Y[2]$.*

The relative pair $Y[n]^{\mathrm{rel}} = (Y[n], D[n])$ can be constructed inductively from blowing $Y[n-1] \times \mathbf{A}^1$ along $D[n-1] \times 0$. It is a family over $\mathbf{A}^n$. Clearly, the central fiber $Y[n]_0$ over $0 \in \mathbf{A}^n$ is the union of Y with n copies of Δ. As a convention, we will index the n coordinate lines of $\mathbf{A}^n$ so that the added $\mathbf{A}^1$ in the construction of $Y[l]$ from $Y[l-1]$ is t_l. Also we index the Δ's so that the first intersects Y and the rest are indexed consecutively. We also call $\Delta_{l-1} \cap \Delta_l = D_l$ with $\Delta_0 = Y$ understood. This way, if we let $\mathbf{A}^1_l$ be the l-th coordinate line in $\mathbf{A}^n$, then $Y[n] \times_{\mathbf{A}^n} \mathbf{A}^1_l$, which is the union of all fibers of $Y[n]$ over $\mathbf{A}^1_l$, is a smoothing of $Y[n]_0$ along its l-th singular divisor D_l.

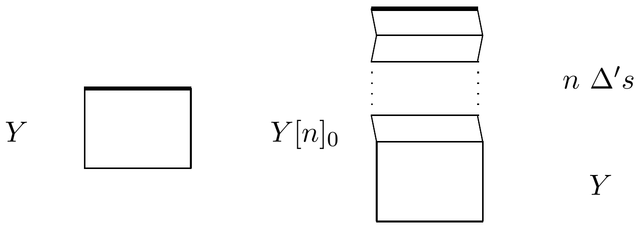

Figure 4: *The space Y and $Y[n]_0$. The thickened lines represent the divisor D and $D[n]_0$.*

We next introduce a group action on $Y[n]$. We let $G[n]$ be the direct product of n copies of $GL(1) = \mathbb{C}^*$. Let $G[1]$ act on $\mathbf{A}^1$ via $t^\sigma = \sigma t$. Then the induced $G[1]$ action on $Y \times \mathbf{A}^1$ lifts to an action on $Y[1]$ that preserves the divisor $D[1]$. In general, the induced action $G[n]$ on $\mathbf{A}^n$ via

$$(t_1, \cdots, t_n)^\sigma = (\sigma_1 t_1, \cdots, \sigma_n t_n)$$

lifts to an action on $Y[n]$ that preserves the divisor $D[n]$.

The $G[n]$-action on the central fiber $Y[n]_0$ can be described explicitly. First of all, each Δ_l is a ruled variety over D with two sections $D_\pm$ that are the intersection of Δ_l with Δ_{l-1}. (In case $l = n$, D_- is $D[n]$.) There is a unique $\mathbb{C}^*$ action on Δ_l that preserves and acts linearly on each fiber and fixes the two sections $D_\pm$. The l-th copy of $\mathbb{C}^*$ in $G[n]$ acts on $\Delta_l \subset Y[n]_0$ via this action and acts on all other irreducible components trivially. This is the $G[n]$ action on $Y[n]_0$.

Intuitively, $Y[n]$ is a family that combines all members $Y[k]_0$ for $k \le n$. Put it differently, $Y[n]$ is the universal family of all possible smoothings of the singular loci of $Y[n]_0$ relative to the projection $Y[n] \to Y$. For instance, along each coordinate axis the fiber $Y[n]_0$ is deformed to $Y[n-1]_0$. On the other hand, if we let $V_{1=2}$ be the line $t_1 = t_2$ while all other $t_i = 0$, then the total family over $V_{1=2}$ is a smoothing of the first two singular divisors of $Y[n]_0$ simultaneously. Unlike the first case, this time the total space of the smoothing is not smooth. Another important observation is that if we let $I \subset [n]$ be any subset[1] and let

$$V_I = \{t \in \mathbf{A}^n \mid t_i = 1 \text{ if } i \notin I \text{ and } t_i = 0 \text{ otherwise}\},$$

then the total family

$$Y[n]^{\mathrm{rel}} \times_{\mathbf{A}^n} V_I \cong Y[k]^{\mathrm{rel}},\ k = |I|. \tag{2.1}$$

Indeed, more is true. We let

$$G[n]_I = \{\sigma \in G[n] \mid \sigma_i = 1 \text{ if } i \notin I\},$$

which is a subgroup of $G[n]$ and is isomorphic to $G[k]$. Then the $G[n]_I$ action on $Y[n]^{\mathrm{rel}}$ preserves $Y[n]^{\mathrm{rel}} \times_{\mathbf{A}^n} V_I$ and the isomorphism (2.1) is $G[n]_I \cong G[k]$ equivariant. In other words, all pairs $(Y[k]^{\mathrm{rel}}, G[k])$ are equivariant sub-pairs of $(Y[n]^{\mathrm{rel}}, G[n])$ for $k \le n$.

The stack of expanded relative pairs alluded to earlier is the limit stack

$$\mathfrak{Y}^{\mathrm{rel}} = \lim Y[n]^{\mathrm{rel}}/G[n],$$

based on the inclusions

$$Y[m]^{\mathrm{rel}}/G[m] \subset Y[n]^{\mathrm{rel}}/G[n], \quad m \le n.$$

Because the above inclusions are not unique, we need refine the definition of $\mathfrak{Y}^{\mathrm{rel}}$.

[1] We use the convention $[n] = \{1, \cdots, n\}$.

Definition 2. *Let S be any scheme. An expanded relative pair of Y^{rel} over S is a family of relative pairs $\mathcal{Y}^{rel}$ over S coupled with a projection $\mathcal{Y} \to Y$ so that there is an open covering S_α of S such that for each α there is a Y-isomorphism*

$$\mathcal{Y}^{rel} \times_S S_\alpha \cong Y[n]^{rel} \times_{\mathbf{A}^n} S_\alpha$$

for some $\eta_\alpha : S_\alpha \to \mathbf{A}^n$. Let $(\mathcal{Y}_1^{rel}, \rho_1)$ and $(\mathcal{Y}_2^{rel}, \rho_2)$ be two expanded relative pairs over S_1 and S_2 respectively. An arrow $\mathcal{Y}_1^{rel} \to \mathcal{Y}_2^{rel}$ consists of a morphism $S_1 \to S_2$ and an S_1-isomorphism $\mathcal{Y}_1^{rel} \to \mathcal{Y}_2^{rel} \times_{S_2} S_1$ compatible to their projections to Y.

We define the groupoid of the expanded relative pairs of Y^{rel} to be the category $\mathfrak{Y}^{\mathrm{rel}}$ whose objects are expanded relative pairs of Y^{rel} over S, where S is any scheme. In the following, we will call expanded relative pairs of Y^{rel} simply relative pairs if their meanings are clear from the context. The functor $\mathfrak{p} : \mathfrak{Y}^{\mathrm{rel}} \to (\mathrm{Sch}/C)$ is the one that sends relative pairs over S to S. Arrows between two objects are those defined in the above Definition. Obviously, if $\xi \in \mathfrak{Y}^{\mathrm{rel}}$ with $\mathfrak{p}(\xi) = S$ and $i : T \to S$ is a morphism, then there is a unique pull-back family $i^*\xi \in \mathfrak{Y}^{\mathrm{rel}}$ with an arrow $i^*\xi \to \xi$. In this way, $(\mathfrak{Y}^{\mathrm{rel}}, \mathfrak{p})$ forms a groupoid. For any $S \in (\mathrm{Sch}/C)$, we define $\mathfrak{Y}^{\mathrm{rel}}(S)$ to be the category of all degenerations over S.

Proposition 3. *The groupoid $\mathfrak{Y}^{rel}$ is a stack.*

Proof. This is straightforward and will be left to the readers. □

3 Relative stable morphisms

We will define relative stable morphisms in this section.

For simplicity, we fix an ample line bundle on Y. For $f : X \to Y[n]_0$, we define its degree to be the degree of the pull back via $X \to Y[n]_0 \to Y$ of the ample line bundle on Y.

For Gromov-Witten invariants we usually work with stable morphisms with connected domains. To establish the degeneration formula we need to work with relative stable morphisms with more general domains. We will use graphs to classify the topology of the domain curves.

In this note, by a graph Γ we mean a finite collection of vertices, edges, legs and roots. An edge is a line segment with both ends attached to vertices of Γ. A leg or a root is a line segment with only one end attached to a vertex of Γ. A weighted root is a root with an integer assigned to it. We will denote by $V(\Gamma)$ the set of vertices of Γ.

Definition 4. *An admissible weighted graph Γ is a graph without edges coupled with the following additional data: An ordering of legs, an ordering of weighted roots and two weight functions $g : \mathrm{V}(\Gamma) \to \mathbb{Z}_{\geq 0}$ and $d : V(\Gamma) \to \mathbb{Z}_{\geq 0}$. We say the graph is relatively connected if either $\mathrm{V}(\Gamma)$ consists of a single element or each vertex in $\mathrm{V}(\Gamma)$ has at least one root attached to it.*

Now let (X, p_i, q_j) be a nodal curve with $(k+r)$ marked points $p_1, \cdots, p_k$ and $q_1, \cdots, q_r$ and let $f : X \to Y[n]_0$ be a morphism. We associate to such (f, X) a weighted graph Γ as follows. First, we let $V(\Gamma)$ be the set of all connected components of X. Namely each element in $V(\Gamma)$ represents a connected component of X. For the i-th marked point p_i (resp. q_i) on the connected component C we attach the i-th leg (resp. the i-th root) to the vertex $[C]$ associated to the component C. We define the weight function $f : V(\Gamma) \to \mathbb{Z}_{\geq 0}$ to be the one assigning each $[C]$ the arithmetic genus $g_a(C)$ of C. In case there is a morphism $f : X \to Y[n]$, we define the weight function $d : V(\Gamma) \to \mathbb{Z}_{\geq 0}$ to be the one assigning each $[C]$ the degree of f along C. In case f is a morphism so that $f^{-1}(D[n]) = \sum \mu_i q_i$, we assign μ_i to be the weight of the i-th root of Γ.

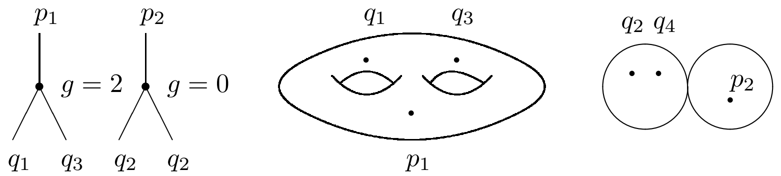

Figure 5: *This is an example of a domain with two connected components and its associated graph. Note that we omitted the multiplicities μ_i and the degree d.*

We now define relative stable morphisms to $Y[n]_0^{\mathrm{rel}}$.

Definition 5. *Let Γ be an admissible graph with r roots, k legs and l vertices $v_1, \cdots, v_l$. A (pre-stable) relative morphism to $Y[n]_0^{rel}$ of type Γ is a quadruple (f, X, q_i, p_j) such that*

1. X is a disjoint union $X_1 \cup \cdots \cup X_l$ of which each X_i is a pre-stable connected curve of arithmetic genus $g(v_i)$.

2. $q_i \in X$, $i = 1, \cdots, r$, and $p_j \in X$, $j = 1, \cdots, k$, are distinct points away from the singular loci of X so that $q_i \in X_j$ (resp. $p_i \in X_j$) if the i-th root

(resp. i-th leg) is attached to the vertex $v_j \in V(\Gamma)$.
3. $f : X \to Y[n]_0$ is a morphism so that $f^{-1}(D[n]_0) = \sum_{i=1}^r \mu_i q_i$ as Cartier divisors and that $\deg f|_{X_i} = d(v_i)$.

We next define the notion of stability of relative morphisms. Let $f : X \to Y[n]_0$ be a relative morphism and let $D_1, \cdots, D_n \subset Y[n]_0$ be its n singular divisors.

Definition 6. *We say f is admissible if the following holds:*
1. $f^{-1}(D_i) = \{z_{i,1}, \cdots, z_{i,j_i}\}$ is a discrete set;
2. Each $z_{i,j}$ is a node of X and is the intersection of two irreducible components, say A_- and A_+, of X so that $f(A_+) \subset \Delta_i$ and $f(A_-) \subset \Delta_{i-1}$. (Here we agree that $\Delta_0 = Y$.) Further, if we let $e_{+,j}$ and $e_{-,j}$ be the contact orders of $f|_{A_+}$ and $f|_{A_-}$ with D_i at $z_{i,j}$, then $e_{+,j} = e_{-,j}$.

This requirement is necessary, as can be illustrated by the following example. We let $W = \mathbf{A}^2$ be a family over $\mathbf{A}^1$ and $f_t : X_t \to W$ be a family of morphisms, as shown in the diagram

$$\begin{array}{ccccccc}
\cup X_t = \mathbf{A}^2 & \xrightarrow{f} & W = \mathbf{A}^2 & \qquad & (z_1, z_2) & \xrightarrow{w_i = z_i^\mu} & (w_1, w_2) \\
\downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle s = z_1 z_2} & & \downarrow{\scriptstyle t = w_1 w_2} \\
\mathbf{A}^1 & \longrightarrow & \mathbf{A}^1 & & s & \xrightarrow{t = s^\mu} & t
\end{array}$$

Note that the central fiber $X_0 = \{z_1 z_2 = 0\} \subset \mathbf{A}^2$ consists of two irreducible components $X_1 = \{z_2 = 0\}$ and $X_2 = \{z_1 = 0\}$ which are mapped to $Y_1 = \{w_2 = 0\}$ and $Y_2 = \{w_1 = 0\}$ under f respectively. Furthermore, the contact orders of $f|_{X_1}$ and $f|_{X_2}$ with respect to the divisor $0 \in Y_1$ and $0 \in Y_2$ are both μ. It can be proved that this is true in general:

If we have a relative morphism $f_0 : X_0 \to Y[n]_0$ that can be deformed to relative morphisms to $Y[n]_t$ for a general $t \in \mathbf{A}^n$, then f_0 has to satisfy the admissible condition defined above.

Next we define the automorphism group of f.

$$\mathfrak{Aut}(f) = \{(h, \sigma) \mid \sigma \in G[n],\ h : X \xrightarrow{\cong} X,\ \sigma \circ f = f \circ h\}.$$

Definition 7. *A pre-stable relative morphism f is stable if f is admissible and $\mathfrak{Aut}(f)$ is finite.*

Here are examples of stable and non-stable relative morphisms.

We let $Y = \mathbf{P}^1 \times \mathbf{P}^1$ with homogeneous coordinate $([1, z_1], [1, z_2])$ and let $D = \{z_1 = 0\}$. Then $Y[1]_0 = Y \cup \Delta$, where $\Delta = \mathbf{P}^1 \times \mathbf{P}^1$ as well. We endow Δ with the homogeneous coordinate $([1, w_1], [1, w_2])$ so that $D_+ = \{w_1 = 0\}$ and $D_- = \{w_1 = \infty\}$. Here $Y[1]_0$ is the gluing of Y and Δ along $D \subset Y$ and $D_+ \subset \Delta$ via $z_2 = w_2$. We let X be a connected curve with two irreducible components $X_1 \cong \mathbf{P}^1$ and $X_2 \cong \mathbf{P}^1$. We assign X_1 (resp. X_2) the homogeneous coordinate $[u_0, u_1]$ (resp. $[v_0, v_1]$) so that the node $\xi \in X$ is $[1, 0] \in X_1$ and $[0, 1] \in X_2$.

Example 1. The morphism $f\colon X \to Y[1]_0$ is defined via

$$f([1, u_1]) = ([1, u_1], [1, u_1]) \in Y f([1, v_1]) = ([1, v_1^2], [v_1, 1]) \in Y.$$

This f is not admissible because the node $\xi \in X$ is mapped to the singular locus of $Y[1]_0$ while the two irreducible components X_1 and X_2 containing this node are all mapped to Y.

Example 2. We define the morphism $f\colon X \to Y[1]_0$ via

$$f([1, u_1]) = ([1, u_1], [1, u_1]) \in Y f([1, v_1]) = ([1, v_1^2], [v_1^2, 1]) \in \Delta.$$

This f is not admissible because at the node $\xi \in X$ that is mapped to the singular locus of $Y[1]_0$, the contact order of $f|_{X_1}$ with D is 1 while the contact order of $f|_{X_2}$ is 2, which are different.

Example 3. We define the morphism $f\colon X \to Y[1]_0$ via

$$f([1, u_1]) = ([1, u_1], [1, u_1]) \in Y f([1, v_1]) = ([1, v_1], [1, 0]) \in \Delta.$$

This f is admissible but not stable because the automorphism group $\mathfrak{Aut}(f) = \mathbb{C}^*$.

Example 4. We define $f\colon X \to Y[1]_0$ be the same as in example 3. However, we add an ordinary marked point $p_1 = [1, 1] \in X_2$. Then f is stable since the extra marked point makes $\mathfrak{Aut}(f) = \{1\}$.

Example 5. We define the morphism $f\colon X \to Y[1]_0$ via

$$f([1, u_1]) = ([1, u_1^3], [1, u_1]) \in Y f([1, v_1]) = ([1, v_1^3/(1 + v_1)], [1, 0]) \in \Delta.$$

This f is admissible and has automorphism group $\mathfrak{Aut}(f) = \{1\}$. Hence f is stable.

In the end, we define the equivalence relation of relative stable morphisms.

Definition 8. *We say two $f:X\to Y[n]_0$ and $f':X'\to Y[n]_0$ are equivalent if there is an isomorphism h and an $\sigma\in G[n]$ so that the following diagram is commutative:*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y[n]_0 \\ \downarrow h & & \downarrow \sigma \\ X' & \xrightarrow{f'} & Y[n]_0. \end{array}$$

4 Moduli stack of relative stable morphisms

We list the main theorem concerning the relative stable morphisms.

Theorem 9. *The groupoid $\mathfrak{M}(\mathfrak{Y}^{rel},\Gamma)$ of relative stable morphisms to $\mathfrak{Y}^{rel}$ of a fixed topological type Γ is a proper, separated DM stack.*

We now sketch the proof in this section. We let k and l be the number of legs and roots in Γ.

The first step is to define the groupoid $\mathfrak{M}(\mathfrak{Y}^{\mathrm{rel}},\Gamma)$ of relative stable morphisms to $\mathfrak{Y}^{\mathrm{rel}}$ of the topological type Γ. To this end, we need to make sense of the notion of families of relative stable morphisms over any scheme S. Apparently, a family of relative stable morphisms over S should consist of a family of expanded pairs $\mathcal{Y}^{\mathrm{rel}}/S\in\mathfrak{Y}^{\mathrm{rel}}(S)$, a flat family of nodal curves $\mathcal{X}/S$ over S, $(k+r)$-disjoint sections p_i and q_j of smooth points of the fibers of $\mathcal{X}/S$, an S-morphism

$$\mathcal{F}:\mathcal{X}\longrightarrow\mathcal{Y}$$

so that for each closed $s\in S$ the fiber $\mathcal{F}_s:\mathcal{X}_s\to\mathcal{Y}_s$ is a relative stable morphism of type Γ, with p_i the ordinary marked points and q_j the distinguished marked points.

However, there is a technical issue we need to sort out. Of the conditions concerning pre-stable relative morphisms and the stability of relative morphisms, all but one are open conditions, hence it is sufficient if they are satisfied at all closed points. The one exception is the "admissible" requirement. It is a closed condition. Here is an example which demonstrates its subtlety. We consider a local situation. We let $W\subset\mathbf{A}^2$ be the scheme $w_1w_2=0$ and let $\mathcal{X}\subset\mathbf{A}^3$ be defined by $z_1z_2=0$ with $\mathcal{X}\to\mathbf{A}^1$ defined by $(z_1,z_2,z_3)\mapsto s=z_3$. Hence $\mathcal{X}/\mathbf{A}^1$ is a constant family of (open) nodal curves. We define

$$\mathcal{F}:\mathcal{X}\longrightarrow W$$

via

$$w_1 = z_1^5 w_2 = z_2(z_2 - z_3^4)^4.$$

I will leave it to reader to show that the restriction of the family $\mathcal{F}$ to $\{z_3^4 = 0\}$ is a family of admissible maps while its restriction to $\{z_3^5 = 0\}$ is not.

It was shown in [23] that given any family of pre-stable relative morphisms $\mathcal{F}:\mathcal{X} \to \mathcal{Y}^{\mathrm{rel}}$ over a scheme S, there is a canonical maximal possible closed subscheme $S_0 \subset S$ so that the restriction of $\mathcal{F}$ to S_0 is a family of stable relative morphisms.

With this understood, we can make sense of families of stable relative morphisms. We can define the groupoid of relative stable morphisms in a standard way. For any scheme S we define $\mathfrak{M}(\mathfrak{Y}^{\mathrm{rel}},\Gamma)(S)$ to be the set of all relative stable morphisms over S to $\mathfrak{Y}^{\mathrm{rel}}$ of topological type Γ. We define arrows in this groupoid in the obvious way, etc.

The next step is to prove that the groupoid $\mathfrak{M}(\mathfrak{Y}^{\mathrm{rel}},\Gamma)$ is a stack. For this we need to check that the groupoid satisfies certain sheaf-theoretic properties. This turns out to be straightforward, by going through the definition of relative stable morphisms.

The next issue is to check that the stack $\mathfrak{M}(\mathfrak{Y}^{\mathrm{rel}},\Gamma)$ is separated. Let $0 \in S$ be a smooth curve and $\mathcal{F}_i : \mathcal{X}_i \to \mathcal{Y}_i^{\mathrm{rel}}$, $i = 1, 2$, be two families of relative stable morphisms over S so that their restrictions to $S^* = S - 0$ are isomorphic. To show that $\mathfrak{M}(\mathfrak{Y}^{\mathrm{rel}},\Gamma)$ is separated we need to check that the two families $\mathcal{F}_1$ and $\mathcal{F}_2$ are isomorphic as well. This is proved in [23], using the stability condition on stable relative morphisms.

Another issue is to check that the stack $\mathfrak{M}(\mathfrak{Y}^{\mathrm{rel}},\Gamma)$ is proper. For this, we need to show that if $0 \in S$ is a smooth curve and $\mathcal{F}^\circ : \mathcal{X}^\circ \to \mathcal{Y}^\circ$ is a family of stable relative morphisms over S°, then possibly after a flat base change, we can extend $\mathcal{Y} \in \mathfrak{Y}(S^\circ)$ to a family $\tilde{\mathcal{Y}} \in \mathfrak{Y}^{\mathrm{rel}}(S)$ and extend $\mathcal{F}^\circ$ to a family of stable relative morphisms $\mathcal{F}:\mathcal{X} \to \mathcal{Y}$ over S. We now sketch the proof of this. For simplicity we assume that $\mathcal{Y}^\circ = Y \times S^\circ$, hence we can write the family as $\mathcal{F}^\circ : \mathcal{X}^\circ \to Y$. Since $\mathcal{F}^\circ$ is a family of ordinary stable morphisms, possibly after a flat base change, we can extend $\mathcal{F}^\circ$ to a family of stable morphisms over S. We denote the new family by $\bar{\mathcal{F}}:\bar{\mathcal{X}} \to Y$. In case no ordinary marked points of $\bar{\mathcal{X}}$ and no irreducible components of $\bar{\mathcal{X}}_0$ are mapped to D, then $\bar{\mathcal{F}}$ is a family of relative morphisms, and we are done. (In this case the morphism is automatically admissible and $\mathfrak{Aut}$ is finite since no Δ occurs in the target.) Now assume that there is an irreducible component of $\bar{\mathcal{X}}_0$, say $C \subset \bar{\mathcal{X}}_0$, that is mapped to D. To resolve the degeneracy, we first embed Y into $Y[1]$ via the standard $Y \equiv Y[1] \times_{\mathbf{A}^1} \{1\} \subset Y[1]$. We consider the induced morphism

$$F : \bar{\mathcal{X}} \longrightarrow Y[1].$$

Next we pick a smooth point of $\bar{\mathcal{X}}_0$ in C, say $\xi \in C$. By shrinking S if necessary, we can assume that there is a section $\rho : S \to \bar{\mathcal{X}}$ passing through $\xi \in \bar{\mathcal{X}}$. Clearly, $F \circ \rho : S \to Y[1]$ specialize to a point in $D \times \{1\}$. We then pick a morphism $\sigma : S^\circ \to G[1]$ so that

$$F' \triangleq \sigma \cdot (F \circ \rho) : S^\circ \longrightarrow Y[1]$$

specializes to a point in $\Delta - D_\pm \subset Y[1]_0$. Such σ always exist. After a base change if necessary, we can assume that the family F' extends to a family of ordinary stable morphisms

$$\mathcal{F}' : \mathcal{X}' \longrightarrow Y[1].$$

In general, $\mathcal{F}'$ may fail to be a family of relative morphisms. However, the degeneracy of $\mathcal{F}'_0$ is measurably less than that of $\mathcal{F}^\circ_0$. Then we can embed $Y[1]$ into $Y[2]$ and perform the same trick to get a new family whose central fiber has less degeneracy than before. In the end, we can find an n so that if we embed $Y \subset Y[n]$ in the standard way and let $f^\circ : \mathcal{X}^\circ \to Y[n]$ be the induced morphisms, then possibly after a base change of S, there is a morphism

$$\sigma : S^\circ \longrightarrow G[n]$$

so that

$$\sigma \cdot f^\circ : \mathcal{X}^\circ \longrightarrow Y[n]$$

specializes to a family of relative morphisms

$$f : \mathcal{X} \longrightarrow Y[n].$$

It remains to check the stability condition. Since general members of f have smooth targets, this family is admissible. The only thing we need to check is that $\mathfrak{Aut}(f_0)$ is finite. This may not be true because $\mathcal{X} \to Y[n]$ is constructed only to resolve the degeneracy of the original family. On the other hand, in case $\mathfrak{Aut}(f_0)$ is infinite, then we can contract some Δ in

$$f_0 : \mathcal{X}_0 \longrightarrow Y[n]_0 = Y \cup \Delta \cup \cdots \cup \Delta$$

to get a stable relative morphism. After this we obtain a specialization of $\mathcal{F}^\circ$ that is a stable relative morphism. This proves the properness of the moduli stack.

In the end, we need to show that this is a DM-stack (algebraic stack). For this, we need to check that there is a scheme of finite type $\mathfrak{Z}$ and a surjective étale morphism

$$\mathfrak{Z} \longrightarrow \mathfrak{M}(\mathfrak{Y}^{\mathrm{rel}}, \Gamma).$$

Let $\mathfrak{M}(Y[n], \Gamma)$ be the moduli stack of ordinary stable morphisms from pointed curves to $Y[n]$ of type Γ regardless of the relative divisor condition, the admissible condition and the finite $\mathfrak{Aut}$ groups condition. Clearly, $\mathfrak{M}(Y[n], \Gamma)$ is a DM stack. We then let $\mathfrak{M}(Y[n]^{\mathrm{rel}}, \Gamma)^{st}$ be the substack of $\mathfrak{M}(Y[n], \Gamma)$ consisting of all admissible relative morphisms that have finite $\mathfrak{Aut}$ groups. As we argued before, $\mathfrak{M}(Y[n]^{\mathrm{rel}}, \Gamma)^{st}$ is a closed substack of $\mathfrak{M}(Y[n], \Gamma)$, hence a DM stack. Since members of $\mathfrak{M}(Y[n]^{\mathrm{rel}}, \Gamma)^{st}$ are stable relative morphisms to $Y[n]^{\mathrm{rel}}$, there is a natural morphism $\mathfrak{M}(Y[n]^{\mathrm{rel}}, \Gamma)^{st} \to \mathfrak{M}(\mathfrak{Y}^{\mathrm{rel}}, \Gamma)$. An easy argument shows that there is an integer $\bar{n}$ so that

$$\mathfrak{M}(Y[\bar{n}]^{\mathrm{rel}}, \Gamma)^{st} \longrightarrow \mathfrak{M}(\mathfrak{Y}^{\mathrm{rel}}, \Gamma)$$

is surjective. Further, since $Y[\bar{n}]$ is a $G[\bar{n}]$-variety, $\mathfrak{M}(Y[\bar{n}], \Gamma)$ admits a natural $G[\bar{n}]$-action. By the definition of stable relative morphisms, $\mathfrak{M}(Y[\bar{n}]^{\mathrm{rel}}, \Gamma)^{st}$ is $G[\bar{n}]$-invariant and the $G[\bar{n}]$ action on $\mathfrak{M}(Y[\bar{n}]^{\mathrm{rel}}, \Gamma)^{st}$ has only finite stabilizers. Hence $\mathfrak{M}(Y[\bar{n}]^{\mathrm{rel}}, \Gamma)^{st}/G[\bar{n}]$ is a DM-stack. Finally, it is clear that the induced morphism

$$\mathfrak{M}(Y[\bar{n}], \Gamma)^{st}/G[\bar{n}] \longrightarrow \mathfrak{M}(\mathfrak{Y}^{\mathrm{rel}}, \Gamma)$$

is finite and étale. This proves that $\mathfrak{M}(\mathfrak{Y}^{\mathrm{rel}}, \Gamma)$ is an algebraic (i.e. Deligne-Mumford) stack.

We define the relative Gromov-Witten invariants of Y^{rel} as follows: We let

$$\mathrm{ev} : \mathfrak{M}(\mathfrak{Y}^{\mathrm{rel}}, \Gamma) \longrightarrow Y^n, \quad f \mapsto (f(p_1), \cdots, f(p_k)) \in Y^k$$

and

$$\mathbf{q} : \mathfrak{M}(\mathfrak{Y}^{\mathrm{rel}}, \Gamma) \longrightarrow D^r, \quad f \mapsto (f(q_1), \cdots, f(q_r)) \in D^r$$

be the evaluation maps. The relative GW-invariant is

$$\Psi_\Gamma^{Y^{\mathrm{rel}}} : H^*(Y^k) \longrightarrow H_*(D^r)$$

defined by

$$\Psi_{\Gamma}^{Y^{\mathrm{rel}}}(\alpha) = \mathbf{q}_*\big(\mathrm{ev}^*(\alpha)\,[\mathfrak{M}(\mathfrak{Y}^{\mathrm{rel}},\Gamma)]^{\mathrm{vir}}\big) \in H_*(D^r).$$

Here k (resp. r) is the number of ordinary (resp. distinguished) marked points of the domain curves and

$$[\mathfrak{M}(\mathfrak{Y}^{\mathrm{rel}},\Gamma)]^{\mathrm{vir}} \in \mathbf{A}_*\mathfrak{M}(\mathfrak{Y}^{\mathrm{rel}},\Gamma)$$

is its virtual moduli cycle. (The moduli space $\mathfrak{M}(\mathfrak{Y}^{\mathrm{rel}},\Gamma)$ admits a tautological perfect obstruction theory, and thus carries a virtual cycle.)

5 The expanded degenerations

As mentioned before, the main purpose of introducing the relative stable morphism is to establish a degeneration formula for Gromov-Witten invariants.

Let C be a smooth irreducible curve, $0 \in C$ a closed point and $\pi : W \to C$ a flat and projective family of schemes. We require that π is smooth away from the central fiber W_0 which is the union of two smooth irreducible varieties Y_1 and Y_2 intersecting transversally along a smooth divisor $D \subset W_0$. When we view D as a divisor in Y_i, we will denote it by $D_i \subset Y_i$.

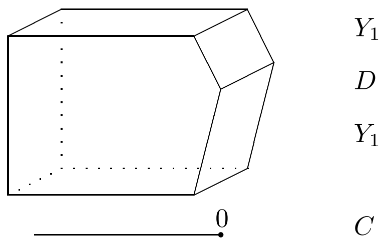

Figure 6. A degeneration of targets.

Then the union

$$\mathfrak{M}_{g,k}(W^\circ/C^\circ, d) = \coprod_{t\in C^\circ} \mathfrak{M}_{g,k}(W_t, d), \quad C^\circ = C - 0,$$

is a proper family over C°. We can compactify this family by inserting $\mathfrak{M}_{g,k}(W_0, d)$ as its central fiber. However, the natural obstruction theory of this new family is no longer perfect near degenerate stable morphisms.

The solution we propose is to work out a new degeneration of $\mathfrak{M}_{g,k}(W_{C^\circ}/C^\circ, d)$ that will allow us to apply the machinery of virtual moduli cycle. The construction is parallel to the construction of relative stable morphisms. Namely, whenever we encounter degenerate stable morphisms

$$f : X \longrightarrow W_0,$$

we should replace them by regular stable morphisms

$$\tilde{f} : \tilde{X} \longrightarrow W[n]_0$$

for some n. Here $W[n]_0$ is a semistable model of W_0 that is the result of inserting a chain of ruled varieties Δ (over D) in the singular locus of W_0. As before, the collection $W[n]_0$ forms an expanded degeneration of W.

We now construct this class of expanded degenerations of W. We let Δ be the projective bundle $\mathbf{P}(\mathbf{1}_D \oplus N_{D_2/Y_2})$ over D, where $\mathbf{1}_D$ is the trivial holomorphic line bundle on D and N_{D_2/Y_2} is the normal bundle of D_2 in Y_2. The variety Δ has two distinguished divisors

$$D_- = \mathbf{P}(\mathbf{1}_D \oplus 0) \quad \text{and} \quad D_+ = \mathbf{P}(0 \oplus N_{D_2/Y_2}).$$

For convenience, we call D_- the *lower* and D_+ the *upper* distinguished divisors. Clearly, there are canonical isomorphisms $D_- \equiv D_+ \equiv D$. Using such identifications, we can *glue* an ordered chain of n Δ's by identifying the upper distinguished divisor in the k-th Δ with the lower distinguished divisor of the $(k-1)$-th Δ. We denote the gluing of n-such Δ's by $\Delta[n]$. It is connected and has normal crossing singularities. We then *glue* Y_1 to $\Delta[n]$ by identifying D_1 in Y_1 with the top distinguished divisor in the first Δ of $\Delta[n]$ and then *glue* Y_2 to this scheme by identifying the lower distinguished divisor of the last Δ in $\Delta[n]$ with D_2 in Y_2. We denote the resulting scheme by $W[n]_0$. Note that $W[n]_0$ has $(n+2)$-irreducible components. The singular locus of $W[n]_0$ consists of $(n+1)$-disjoint divisors, each canonically isomorphic to D. We index them by $D_0, \cdots, D_n$, as shown in the picture below.

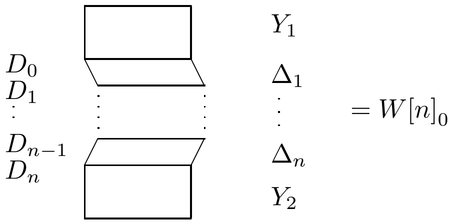

Figure 7. *The picture of* $W[n]_0$.

Our first task is to construct a stack $\mathfrak{W}$ representing all expanded degenerations which include all W_t and all $W[n]_0$ as its individual members. For simplicity, we assume $C = \mathbf{A}^1$. (The general case can be treated by picking an étale map $C \to \mathbf{A}^1$ so that $0 \in C$ is the only point lying over $0 \in \mathbf{A}^1$.)

We construct $W[n]$ inductively. For $n = 0$, $W[0] = W$. For $n = 1$, we first form $W_1 = W \times_{\mathbf{A}^1} \mathbf{A}^2$, where $\mathbf{A}^2 \to \mathbf{A}^1$ is defined via $(t_0, t_1) \mapsto t_0 t_1$. Clearly, W_1 is smooth away from $D \times_{\mathbf{A}^1} 0$, which is a smooth codimension 3 subscheme in W_1. After blowing up the subscheme $D \times_{\mathbf{A}^1} 0$ in $W \times_{\mathbf{A}^1} \mathbf{A}^2$ we obtain the scheme $\tilde{W}_1$ whose exceptional divisor is isomorphic to a $\mathbf{P}^1 \times \mathbf{P}^1$-bundle over D. It is easy to see that we can contract one of the two factors of this $\mathbf{P}^1 \times \mathbf{P}^1$-bundle to obtain a smooth scheme $W[1]$ over $\mathbf{A}^2$. Note that by contracting either factor, the fiber of $W[1]$ over $0 \in \mathbf{A}^2$ is the $W[1]_0$ introduced before. For us, we choose the contraction so that the fibers of $W[1]$ over the first and the second coordinate line of $\mathbf{A}^2$ is a smoothing of the singular divisor D_0 and D_1 of $W[1]_0$.

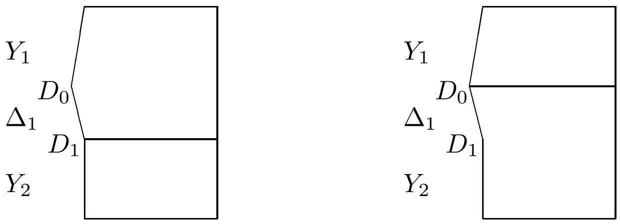

Figure 8: *The first is the fiber over the first coordinate line, which is a smoothing of the divisor* D_0. *The second is the fibers over the second coordinate line, which is a smoothing of* D_1.

We next construct $W[2]$. We let W_2 be the fiber product $W[1] \times_{\mathbf{A}^2} \mathbf{A}^3$ with $\mathbf{A}^3 \to \mathbf{A}^2$ the morphism $(t_0, t_1, t_2) \mapsto (t_0, t_1 t_2)$. We let $W[2] \to \mathbf{A}^3$ be a small resolution of W_2. We choose the small resolution so that the fiber of $W[2]$ over $0 \in \mathbf{A}^3$ is $W[2]_0$, the fibers over the i-th coordinate line is a smoothing of the i-th singular divisor $D_i \subset W[2]_0$.

We construct $W[n] \to \mathbf{A}^{n+1}$ inductively following the same rule.

The next step is to introduce the group action on $W[n]/\mathbf{A}^{n+1}$. We let $G[n]$ act on $\mathbf{A}^{n+1}$ via

$$\mathbf{t}^\sigma = (\sigma_1 t_0, \sigma_1^{-1}\sigma_2 t_1, \cdots, \sigma_{n-1}^{-1}\sigma_n t_{n-1}, \sigma_n^{-1} t_n). \tag{5.1}$$

Note that the projection

$$\mathbf{p} : \mathbf{A}^{n+1} \to \mathbf{A}^1, \quad \mathbf{p}(\mathbf{t}) = t_0 \times \cdots \times t_n$$

is $G[n]$-equivariant with the trivial $G[n]$-action on $\mathbf{A}^1$. The $G[n]$ action on $\mathbf{A}^{n+1}$ lifts uniquely to a $G[n]$ action on $W[n] \to \mathbf{A}^{n+1}$. Its action on $W[n]_0$ is the product of the $\mathbb{C}^*$ action on Δ leaving $D_\pm$ fixed.

Note that for any $m \le n$ there are group homomorphisms $G[m] \to G[n]$ and $G[m]$-equivariant embeddings $W[m] \subset W[n]$. Consequently, we take the limit

$$\mathfrak{W} = \lim W[n]/G[n].$$

Formally, $\mathfrak{W}$ should be defined as a groupoid over $\mathbf{A}^1$ as follows: We associate to each scheme $S/\mathbf{A}^1$ the collection $\mathfrak{W}(S)$ of all families $\mathcal{W}/S$ with $\mathbf{A}^1$-projections $\rho : \mathcal{W} \to W$ so that locally $\mathcal{W}/S$ are of the forms $W[n] \times_{\mathbf{A}^{n+1}} S$ for some $\mathbf{A}^1$-morphisms $S \to \mathbf{A}^{n+1}$ and that the projections $\mathcal{W} \to W$ are those induced by $W[n] \to W$. It is direct to check that such $\mathfrak{W}$ forms a stack.

We call $\mathfrak{W}/\mathbf{A}^1$ the stack of expanded degenerations of $W/\mathbf{A}^1$. We define $\mathfrak{W}_0 = \mathfrak{W} \times_{\mathbf{A}^1} 0$.

6 The degeneration of the moduli spaces

We need to construct the new central fiber of the family of the moduli spaces.

Definition 10. *A stable map to $\mathfrak{W}_0$ is an ordinary stable map $f : X \to W[n]_0$ for some n such that it is admissible at each point in $f^{-1}(D_l)$, $l = 0, \cdots, n$, and such that $\mathfrak{Aut}(f)$ is finite.*

The notion of admissible is similar to that of relative stable maps. First of all, for each singular divisor $D_l \subset W[n]_0$ the pre-image $f^{-1}(D_l)$ is discrete; Secondly, each $\xi \in f^{-1}(D_l)$ is necessarily a node of X and is the intersection of two irreducible components, say C_- and C_+, of X so that $f(C_-) \subset \Delta_l$ and $f(C_+) \subset \Delta_{l+1}$; Thirdly, the contact order of $f|_{C_-}$ with D_l at ξ is identical to that of $f|_{C_+}$ at ξ.

Similarly, we define;

An equivalence relation between two relative stable morphisms $f : X \to W[n]_0$ and $f' : X' \to W[n]_0$ is a pair (h, σ), where h an isomorphism and $\sigma \in G[n]$, so that the following diagram is commutative.

$$\begin{array}{ccc} X & \xrightarrow{f} & W[n]_0 \\ \downarrow h & & \downarrow \sigma \\ X' & \xrightarrow{f'} & W[n]_0. \end{array}$$

The automorphism group $\mathfrak{Aut}(f)$ is the set of all equivalences (h, σ) between f and itself.

We have the following existence theorem.

Theorem 11. *Given integers g, k and degree d, the groupoid $\mathfrak{M}_{g,k}(\mathfrak{W}, d)$ of all equivalence classes of stable morphisms to $\mathfrak{W}$ of genus g, with k marked points and of degree d is an $\mathbf{A}^1$-proper, separated DM stack.*

Since for $t \neq 0 \in \mathbf{A}^1$ the stack $\mathfrak{W} \times_{\mathbf{A}^1} t$ consists of the single element W_t,

$$\mathfrak{M}_{g,k}(\mathfrak{W}, d) \times_{\mathbf{A}^1} t \equiv \mathfrak{M}_{g,k}(W_t, d).$$

Hence $\mathfrak{M}_{g,k}(\mathfrak{W}, d)$ is a degeneration of

$$\mathfrak{M}_{g,k}(W^\circ, d) = \coprod_{t \in \mathbf{A}^1 - 0} \mathfrak{M}_{g,k}(W_t, d).$$

We now sketch the proof of the theorem. As usual, the first step is to define the corresponding groupoid, which is straightforward. To each scheme S over $\mathbf{A}^1$, we define

$$\mathfrak{M}_{g,k}(\mathfrak{W}, d)(S)$$

to be the collection of all triples $(\mathcal{F}, \mathcal{X}, \mathcal{W})$ of which $\mathcal{W} \in \mathfrak{W}(S)$, $\mathcal{X}$ is a flat family of k-pointed genus g curves and

$$\mathcal{F} : \mathcal{X} \to \mathcal{W}$$

is an S-family of ordinary stable morphisms so that it is admissible and that for each closed $s \in S$ the fiber $\mathcal{F}_s$ has finite automorphism group $\mathfrak{Aut}(\mathcal{F}_s)$.

The proof that $\mathfrak{M}_{g,k}(\mathfrak{W},d)$ is a separated, $\mathbf{A}^1$-proper DM stack is parallel to that of the moduli of relative stable morphisms, though technically more involved. For details of this proof please see [23].

We comment that there are tautological obstruction theories of the moduli stacks $\mathfrak{M}_{g,k}(\mathfrak{W},d)$ and $\mathfrak{M}_{g,k}(\mathfrak{W}_0,d)$ $(= \mathfrak{M}_{g,k}(\mathfrak{W},d) \times_{\mathbf{A}^1} 0)$, and both are perfect. Hence they admit virtual moduli cycles $[\mathfrak{M}_{g,k}(\mathfrak{W},d)]^{\mathrm{vir}}$ and $[\mathfrak{M}_{g,k}(\mathfrak{W}_0,d)]^{\mathrm{vir}}$, respectively. Accordingly, we can define their respective GW-invariants.

The GW-invariant of W_t, including $t=0$, is the homomorphism

$$\Psi^{W_t}_{g,k,d} : H^*(W_t)^k \longrightarrow \mathbb{Q}$$

defined by

$$\Psi^{W_t}_{g,k,d}(\alpha) = \deg\big[\mathrm{ev}^*(\alpha)[\mathfrak{M}_{g,k}(W_t,d)]^{\mathrm{vir}}\big]_0,$$

where

$$\mathrm{ev} : \mathfrak{M}_{g,k}(W_t,d) \longrightarrow W_t^k$$

is the ordinary evaluation morphism and $[\cdot]_0$ is the dimension 0 part of the cycle.

The Gromov-Witten invariant of W is the homomorphism

$$\Psi^{W/\mathbf{A}^1}_{g,k,d} : H^0(R^*\pi_*\mathbb{Q}_W)^{\times k} \longrightarrow H_2^{\mathrm{BM}}(\mathbf{A}^1) \cong \mathbb{Q}$$

defined by

$$\Psi^{W/\mathbf{A}^1}_{g,k,d}(\alpha) = \mathbf{q}_{*1}\big[\mathrm{ev}^*(\alpha)\,[\mathfrak{M}_{g,k}(\mathfrak{W},d)]^{\mathrm{vir}}\big]_1.$$

Here $\mathbb{Q}_W$ is the sheaf of locally constant functions on W, $\pi : W \to \mathbf{A}^1$ is the tautological projection, H_2^{BM} is the Borel-Moore homology of the open complex curve $\mathbf{A}^1$ and $\mathbf{q}_{*1}$ is the projection

$$\mathbf{q}_{*1} : \mathbf{A}_1\mathfrak{M}_{g,k}(\mathfrak{W},d) \longrightarrow H_2^{\mathrm{BM}}(\mathbf{A}^1).$$

The invariance of the GW-invariants can be stated as the commutativity of the following diagram. Let $t \in \mathbf{A}^1$ be any closed point and let $H^0(R^*\pi_*\mathbb{Q}_W) \to H^*(W_t)$ be induced by $W_t \to W$. For $\alpha \in H^0(R^*\pi_*\mathbb{Q}_W)$ we denote by α_t its image in $H^*(W_t)$. We let $H_2^{\mathrm{BM}}(\mathbf{A}^1) \to \mathbb{Q}$ be the Gysin homomorphism defined by intersecting with the divisor $t \in \mathbf{A}^1$. Then we have the commutative diagram

$$\begin{array}{ccc} H^0(R^*\pi_*\mathbb{Q}_W)^{\times k} & \xrightarrow{\Psi^{W/\mathbf{A}^1}_{g,k,d}} & H_2^{\mathrm{BM}}(\mathbf{A}^1) \cong \mathbb{Q} \\ \Big\downarrow & & \Big\downarrow \\ H^*(W_t)^{\times k} & \xrightarrow{\Psi^{W_t}_{g,k,d}} & \mathbb{Q}. \end{array} \tag{6.1}$$

The proof is quite involved, relying on the comparison of the obstruction theory of $\mathfrak{M}_{g,k}(\mathfrak{W}, d)$ and of $\mathfrak{M}_{g,k}(\mathfrak{W}_t, d)$, and will be omitted.

7 The degeneration formula

We consider a stable $f \in \mathfrak{M}_{g,k}(\mathfrak{W}_0, d)$ represented by an

$$f : X \longrightarrow W[n]_0.$$

We let $D_l \subset W[n]_0$ be its l-th singular divisor. (The D_i is indexed from $i = 0, \cdots, n$.) Then $W[n]_0$ is the gluing of $Y_1[l]_0$ and $Y_2[n-l]_0$ along their distinguished divisors.

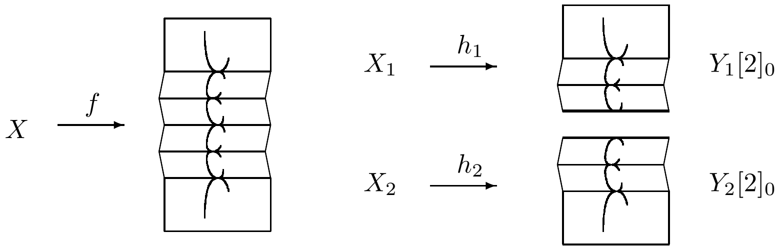

Figure 9: *This is an example of decomposing a stable morphism to $W[4]_0$ into a pair of relative stable maps.*

By the stability condition on f, the nodes $f^{-1}(D_l)$ split X into two parts:

$$X_1 = f^{-1}(Y[l]_0) \quad X_2 = f^{-1}(Y_2[n-l]_0),$$

each endowed with the reduced scheme structure. After ordering the set $f^{-1}(D_l)$, say $\{q_1, \cdots, q_r\}$, we demand that both X_1 and X_2 have distinguished marked points $f^{-1}(D_l) \subset X_1, X_2$, labelled as $\{q_{1,i}\}$ and $\{q_{2,i}\}$. We let $h_i = f|_{X_i}$. Then $(h_i, X_i, q_{i,j})$ is a relative stable morphism to $Y_i[\cdot]_0$. (For simplicity, we ignore the issue of labelling the corresponding ordinary marked points on X_1 and X_2. It can be treated in the obvious way.) Obviously, we can reconstruct the original f by gluing X_1 and X_2 along $q_{1,i} = q_{2,i}$ for all i and then the morphism f is simply the one induced by the pair (h_1, h_2).

One word of caution. In case f is a stable morphism to W_0, then there is a unique way to split it into one pair of relative stable morphisms. However, when f is a stable morphism to $W[n]_0$, then there are $(n+1)$ different ways to split it. This may at first sound perplexing. However, this is exactly what should be expected, since the domain W has normal crossing singularity along W_0 so that the central fiber $\mathfrak{M}_{g,k}(\mathfrak{W}_0, d)$ of $\mathfrak{M}_{g,k}(\mathfrak{W}, d)$ should in principle behave like a scheme with normal crossing singularity. The point represented by $f : X \to W[n]_0$ with $n > 0$ is a singular point which in principle lies in the intersection of $(n+1)$ irreducible components of $\mathfrak{M}_{g,k}(\mathfrak{W}_0, d)$. Each decomposition of f into a pair of relative stable morphisms is then canonically associated to a particular irreducible "component" of $\mathfrak{M}_{g,k}(\mathfrak{W}_0, d)$ containing ξ. This is the basis of the degeneration formula we are about to establish.

Proposition 12. *Any stable map $f \in \mathfrak{M}_{g,k}(\mathfrak{W}_0, d)$ is a gluing of at least one pair of relative stable morphisms to $\mathfrak{Y}_1$ and $\mathfrak{Y}_2$. If the target of f is $W[n]_0$, then there are exactly $n+1$ different ways of such decompositions.*

Let $f : X \to W[n]_0$ be a stable morphism to $\mathfrak{W}_0$. For each integer $l \in [0, n]$ we can decompose f into a pair of relative stable morphisms:

$$h_1^l : X_1^l \longrightarrow Y_1[l]_0 h_2^l : X_2^l \longrightarrow Y_2[n-l]_0.$$

Let Γ_i^l be the associated graph of h_i^l.

Lemma 13. *The $n+1$ pairs of graphs (Γ_1^j, Γ_2^j) are all distinct.*

This is true because of the stability condition imposed on f.

Now we show how to glue families of pairs of relative stable morphisms to obtain families of stable morphisms to $\mathfrak{W}$. Let $\gamma = (\Gamma_1, \Gamma_2)$ be a pair that arises from a decomposition just described. We consider a pair of relative stable morphisms

$$(h_1, h_2) \in \mathfrak{M}(\mathfrak{Y}_1^{\mathrm{rel}}, \Gamma_1) \times \mathfrak{M}(\mathfrak{Y}_2^{\mathrm{rel}}, \Gamma_2).$$

As before, we denote by X_i the domain of h_i and denote by $q_{i,1}, \cdots, q_{i,r}$ the r distinguished marked points of h_i. Because γ is derived from a splitting of some $f \in \mathfrak{M}_{g,k}(\mathfrak{W}_0, d)$, X_1 and X_2 have identical number of distinguished marked points. Further, the multiplicity $\mu_{1,i}$ of $q_{1,i}$ is identical to the multiplicity $\mu_{2,i}$ of $q_{2,i}$. We glue X_1 and X_2 by identifying $q_{1,i}$ with $q_{2,i}$ for all i. The resulting curve $X_1 \sqcup X_2$ is a connected nodal curve of genus g. In case

$$h_1(q_{1,i}) = h_2(q_{2,i}) \in D$$

for all i, then the pair (h_1, h_2) induces a stable morphism

$$h_1 \sqcup h_2 : X_1 \sqcup X_2 \longrightarrow Y_1[l_1]_0 \sqcup Y_2[l_2]_0 = W[l_1 + l_2]_0.$$

Put differently, we let $\mathfrak{M}(\mathfrak{Y}_i^{\mathrm{rel}}, \Gamma_i)$ be the moduli space of relative stable morphisms to Y_i^{rel} of topological type Γ_i and let

$$\mathfrak{M}(\mathfrak{Y}_i^{\mathrm{rel}}, \Gamma_i) \longrightarrow D^r$$

be the evaluation of the distinguished marked points, then the pair (h_1, h_2) can be glued to form a stable morphism $h_1 \sqcup h_2$ if and only if

$$(h_1, h_2) \in \mathfrak{M}(\mathfrak{Y}_1^{\mathrm{rel}}, \Gamma_1) \times_{D^r} \mathfrak{M}(\mathfrak{Y}_2^{\mathrm{rel}}, \Gamma_2).$$

Therefore, after generalizing the above gluing construction to families, we define a morphism

$$\Phi_\gamma : \mathfrak{M}(\mathfrak{Y}_1^{\mathrm{rel}}, \Gamma_1) \times_{D^r} \mathfrak{M}(\mathfrak{Y}_2^{\mathrm{rel}}, \Gamma_2) \longrightarrow \mathfrak{M}_{g,k}(\mathfrak{W}_0, d).$$

It is easy to show that the above morphism is a local immersion.

Since every stable morphism in $\mathfrak{M}_{g,k}(\mathfrak{W}_0, d)$ is the result of gluing pairs of relative stable morphisms, we have the following identity as sets

$$\mathfrak{M}_{g,k}(\mathfrak{W}_0, d) = \cup_\gamma \Phi_\gamma\big(\mathfrak{M}(\mathfrak{Y}_1^{\mathrm{rel}}, \Gamma_1) \times_{D^{r_\gamma}} \mathfrak{M}(\mathfrak{Y}_2^{\mathrm{rel}}, \Gamma_2)\big). \tag{7.1}$$

Here the summation is over all possible $\gamma = (\Gamma_1, \Gamma_2)$ that arise from such splitting and the r_γ is the number of roots in γ.

Because the decomposition is not unique, the above is not a disjoint union. In principle, one should view each summand in the union as an "irreducible component" of $\mathfrak{M}_{g,k}(\mathfrak{W}_0, d)$. With this picture in mind, it is fairly easy to understand the degeneration formula we shall derive momentarily. To avoid involving the notion of virtual cycle, we assume for the moment that the family $\mathfrak{M}_{g,k}(\mathfrak{W}, d)$ is flat over $\mathbf{A}^1$ and has the expected dimension. In this case, the Gromov-Witten invariant $\Psi^{W_t}_{g,k,d}(\cdot)$ is the integration of some cohomology class β on $\mathfrak{M}_{g,k}(\mathfrak{W},d)$ over the closed fiber $\mathfrak{M}_{g,k}(\mathfrak{W},d)\times_{\mathbf{A}^1} t$. Since the family $\mathfrak{M}_{g,k}(\mathfrak{W},d)$ is flat,

$$\Psi^{W_t}_{g,k,d}(\cdot)=\Psi^{W_0}_{g,k,d}(\cdot)=\int_{[\mathfrak{M}_{g,k}(\mathfrak{W}_0,d)]}\beta.$$

Then since $\mathfrak{M}_{g,k}(\mathfrak{W}_0,d)$ decomposes into different components, as shown in (7.1),

$$\int_{[\mathfrak{M}_{g,k}(\mathfrak{W}_0,d)]}\beta=\sum_{\gamma}\int_{[\mathfrak{M}(\mathfrak{Y}_1^{\mathrm{rel}},\Gamma_1)\times_{D^{r_\gamma}}\mathfrak{M}(\mathfrak{Y}_2^{\mathrm{rel}},\Gamma_2)]}c_\gamma\Phi_\gamma^*(\beta).$$

Here c_γ are positive rational numbers, depending on the local immersion Φ_γ and the inclusion $\operatorname{Im}\Phi_\gamma\subset\mathfrak{M}_{g,k}(\mathfrak{W}_0,d)$. The degeneration formula will follow once we have determined the constants c_γ and reformulated the integrals in the summation.

In the next part of this section, we will demonstrate how to decompose

$$\mathfrak{M}_{g,k}(\mathfrak{W}_0,d)\subset\mathfrak{M}_{g,k}(\mathfrak{W},d)$$

into unions of Cartier divisors so each component is one of $\operatorname{Im}\Phi_\gamma$, with possibly different multiplicity. To this end, we need to look at the deformation of f in $\mathfrak{M}_{g,k}(\mathfrak{W},d)$. Let $f\in\mathfrak{M}_{g,k}(\mathfrak{W}_0,d)$ be any element. We know that the deformation of $f\in\mathfrak{M}_{g,k}(\mathfrak{W},d)$ involves possibly the smoothing of the singular loci of the target of f. Let f be in the image of Φ_γ with $W[n]_0$ its target. It is easy to see that the deformations of f within

$$\operatorname{Im}(\Phi_\gamma)\subset\mathfrak{M}_{g,k}(\mathfrak{W},d)$$

are those that allow smoothing of all singular divisors $D_{l'}$ except D_l. In other words, if we let $\mathbf{H}_l\subset\mathbf{A}^{n+1}$ be the coordinate hyperplane defined by $t_l=0$, then all such deformations of f will have targets in the family

$$W[n]\times_{\mathbf{A}^{n+1}}\mathbf{H}_l.$$

Here is the key to our next step:

Lemma 14. *There is a $G[n]$-linearized pair of a line bundle with a section (L_l,s_l) on $\mathbf{A}^{n+1}$ so that $s_l^{-1}(0)=\mathbf{H}_l$.*

This can be constructed by hand.

Now let us consider a family in $\mathfrak{M}_{g,n}(\mathfrak{W},d)(S)$ represented by a $\rho\colon S\to\mathbf{A}^{n+1}$ and

$$F:\mathcal{X}\longrightarrow\mathcal{W}_S=W[n]\times_{\mathbf{A}^{n+1}}S.$$

then after pull back (L_l, s_l) we obtain a line bundle with a section

$$(\rho^* \mathbf{H}_l, \rho^* s_l) \tag{7.2}$$

whose vanishing locus defines a close subscheme $S_l = \rho^*(s_l)^{-1}(0) \subset S$. We can decompose the subfamily of F over S_l as follows. We first look at the total space

$$W[n] \times_{\mathbf{A}^{n+1}} \mathbf{H}_l$$

that has two smooth irreducible components and whose singular locus $\mathbf{D}_l$ is isomorphic to $D \times \mathbf{H}_l$. It is easy to see that for each $z \in \mathbf{H}_l$ the fiber of $\mathbf{D}_l$ over z (namely $W[n]_z \cap \mathbf{D}_l$) is one of the D_i in $W[n]_z$ with i possibly different from l. We denote this i by l_z. For simplicity we assume S_l is connected. Then the union

$$\coprod_{s \in S_l} D_{l_{\rho(s)}} \subset W[n] \times_{\mathbf{A}^{n+1}} S_l$$

is a divisor and is isomorphic to $D \times S_l$. Further, if we decompose each F_s along $D_{l_{\rho(s)}}$ to form a pair of relative stable morphisms $(h_{s,1}, h_{s,2})$, then since S is connected, the topological type $\Gamma_{s,i}$ of $h_{s,i}$ is independent of s. Let $\gamma = (\Gamma_1, \Gamma_2)$ be their common type, then the pair (7.2) depends only on γ. Thus it is natural to write (7.2) as

$$(\mathbf{L}_\gamma, \mathbf{s}_\gamma).$$

Now we cover $\mathfrak{M}_{g,n}(\mathfrak{W}, d)$ by S_α with F_α the tautological families over S_α. For each γ we obtain a pair of a line bundle and a section $(\mathbf{L}_{\alpha,\gamma}, \mathbf{s}_{\alpha,\gamma})$ as before. Because the pairs (L_l, s_l) are $G[n]$-equivariant, a moment of thought proves to us

Lemma 15. *For each γ appeared in the morphism Φ_γ, there is pair of a line bundle and a section $(\mathbf{L}_\gamma, \mathbf{s}_\gamma)$ over $\mathfrak{M}_{g,n}(\mathfrak{W}, d)$ so that as sets*

$$\mathrm{Im}(\Phi_\gamma) = \mathbf{s}_\gamma^{-1}(0).$$

Proof. Locally over each $S_\alpha \subset \mathfrak{M}_{g,k}(\mathfrak{W}, d)$ the restriction of $(\mathbf{L}_\gamma, \mathbf{s}_\gamma)$ to S_α is $(\mathbf{L}_{\alpha,\gamma}, \mathbf{s}_{\alpha,\gamma})$. Because the pairs (L_l, s_l) are $G[n]$-equivariant, the pairs $(\mathbf{L}_{\alpha,\gamma}, \mathbf{s}_{\alpha,\gamma})$ patch together to form a line bundle with a section on $\mathfrak{M}_{g,k}(\mathfrak{W}, d)$.

The last statement is easy to check, and will be omitted. □

It turns out that

$$\mathrm{Im}(\Phi_\gamma) \subset \mathbf{s}_\gamma^{-1}(0)$$

as stacks and they are identical as sets and usually they differ as stacks. The following example is an easy way to see this phenomenon. We let $W/\mathbf{A}^1$ be locally the family $\mathbf{A}^2/\mathbf{A}^1$ via $(w_1, w_2) \mapsto t = w_1 w_2$. We let $\mathcal{X}/\mathbf{A}^1$ be the family with $\mathcal{X} = \mathbf{A}^2$ and $\mathcal{X} \to \mathbf{A}^1$ defined by $(z_1, z_2) \mapsto s = z_1 z_2$. We define the map $F: \mathcal{X} \to W$ by

$$(w_1, w_2) = F(z_1, z_2) = (z_1^\mu, z_2^\mu).$$

Clearly, the scheme $\operatorname{Im} \Phi_\gamma$ is associated to the subscheme $\{s = 0\} \subset S$ while the scheme $\mathbf{s}_\gamma^{-1}(0)$ is the pull back of $\{t = 0\}$, which is $\{s^\mu = 0\} \subset S$. They differ by a multiple of μ, which is exactly the order associated to the node where the decomposition of the domain is taking place.

Lemma 16. *Assume $\mathfrak{M}_{g,k}(\mathfrak{W}, d)$ is flat over $\mathbf{A}^1$ and has the expected dimension. Then both $\operatorname{Im}(\phi_\gamma)$ and $\mathbf{s}_\gamma^{-1}(0)$ are Cartier divisors of the moduli space $\mathfrak{M}_{g,n}(\mathfrak{W}, d)$, and as divisors they are related by*

$$m(\gamma) \operatorname{Im}(\phi_\gamma) = \mathbf{s}_\gamma^{-1}(0).$$

Here $m(\gamma)$ is the product of all the orders of the distinguished marked points of $\gamma = (\Gamma_1, \Gamma_2)$. In case $\mathfrak{M}_{g,k}(\mathfrak{W}, d)$ is not flat or is not of the expected dimension, this lemma still holds in terms of virtual cycles.

The bridge between $\mathfrak{M}_{g,n}(\mathfrak{W}, d)$ and $\mathfrak{M}(\mathfrak{Y}_i^{\mathrm{rel}}, \Gamma_i)$ is via the morphisms Φ_γ and the isomorphisms of line bundles with sections on $\mathfrak{M}_{g,k}(\mathfrak{W}, d)$:

$$\otimes(\mathbf{L}_\gamma, \mathbf{s}_\gamma) \cong (\mathbf{L}_0, \mathbf{s}_0).$$

Here the tensor product runs over all γ appearing in the decompsotion and $\mathbf{L}_0$ is the trivial line bundle on $\mathfrak{M}_{g,n}(\mathfrak{W}, d)$ with $\mathbf{s}_0$ the pull-back of $t \in \Gamma(\mathcal{O}_{\mathbf{A}^1})$. The proof of the above identity relies on the identity $\otimes_{l=0}^n (L_l, s_l) \cong \pi^*(\mathbf{1}_{\mathbf{A}^1}, t)$ in which $\pi: \mathbf{A}^{n+1} \to \mathbf{A}^1$ is the projection.

We are now ready to derive the degeneration formula of the Gromov-Witten invariants. First of all, we have the identity

$$c_1(\mathbf{L}_0, \mathbf{s}_0)[\mathfrak{M}(\mathfrak{W}, \Gamma)]^{\mathrm{vir}} = [\mathfrak{M}(\mathfrak{W}_0, \Gamma)]^{\mathrm{vir}} \in A_* \mathfrak{M}_{g,n}(\mathfrak{W}_0, d).$$

Here $c_1(\mathbf{L}_0, \mathbf{s}_0)$ is the localized Chern class of the $\mathbf{L}_0$ and the section $\mathbf{s}_0$ that is a homomorphism

$$c_1(\mathbf{L}_0, \mathbf{s}_0) : \mathbf{A}_* \mathfrak{M}_{g,k}(\mathfrak{W}, d) \longrightarrow \mathbf{A}_* \mathfrak{M}_{g,k}(\mathfrak{W}_0, d).$$

Then by analyzing the obstruction theories of related moduli stacks, we obtain

$$[\mathbf{s}_\gamma^{-1}(0)]^{\mathrm{vir}} = c_1(\mathbf{L}_\gamma, \mathbf{s}_\gamma)[\mathfrak{M}(\mathfrak{W}, \Gamma)]^{\mathrm{vir}} \in A_*\mathfrak{M}_{g,n}(\mathfrak{W}_0, d),$$

and

$$m(\gamma)[\mathrm{Im}(\Phi_\gamma)]^{\mathrm{vir}} = [\mathbf{s}_\gamma^{-1}(0)]^{\mathrm{vir}} \in A_*\mathfrak{M}_{g,n}(\mathfrak{W}_0, d).$$

Combined, we have

$$[\mathfrak{M}(\mathfrak{W}_0, \Gamma)]^{\mathrm{vir}} = \sum_\gamma c_1(\mathbf{L}_\gamma, \mathbf{s}_\gamma)[\mathfrak{M}(\mathfrak{W}, \Gamma)]^{\mathrm{vir}} = \sum_\gamma m(\gamma)[\mathrm{Im}(\Phi_\gamma)]^{\mathrm{vir}}.$$

The next task is to relate the virtual cycle $[\mathrm{Im}(\Phi_\gamma)]^{\mathrm{vir}}$ with $[\mathfrak{M}(\mathfrak{Y}_i^{\mathrm{rel}}, \Gamma_i)]^{\mathrm{vir}}$. Based on the evaluations $\mathbf{q}_i : \mathfrak{M}(\mathfrak{Y}_i^{\mathrm{rel}}, \Gamma_i) \to D^r$ and the diagonal Δ, we can form the Cartesian diagram

$$\begin{array}{ccc}
\mathfrak{M}(\mathfrak{Y}_1^{\mathrm{rel}}, \Gamma_1) \times_{D^r} \mathfrak{M}(\mathfrak{Y}_2^{\mathrm{rel}}, \Gamma_2) & \longrightarrow & \mathfrak{M}(\mathfrak{Y}_1^{\mathrm{rel}}, \Gamma_1) \times \mathfrak{M}(\mathfrak{Y}_2^{\mathrm{rel}}, \Gamma_2) \\
\downarrow & & \downarrow \\
D^r & \xrightarrow{\Delta} & D^r \times D^r
\end{array}$$

which proves to us the identity

$$\frac{1}{|\operatorname{Aut}(\gamma)|}\Phi_{\gamma*}\Delta^!\bigl([\mathfrak{M}(\mathfrak{Y}_1^{\mathrm{rel}}, \Gamma_1)]^{\mathrm{vir}} \times [\mathfrak{M}(\mathfrak{Y}_2^{\mathrm{rel}}, \Gamma_2)]^{\mathrm{vir}}\bigr) = [\mathrm{Im}(\Phi_\gamma)]^{\mathrm{vir}}.$$

The main degeneration formula of the Gromov-Witten invariants of $W/\mathbf{A}^1$ follows immediately from these identities.

Theorem 17. *Let the notation be as before. Then*

$$[\mathfrak{M}_{g,n}(\mathfrak{W}_0, d)]^{\mathrm{vir}} = \sum_\gamma \frac{m(\gamma)}{|\operatorname{Aut}(\gamma)|}\Phi_{\gamma*}\Delta^!\bigl([\mathfrak{M}(\mathfrak{Y}_1^{rel}, \Gamma_1)]^{\mathrm{vir}} \times [\mathfrak{M}(\mathfrak{Y}_2^{rel}, \Gamma_2)]^{\mathrm{vir}}\bigr).$$

Here are a few words on $\operatorname{Aut}(\gamma)$. When we decompose $f : X \to W[n]_0$, we assign an ordering to the distinguished marked points. This assignment is artificial, and has to be quotiented out to get the right answer. Also, the automorphism of the decomposed curves will introduce extra quotient. The term $\operatorname{Aut}(\gamma)$ is a combination of all these.

Finally, we state the numerical corollary of this theorem. Let $\jmath_i : Y_i \to W$ be the inclusion and let

$$\jmath_i^* : H^0_{\mathbf{A}^1}(R^*\pi_*\mathbb{Q}_W)^{\times k_i} \to H^*(Y_i, \mathbb{Q})^{\times k_i}$$

be the induced pull back homomorphism.

Corollary 18. *Let $W/\mathbf{A}^1$ be the family and let $\Gamma = (g, b, k)$ be as before. Then for any closed point $\xi \neq 0 \in \mathbf{A}^1$, $d \in H^0_{\mathbf{A}^1}(R^*\pi_*\mathbb{Q}_W)^{\times k}$ and $\beta \in H^*(\mathfrak{M}_{g,k})$ as before,*

$$\Phi_\Gamma^{W_\xi}(\alpha(\xi)) = \sum_\gamma \frac{m(\gamma)}{|\operatorname{Aut}(\gamma)|} \big[\Psi_{\Gamma_1}^{Y_1^{rel}}(\jmath_1^* d) \bullet \Psi_{\Gamma_2}^{Y_2^{rel}}(\jmath_2^* d)\big]_0.$$

Here $\bullet$ is the intersection of the homology groups

$$H_*(D^r) \times H_*(D^r) \xrightarrow{\cap} H_*(D^r)$$

and $[\gamma]_0$ is the degree of the degree 0 part of the homology class $\gamma \in H_*(D^r)$.

8 The Obstruction theory

One of the main advantages of our algebraic version over the symplectic version developed in [9, 18, 26] is that one can apply virtual localization to the moduli space of relative stable morphisms. For this a good understanding of the deformation of the moduli of relative GW-invariants is in order. In this part, we give a brief description of such obstruction. For details see [23, Appendix], and the more recent preprint [14].

Let $Y^{\mathrm{rel}} = (Y, D)$ be as before and let $f: X \to Y[n]_0$ be a relative stable morphism. We like to describe the tangent space $T^1 = T_f\mathfrak{M}(\mathfrak{Y}^{\mathrm{rel}}, \Gamma)$ and the obstruction space T^2 to deformation of f.

Obviously, a first order deformation of f is an equivalence class that is induced by the $G[n]$ action of a first order deformation of $f: X \to Y[n]$ that observes the admissible condition and the contact order requirement. To be more precise, we let $q_1, \cdots, q_r$ be the distinguished marked points of X with contact orders $\mu_1, \cdots, \mu_r$, let $E \subset X$ be the divisor of all marked points of f, and let $\xi_1, \cdots, \xi_l$ be the nodes of X in $f^{-1}(D_1)$. Here for simplicity we consider the case where $n = 1$. We also denote by $\Omega_{Y[1]^\dagger/\mathbf{A}^{1\dagger}}$ the sheaf of log-differentials of log-schemes (cf. [19, 20]). Then there are two exact sequences relating T^1 and T^2 to some known cohomology groups:

$$\begin{aligned} 0 \longrightarrow \operatorname{Ext}^0_X(\Omega_X(E), \mathcal{O}_X) \longrightarrow A^0 \longrightarrow T^1 \oplus \mathbb{C} \longrightarrow \\ \longrightarrow \operatorname{Ext}^1_X(\Omega_X(E), \mathcal{O}_X) \longrightarrow A^1 \longrightarrow T^2 \longrightarrow 0 \end{aligned}$$

and

$$0 \longrightarrow H^0(f^*\Omega^\vee_{Y[1]^\dagger/\mathbf{A}^{1\dagger}}) \longrightarrow A^0 \xrightarrow{b_0} \mathbb{C}^{\oplus l} \longrightarrow$$
$$\xrightarrow{\delta} H^1(f^*\Omega^\vee_{Y[1]^\dagger/\mathbf{A}^{1\dagger}}) \longrightarrow A^1 \xrightarrow{b_1} \mathbb{C}^{\oplus l}/\mathbb{C} \longrightarrow 0.$$

It is instructive to briefly outline the derivation of these exact sequences. First, the term $T^1 \oplus \mathbb{C}$ is the space of the first order deformations of $f\colon X \to Y[1]$, as admissible map. Since $f \in \mathfrak{M}_{g,n}(\mathfrak{Y}^{\mathrm{rel}}, \Gamma)$ is the equivalence class of f via the $\mathbb{C}^*$ action, the tangent space $T_f\mathfrak{M}_{g,n}(\mathfrak{Y}^{\mathrm{rel}}, \Gamma) = (T^1 \oplus \mathbb{C})/\mathbb{C} = T^1$. Next let $f_s : X_s \to Y[1]$ be a small deformation of f. Then X_s induces a deformation of the domain curve X, which reflects the arrow

$$T^1 \oplus \mathbb{C} \to \mathrm{Ext}^1_X(\Omega_X(E), \mathcal{O}_X).$$

In case X_s is a constant family, then the deformation becomes a deformation of maps $f_s\colon X \to Y[1]$. In this case, we let q be a node in $f^{-1}(D_1)$, we pick an (analytic) chart of $q \in X$ with (z_1, z_2) its analytic coordinate subject to $z_1z_2 = 0$, and pick a chart of $f(q) \in Y[1]$ with $(w_1, w_2, \cdots)$ its analytic coordinate with $Y[1] \to \mathbf{A}^1$ defined by $t = w_1w_2$. Without lose of generality we can assume the map f is given by $w_i = z_i^\mu$. Then the deformation $\dot{f}_0$ induced by f_s will be of the form

$$w_i = z_i^\mu(1 + c_i(z, s))$$

subject to the condition

$$(1 + c_1(z, s))(1 + c_2(z, s)) = 1 + c(s).$$

The arrow b_0 then send $\dot{f}_0$ to $\dot{c}(0) \in \mathbb{C}$. Clearly, if $b_0(\dot{f}_0) = 0$, then the first order deformation $\dot{f}_0$ is given by a (local) section of the log differential $f^*\Omega^\vee_{Y[1]^\dagger/\mathbf{A}^{1\dagger}}$. This shows that the kernel of b_0 is $H^0(f^*\Omega^\vee_{Y[1]^\dagger/\mathbf{A}^{1\dagger}})$. After cancelling out those in A^0 that arise from vector fields of X, we obtain

$$\ker\{T^1 \oplus \mathbb{C} \to \mathrm{Ext}^1_X(\Omega_X(E), \mathcal{O}_X)\} \cong \mathrm{Coker}\{\mathrm{Ext}^0_X(\Omega_X(E), \mathcal{O}_X) \to A^0\}.$$

This explains the part of the exact sequences that relates to T^1.

We will skip the parts of the exact sequences that are related to the obstruction space T^2.

For amusement, let us determine the obstruction bundle of an example of the moduli of relative stable morphisms. Let $Y_{\mathrm{rel}} = (Y, D)$ be a pair of smooth variety and a smooth divisor. We let Γ be the graph consisting of one vertex and one leg. We assign the weights of the vertex to be $g = 1$ and $d = 0$. Thus $\mathfrak{M}(\mathfrak{Y}^{\mathrm{rel}}, \Gamma)$ is the moduli of degree 0 relative stable morphisms to Y from 1-pointed genus 1 curves to Y. Since $d = 0$, all $f: X \to Y$ in $\mathfrak{M}(\mathfrak{Y}^{\mathrm{rel}}, \Gamma)$ are constant maps. Hence $\mathfrak{M}(\mathfrak{Y}^{\mathrm{rel}}, \Gamma)$ is isomorphic to $\mathfrak{M}_{1,1} \times Y$. We now show that its obstruction sheaf is

$$\mathcal{O}b = \pi_2^* \Omega_Y(\log D)^\vee$$

with $\pi_2 : \mathfrak{M}_{1,1} \times Y \to Y$ the second projection.

Let $f_0 \in \mathfrak{M}(\mathfrak{Y}^{\mathrm{rel}}, \Gamma)$ be a relative stable morphism. Since $d = 0$, we can always represent f_0 by a morphism $f_0 : X \to Y[1]^0$, where $Y[1]^0 = Y[1] - D[1] \cup Y[1]_{0,\mathrm{sing}}$ with $Y[1]_{0,\mathrm{sing}}$ the singular locus of $Y[1]_0$. Then $Y[1]^0/\mathbb{C}^* \equiv Y$. The obstruction space to deforming f_0 as morphism to $Y[1]^0$ is

$$H^1(f_0^* T_{Y[1]^0/\mathbf{A}^1}) \equiv T_{Y[1]^0/\mathbf{A}^1}|_{f_0(X)},$$

where $Y[1]^0 \to \mathbf{A}^1$ is the tautological projection and $T_{Y[1]^0/\mathbf{A}^1}$ is the relative tangent bundle. We let $f : \mathcal{X} \to Y[1]^0$ be the family over $\mathfrak{M}_{1,1} \times Y[1]^0$ so that $\mathcal{X}$ is the pull back of the universal family over $\mathfrak{M}_{1,1}$ while the morphism f is the composite of the projection $\mathcal{X} \to \mathfrak{M}_{1,1} \times Y[1]^0$ with the second projection $\mathfrak{M}_{1,1} \times Y[1]^0 \to Y[1]^0$. Clearly, f is the universal family of $\mathfrak{M}(Y[1]^{\mathrm{rel}}, \Gamma)$. The obstruction bundle to the moduli space $\mathfrak{M}(Y[1]^{\mathrm{rel}}, \Gamma)$ restricting to the open subset $\mathfrak{M}_{1,1} \times Y[1]^0$ is $p_2^* T_{Y[1]^0/\mathbf{A}^1}$, where p_2 is the second projection of $\mathfrak{M}_{1,1} \times Y[1]^0$. The $\mathbb{C}^*$-action lifts canonically to $p_2^* T_{Y[1]^0/\mathbf{A}^1}$ and the obstruction sheaf of $\mathfrak{M}(\mathfrak{Y}^{\mathrm{rel}}, \Gamma)$ is the descent of $p_2^* T_{Y[1]^0/\mathbf{A}^1}$. It is direct to check that under the quotient map $\mathfrak{M}_{1,1} \times Y[1]^0/\mathbb{C}^* \cong \mathfrak{M}_{1,1} \times Y$, the equivariant part $(p_2^* T_{Y[1]^0/\mathbf{A}^1})^{\mathbb{C}^*}$ is canonically isomorphic to $\pi_2^* \Omega_Y(\log D)^\vee$. This proves the identity.

9 Application I

We now give a few examples of the relative Gromov-Witten invariants and the degeneration formula.

The first example concerns the classical Hurwitz numbers. We first fix a genus g and a degree d, and then pick a partition

$$\alpha = (\alpha_1, \cdots, \alpha_m),$$

of d into m parts, namely $d = \sum \alpha_i$. Let $b = 2d + 2g - 2$, which is the expected number of branch points of a degree d genus g cover of $\mathbf{P}^1$. Let

$$r = d + m + 2(g - 1).$$

Then a branched cover of $\mathbf{P}^1$ with monodromy above $w_0 = \infty \in \mathbf{P}^1$ given by α and r other specified simple branch points (and no other branching) has genus g. We fix r general points $w_1, \cdots, w_r \in \mathbf{P}^1$. The Hurwitz number H^g_α is the number of branched covers

$$f : X_g \longrightarrow \mathbf{P}^1$$

so that X_g are smooth connected genus g curves, f are ramified at $D = w_0 + \cdots + w_r$ so that they are simply ramified at each $w_1, \cdots, w_r$ while their monodromies above $w_0 \in \mathbf{P}^1$ are given by α.

Let $k = \sum(\alpha_i - 1)$, so $r = b - k$. The Hurwitz number H^g_α can be defined using the moduli of relative stable morphisms. Let $Y = \mathbf{P}^1$ and $D \subset \mathbf{P}^1$ be the smooth divisor w_0, which then for a relative pair Y^{rel}. In this case, any branched cover f as before is a relative stable morphism to Y^{rel}. Since the domain curves are connected, the graph Γ of the topology of f has only one vertex v, has no legs, since the domain curves have no ordinary marked points, and has m-roots attached to it, representing the ramifications over w_0. The weight μ is given by

$$\mu_j = \alpha_j, \quad j = 1, \cdots, m.$$

The genus g is given by the relation above and the degree of the maps are d. We then form the moduli $\mathfrak{M}(\mathfrak{Y}^{\mathrm{rel}}, \Gamma)$ of stable relative morphisms of topological type Γ. Let $\mathbf{P}^b = \mathrm{Symm}^b \mathbf{P}^1$. Let

$$\iota : \mathfrak{M}(\mathfrak{Y}^{\mathrm{rel}}, \Gamma) \longrightarrow \mathrm{Symm}^b \mathbf{P}^1$$

be the morphism that sends any f to the divisor of the branched loci of f away from the distinguished marked points $q_1, \cdots, q_m$. It is direct to check that ι is a morphism. Using the virtual moduli cycle, the Hurwitz number H^g_α can be defined as

$$H^k_\alpha = \frac{1}{|\operatorname{Aut}(\alpha)|} \int_{[\mathfrak{M}(\mathfrak{Y}^{\mathrm{rel}}, \Gamma)]^{\mathrm{vir}}} \iota^*[p]^\vee,$$

where $[p]^\vee \in \mathbf{A}^* \mathbf{P}^b$ is the Poincare dual of the point class.

I should add that using the moduli space of relative stable morphisms as formulated, one can apply the localization technique to evaluate these integrals and prove the formula of Ekedahl-Lando-Shapiro-Vainshtein [8]:

$$H^g_\alpha = \frac{r!}{|\operatorname{Aut}(\alpha)|} \prod_{i=1}^m \frac{\alpha_i^{\alpha_i}}{\alpha_i!} \int_{\mathfrak{M}_{g,m}} \frac{1 - \lambda_1 + \cdots \pm \lambda_g}{\prod(1 - \alpha_i \psi_i)},$$

where $\lambda_i = c_i(\mathfrak{E})$ ($\mathfrak{E}$ is the Hodge bundle). For the details of this proof, please see [13].

10 Application II

In this section, we will show how to use the moduli of relative stable morphisms to investigate open string problems in Super-String theories.

An important research area in Super-String theories is the study of open strings in a Calabi-Yau manifold. In terms of mathematics, it is to enumerate the set of holomorphic maps from Riemann surfaces with boundaries to Calabi-Yau manifolds that map the boundaries of the Riemann surfaces to pre-chosen special Lagrangian submanifolds in the Calabi-Yau manifolds. More precisely, for a Calabi-Yau threefold W and a special Lagrangian submanifold $L \subset W$, we form the moduli space of holomorphic maps

$$f : (X, \partial X) \longrightarrow (W, L)$$

from smooth Riemann surfaces of genus g with r boundary components that have topological types $\beta \in H_2(W, L)$. The next step is to show that there is a way to define the "virtual cycle" of such moduli space and then evaluate the degree of this cycle. There are a few technical difficulties in carrying this out, not least the fact that the moduli spaces usually have boundaries of real codimension one. This makes the task of making sense of the degree of its virtual cycle more challenging.

In this section, we will consider a simple case, worked out in [21, 24]. The material of this section is taken from [24]. We let q_0 and $q_\infty \in \mathbf{P}^1$ be two points in $\mathbf{P}^1$, and let $[w_0, w_1]$ be the homogeneous coordinate of $\mathbf{P}^1$ so that q_0 (resp. q_∞) is the point $[0, 1]$ (resp. $[1, 0]$). We will use $w = w_0/w_1$ to denote the standard coordinate of $\mathbf{C} = \mathbf{P}^1 - q_\infty$. Hence q_0 and q_∞ become 0 and ∞, respectively. We let W be the total space of the vector bundle $\mathbf{1_C} \oplus \mathbf{1_C}$ over $\mathbf{C} = \mathbf{P}^1 - q_\infty$. (We use $\mathbf{1}_X$ to denote the trivial holomorphic line bundle over X.) For any $r \in \mathbb{R}^+$ we denote by $D_r \subset \mathbf{C}$ the closed disk $|w| \le r^d$. Furthermore, we let $W_r = \pi^{-1}(D_r)$, where $\pi : W \to \mathbf{C}$ is the projection.

We now describe the Lagrangian submanifold $L \subset W$, following Ooguri and Vafa [27]. We let e_1 and e_2 be the constant 1 section of the first and the second factor of $\mathbf{1_C} \oplus \mathbf{1_C}$. We then assign the element $\xi \in \mathbf{1_C} \oplus \mathbf{1_C}$ over $w \in \mathbf{C}$ the coordinate (w, u, v). The Lagrangian submanifold introduced by

Ooguri and Vafa in [27] is

$$L_r = \{(w,u,v) \mid |w| = r^d,\ u = \bar w \bar v\}. \tag{10.1}$$

We let $g \geq 0$ be an integer and $\mu = (d_1, \cdots, d_n)$ an n-tuple of positive integers. We then define the moduli space $\mathcal{M}_{g,\mu}(W_r, B_r)$ of relative holomorphic maps from genus-g Riemann surfaces with boundaries to (W_r, B_r), with prescribed winding numbers along the boundaries of the domain Riemann surfaces. In this section, we will call connected, holomorphic nodal curves with ordered smooth boundaries prestable nodal holomorphic curves with ordered boundaries. When Σ is such a curve we will use $\partial_k\Sigma$ to denote the k-th boundary component of Σ with its induced orientation. Now let Σ be a holomorphic nodal curve with n ordered boundary components and let $f : (\Sigma, \partial\Sigma) \to (W, L_r)$ be a holomorphic map. We say f is a relative holomorphic map of type μ (with $\mu = (d_1, \cdots, d_n)$ implicitly understood) if, for all k, it has winding number d_k along its k-th boundary component. A relative holomorphic map is said to be stable if there is no irreducible component of Σ that is isomorphic to $\mathbf{P}^1$ and contains at most two nodal points that is mapped to a single point in W_r. We define $\mathcal{M}_{g,\mu}(W_r, L_r)$ to be the moduli space of all relative stable maps of type μ from genus g curves to (W_r, L_r), modulo the obvious equivalence relation. Similarly, we define $\mathcal{M}_{g,\mu}(D_r, \partial D_r)$ to be the moduli space of all relative stable maps from genus g curves with ordered boundaries to $(D_r, \partial D_r)$, of type μ, modulo the same equivalence relation. Since $(D_r, \partial D_t) \subset (W_r, L_r)$, $\mathcal{M}_{g,\mu}(D_r, \partial D_r)$ is a subspace of $\mathcal{M}_{g,\mu}(W_r, L_r)$.

We now bring in the moduli space of relative stable morphisms. Following the notation developed in this note, we let Γ be the graph with one vertex, no legs and n-roots, of multiplicities d_i given before. The domain curves have genus g and the total degree of the maps are the sum of all d_i. We then consider the pair $(\mathbf{P}^1, q_\infty)$ and its associated stack of expanded relative pairs $\mathfrak{P}^{\mathrm{rel}}$, and then form the moduli space $\mathfrak{M}(\mathfrak{P}^{\mathrm{rel}}, \Gamma)$ of relative stable morphisms.

We now show that the moduli space $\mathcal{M}_{g,\mu}(W_r, L_r)$ is an (analytic) open subset of $\mathfrak{M}(\mathfrak{P}^{\mathrm{rel}}, \Gamma)$. Let $(f, \Sigma) \in \mathcal{M}_{g,\mu}(W_r, L_r)$ be any relative stable map. Composed with $\pi : W_r \to D_r$, we obtain a new map

$$\tilde f = \pi \circ f : \Sigma \to D_r.$$

Since fibers of $W_r \to D_r$ are vector spaces, $(\tilde f, \Sigma)$ is also relative stable and hence is in $\mathcal{M}_{g,\mu}(D_r, \partial D_r)$. This defines a map

$$\mathcal{M}_{g,\mu}(W_r, L_r) \longrightarrow \mathcal{M}_{g,\mu}(D_r, \partial D_r).$$

On the other hand, by the tautological embedding $D_r \subset W_r$ via the 0-section, any map $[f] \in \mathcal{M}_{g,\mu}(D_r, \partial D_r)$ is automatically a map in $\mathcal{M}_{g,\mu}(W_r, L_r)$. Hence we have an embedding

$$\mathcal{M}_{g,\mu}(D_r, \partial D_r) \longrightarrow \mathcal{M}_{g,\mu}(W_r, L_r).$$

Lemma 19. *The above two maps are inverse to each other. Therefore $\mathcal{M}_{g,\mu}(W_r, L_r)$ is canonically isomorphic to $\mathcal{M}_{g,\mu}(D_r, \partial D_r)$.*

We now show that $\mathcal{M}_{g,\mu}(D_r, \partial D_r)$ is naturally an open subset of $\mathfrak{M}(\mathfrak{P}^{\mathrm{rel}}, \Gamma)$. Let (f, Σ) be a relative holomorphic map in $\mathcal{M}_{g,\mu}(D_r, \partial D_r)$. We let

$$D_r^\epsilon = \{(r-\epsilon)^d < |w| \le r^d\},$$

which is a neighborhood of $\partial D_r \subset D_r$. Since $f(\Sigma) \subset D_r$, $f(\partial\Sigma) \subset \partial D_r$, and since $d_k > 0$, the derivative of f along $\partial\Sigma$ is nowhere zero. Note that f is analytic up to the boundary by the reflection principle. Hence there is a positive $\epsilon > 0$ so that $f^{-1}(D_r^\epsilon) \to D_r^\epsilon$ is a covering space. Furthermore, if we let U_k be the connected component of $f^{-1}(D_r^\epsilon)$ containing $\partial_k\Sigma$, then $U_k \to D_r^\epsilon$ is a d_k-fold covering. Hence, there is a biholomorphic map

$$\rho_k : \{(r-\epsilon)^{-1} > |z_k| > r^{-1}\} \longrightarrow U_k$$

so that $f \circ \rho_k$ is given by $w = z_k^{-d_k}$. We then attach the disk $\{|z_k| < (r-\epsilon)^{-1}\}$ to Σ using the isomorphism ρ_k. We let C be the complete nodal curve resulting from attaching n disks to the n holes, following the procedure described. Then the map $f : \Sigma \to D_r$ extends to a holomorphic map $\bar{f} : C \to \mathbf{P}^1$ whose restriction to the disk $\{|z_k| < (r-\epsilon)^{-1}\}$ is given by $w = z_k^{-d_k}$. Thus $(\bar{f}, C)$ is a relative stable morphism in $\mathfrak{M}(\mathfrak{P}^{\mathrm{rel}}, \Gamma)$. It is straightforward to check that any two such extensions (to (f, C)) are equivalent. This way we obtain a map

$$\mathcal{M}_{g,\mu}(D_r, \partial D_r) \longrightarrow \mathfrak{M}(\mathfrak{P}^{\mathrm{rel}}, \Gamma). \tag{10.2}$$

This map is obviously injective and the image is an open subset in $\mathfrak{M}(\mathfrak{P}^{\mathrm{rel}}, \Gamma)$. On the other hand, this construction works for any analytic family of relative stable maps in $\mathcal{M}_{g,\mu}(D_r, \partial D_r)$. Therefore (10.2) is an open embedding. In the following, we will view $\mathcal{M}_{g,\mu}(D_r, \partial D_r)$ as an open subset of $\mathfrak{M}(\mathfrak{P}^{\mathrm{rel}}, \Gamma)$.

It is straightforward to check that the $\mathcal{M}_{g,\mu}(D_r, \partial D_r)$ form an increasing chain of open subsets of $\mathfrak{M}(\mathfrak{P}^{\mathrm{rel}}, \Gamma)$. When we take the union of all these open subsets, we obtain

$$\bigcup_{r>0} \mathcal{M}_{g,\mu}(D_r, \partial D_r) = \mathfrak{M}(\mathfrak{P}^{\mathrm{rel}}, \Gamma)_0,$$

Which is Zariski open in $\mathfrak{M}(\mathfrak{P}^{\mathrm{rel}}, \Gamma)$.

In the remainder of this section, we outline briefly the identification of the obstruction bundle and the definition of the "expected" number of stable maps in $\mathcal{M}_{g,\mu}(W_r, L_r)$. Namely, the degree

$$\deg[\mathcal{M}_{g,\mu}(W_r, L_r)]_0^{\mathrm{vir}}.$$

We begin with the investigate the obstruction theory of $\mathcal{M}_{g,\mu}(W_r, L_r)$. First, since this moduli space is identical to $\mathcal{M}_{g,\mu}(D_r, \partial D_r)$ as analytic schemes (stacks), the space of the first order deformations of $f \in \mathcal{M}_{g,\mu}(W_r, L_r)$ is naturally identical to the space of the first order deformations of f as an element in $\mathcal{M}_{g,\mu}(D_r, \partial D_r)$. As to the obstruction, since there are no first order deformations of $f \in \mathcal{M}_{g,\mu}(W_r, L_r)$ along the fiber direction of $W_r \to D_r$, the obstruction classes to extending families in $\mathcal{M}_{g,\mu}(W_r, L_r)$ are exactly the obstruction classes to extending them as families in $\mathcal{M}_{g,\mu}(D_r, \partial D_r)$. Furthermore, the obstruction spaces (sheaves) to extend families in $\mathcal{M}_{g,\mu}(W_r, L_r)$ is the direct sum of the obstruction spaces (sheaves) to extend them in $\mathcal{M}_{g,\mu}(D_r, \partial D_r)$ with the obstruction spaces (sheaves) to deform them along the fiber directions of W_r. Let V be the obstruction bundle to extend families along the fiber directions of W_r. According to the theory of virtual moduli cycles [3, 25], $[\mathcal{M}_{g,\mu}(W_r, L_r)]^{\mathrm{vir}}$ is the top Chern class of the obstruction bundle V paired with the virtual moduli cycle $[\mathcal{M}_{g,\mu}(D_r, \partial D_r)]^{\mathrm{vir}}$.

Proposition 20. *Let $f \in \mathfrak{M}(\mathfrak{P}^{rel}, \Gamma)$ be any relative stable morphism, let C be its domain, let $D_\mu = d_1 x_1 + \cdots + d_h x_h$ be the distinguished divisor in C. We let $D = x_1 + \cdots + x_h$ and let V be the vector bundle over $\mathfrak{M}(\mathfrak{P}^{rel}, \Gamma)$ whose fiber over f is*

$$V_f = H^1(\mathcal{O}_C(-D) \oplus \mathcal{O}_C(-D_\mu)).$$

Then the obstruction bundle to deforming maps in $\mathcal{M}_{g,\mu}(W_r, L_r)$ along the fiber directions of W_r is given by the vector bundle $V|_{\mathcal{M}_{g,\mu}(W_r,L_r)}$.

We now briefly outline the derivation of this obstruction bundle V. Let $(f_r, \Sigma_r) \in \mathcal{M}_{g,\mu}(W_r, L_r)$ be any relative map. We let (f, C) be the canonical extension of (f_r, Σ_r) constructed before. For simplicity, we consider the case where $n = 1$ and C is smooth. We pick a sufficiently small $\epsilon > 0$ so that

$$U_{r,1} = f^{-1}(D_r^\epsilon) \subset \Sigma_r$$

is isomorphic to

$$\{r^{-1} \le |z| < r^{-1} + \epsilon\}$$

and that $f|_{U_{r,1}} : U_{r,1} \to \mathbf{C}$ is defined by $w = z^{-d}$. Now we let $U_{r,0} = \Sigma_r - \partial\Sigma_r$. Then $U_{r,0}$ and $U_{r,1}$ form an open covering of Σ_r with

$$U_{r,1} \cap U_{r,0} \cong \{r^{-1} < |z| < r^{-1} + \epsilon^{-1}\}.$$

Clearly, deformations of $f_r|_{U_{r,0}}$ are given by sections in $\Gamma(U_{r,0}, f_r^* L_1 \oplus f_r^* L_2)$ whereas deformations of $f_r|_{U_{r,1}}$ are given by sections in $\Gamma(U_{r,1}, f_r^* L_1 \oplus f_r^* L_2)$ subject to the boundary condition (10.1). We denote the resulting subspace by

$$\Gamma(U_{r,1}, f_r^* L_1 \oplus f_r^* L_2)_r.$$

According to the recipe of the obstruction theory, the obstruction space to deforming

$$f_r : (\Sigma_r, \partial\Sigma_r) \longrightarrow (W_r, L_r)$$

along the fiber directions of W_r is the cokernel of

$$\Gamma(U_{r,1}, f_r^* L_1 \oplus f_r^* L_2)_r \oplus \Gamma(U_{r,0}, f_r^* L_1 \oplus f_r^* L_2) \longrightarrow \Gamma(U_{r,0} \cap U_{r,1}, f_r^* L_1 \oplus f_r^* L_2).$$

To express this cokernel in terms of a known cohomology group, We first extend the two line bundles L_i to the trivial line bundle over C so that the constant sections e_i of L_i pull back to constant sections of $\mathcal{O}_C$. With this choice of extensions, $\Gamma(U_{r,1}, f_r^* L_1 \oplus f_r^* L_2)_r$ is canonically isomorphic to

$$\Gamma(\mathcal{O}_{\mathcal{U}_1}(-x)) \oplus \Gamma(\mathcal{O}_{\mathcal{U}_1}(-dx)).$$

Here $\mathcal{U}_1$ is the disk $\{|z| < \epsilon\} \subset C$, $x \in \bar{U}_{r,1}$ is the point ramified over $q_\infty \in \mathbf{P}^1$ and $\mathcal{U}_0 = C - x$. Hence the above cokernel becomes the cokernel of

$$\Gamma(\mathcal{U}_1, \mathcal{O}_C(-x) \oplus \mathcal{O}_C(-dx)) \oplus \Gamma(\mathcal{U}_0, \mathcal{O}_C(-x) \oplus \mathcal{O}_C(-dx)) \longrightarrow$$

$$\longrightarrow \Gamma(\mathcal{U}_0 \cap \mathcal{U}_1, \mathcal{O}_C(-x) \oplus \mathcal{O}_C(-dx)),$$

which, by the definition of Čech cohomology, is

$$H^1(\mathcal{O}_C(-x)) \oplus H^1(\mathcal{O}_C(-dx)).$$

This shows that the obstruction space V_{f_r} of $f_r \in \mathcal{M}_{g,\mu}(W_r, L_r)$ is the cohomology group mentioned.

The collection of vector spaces V_{f_r} over $f_r \in \mathcal{M}_{g,\mu}(W_r, L_r)$ form a vector bundle over $\mathcal{M}_{g,\mu}(W_r, L_r)$. Since V_{f_r} is independent of r, this vector bundle extends to a vector bundle over $\mathfrak{M}(\mathfrak{P}^{\mathrm{rel}}, \Gamma)_0$ via the same formula. We denote this vector bundle by V.

According to the general principle of virtual moduli cycles, the expected (virtual) number of maps in $\mathcal{M}_{g,\mu}(W_r, L_r)$ should be

$$\int_{[\mathcal{M}_{g,\mu}(W_r,L_r)]^{\mathrm{vir}}} c_{\mathrm{top}}(V).$$

Here $[\mathcal{M}_{g,\mu}(W_r, L_r)]^{\mathrm{vir}}$ is the virtual moduli cycle of $\mathcal{M}_{g,\mu}(W_r, L_r)$ and c_{top} is the top Chern class of a vector bundle. Clearly, this integration is not well defined mathematically because $\mathcal{M}_{g,\mu}(W_r, L_r)$ is not compact. To make sense of this integral we need to compactify $\mathcal{M}_{g,\mu}(W_r, L_r)$ and then extend V. Since $\mathcal{M}_{g,\mu}(W_r, L_r)$ is naturally an open subset of $\mathfrak{M}(\mathfrak{P}^{\mathrm{rel}}, \Gamma)$, we can use $\mathfrak{M}(\mathfrak{P}^{\mathrm{rel}}, \Gamma)$ to compactify $\mathfrak{M}(\mathfrak{P}^{\mathrm{rel}}, \Gamma)_0$.

Extending V to $\mathfrak{M}(\mathfrak{P}^{\mathrm{rel}}, \Gamma)$ poses more of a problem. There is an obvious extension as follows: Let $f \in \mathfrak{M}(\mathfrak{P}^{\mathrm{rel}}, \Gamma)$ be any relative stable morphism with domain C and distinguished divisor $D_{\mathbf{d}}$ as before. Then the vector bundle $\tilde{V}$ whose fiber over f is the vector space V_f is a vector bundle over $\mathfrak{M}(\mathfrak{P}^{\mathrm{rel}}, \Gamma)$. However, there probably are other extensions, and at the moment we have no reason as to which one is a natural choice of extension. Note that different extensions may give different numerical answers to the integral. The choice of the right extension requires a detailed analysis of the obstruction theory of f near the boundary of $\mathcal{M}_{g,\mu}(W_r, L_r)$.

Recently, there are some new insights into how such virtual numbers can possibly be defined. We will not venture into any of these development. The interested readers can find them in [1].

11 Application III

The degeneration formula proved in this project can be applied to recover the recursion formula of Caporaso-Harris for the number of curves in $\mathbf{P}^2$. This was first pointed out in the note of Ionel-Parker [18].

Let

$$\alpha = (\alpha_1, \cdots, \alpha_r) \qquad \beta = (\beta_{r+1}, \cdots, \beta_l)$$

be two finite sequences of positive integers. We let $N^{d,\delta}(\alpha, \beta)$ be the number of degree d curves in $\mathbf{P}^2$ with δ nodes, having contact order m at α_m fixed points on L, and having contact order m at β_m (non pre-chosen) points of L, and passing through the appropriate number of fixed general points in $\mathbf{P}^2$. We now show that these are relative Gromov-Witten invariants of the pair $(\mathbf{P}^2, L)$.

We let Γ be the weighted graph with one vertex, l roots with weights $\alpha_1, \cdots, \beta_l$ and k legs (k to be determined later). The degree of the vertex is d and the genus of the vertex is given by the adjunction formula: $g = g_d - \delta$ where g_d is the genus of smooth degree d curves in $\mathbf{P}^2$. We consider the moduli space $\mathfrak{M}(\mathfrak{P}^{\mathrm{rel}}, \Gamma)$ of relative stable morphisms to the stack $\mathfrak{P}^{\mathrm{rel}}$ of the expanded pairs of $(\mathbf{P}^2, L)$. In plain language, it is the moduli of relative stable morphisms whose domains are of genus g, have l distinguished marked points $q_1, \cdots, q_l$ and k ordinary marked points $p_1, \cdots, p_k$, and whose maps are of degrees d. Assume the moduli space has dimension $k + s + r$. Then by changing k if necessary, we can assume that its dimension is $2k + r$. Now let

$$\mathrm{ev} : \mathfrak{M}(\mathfrak{P}^{\mathrm{rel}}, \Gamma) \longrightarrow (\mathbf{P}^2)^{\times k} \mathbf{q} : \mathfrak{M}(\mathfrak{P}^{\mathrm{rel}}, \Gamma) \longrightarrow L^{\times l}$$

be the evaluation maps. We let $a \in H^2((\mathbf{P}^2)^{\times k})$ be the product of the point classes and let $b \in H^*(L^{\times l})$ be the product of r copies of point class in $H^2(L)$ and $(l - r)$ copies of $1 \in H^2(L)$. Then the number $N^{d,\delta}(\alpha, \beta)$ is

$$N^{d,\delta}(\alpha, \beta) = \int_{[\mathfrak{M}(\mathfrak{P}^{\mathrm{rel}}, \Gamma)]^{\mathrm{vir}}} \mathrm{ev}^*(a) \cup \mathbf{q}^*(b).$$

On the other hand, we can find a degeneration of the pair $(\mathbf{P}^2, L)$ by blowing up $L \times \{0\}$ in $\mathbf{P}^2 \times \mathbf{A}^1$. In other words, we degenerate $(\mathbf{P}^2, L)$ to the union of two pairs $(\mathbf{F}, L_-, L_+)$ and $(\mathbf{P}^2, L)$ gluing along $L_- \subset \mathbf{F}$ and $L \subset \mathbf{P}^2$. Here $\mathbf{F}$ is the Hirzebruch surface. Then by an easy variation of the degeneration formula, the relative Gromov-Witten invariants

$$N^{d,\delta}(\alpha, \beta)$$

can be expressed in terms of the relative Gromov-Witten invariants of the pair $(\mathbf{P}^2, L)$ and of the pair $(\mathbf{F}, L_-, L_+)$. In simplifying this degeneration formula, one observes that many terms simply vanish for obvious dimension reasons. After a straightforward reorganization, one recovers the recursion formula

$$N^{d,\delta}(\alpha,\beta)=\sum nN^{d',\delta'}(\alpha-\epsilon_n,\beta+\epsilon_n)+\sum|\beta'-\beta|\cdot\binom{\alpha}{\alpha'}\binom{\beta'}{\beta}\cdot N^{d-1,\delta'}(\alpha',\beta')$$

of Caporaso-Harris.

12 Application IV

In this section, we will outline part of J. Bryan and R. Pandharipande's recent work [5] that utilizes the theory of relative Gromov-Witten invariants and the associated degeneration formula. This exposition is based on a note of J. Bryan. The author likes to take this opportunity to thank him and Pandharipande for their generosity.

One aim of their work [4, 5] is to understand the Gromov-Witten invariants of a super rigid curve in a Calabi-Yau manifold. At the moment, the existence of such curves is not entirely clear. Instead, they defined a set of invariants that would agree with the Gromov-Witten invariants of a super-rigid curve should it exist. Let X be a smooth curve of genus g and let $\mathfrak{M}_h(X,d)$ be the moduli of degree d, genus h stable morphisms to X. We let

$$\pi:\mathcal{C}\to\mathfrak{M}_h(X,d)\mathcal{F}:\mathcal{C}\to X$$

be the universal family. We define

$$N_d^{h-g}(g)=\int_{[\mathfrak{M}_h(X,d)]^{\mathrm{vir}}}c\big(R^1\pi_*\mathcal{F}^*(K_X\oplus\mathcal{O}_X)-R^0\pi_*\mathcal{F}^*(K_X\oplus\mathcal{O}_X)\big),$$

where $c(\cdot)$ is the total Chern class of the difference, which is a perfect complex since π is a flat family of nodal curves.

For the convenience of bookkeeping, we look at the space of stable maps to X with possibly disconnected domains and their corresponding invariants. We let $\mathfrak{M}(X,d)^\bullet$ be the moduli space of degree d stable maps $f:C\to X$ from possibly disconnected nodal curve C so that f has positive degree on any connected components of C. We let $\mathcal{F}:\mathcal{C}\to X$ be as before the universal family. We then define the *local Gromov-Witten invariants* to be

$$Z_d^b(g)=\int_{[\mathfrak{M}(X,d)^\bullet]^{\mathrm{vir}}}c_b\big(R^1\pi_*\mathcal{F}^*(K_X\oplus\mathcal{O}_X)-R^0\pi_*\mathcal{F}^*(K_X\oplus\mathcal{O}_X)\big).$$

It is easy to see that this is related to the invariants $N_d^{h-g}(g)$ via

$$\sum_{d>0}\sum_{b\geq 0} Z_d^b(g)t^b q^d = \exp\big(\sum_{d>0}\sum_{b\geq 0} N_d^{h-g}(g)t^b q^d\big),$$

with $h-g=b/2+(d-1)(g-1)$.

In [5], the authors constructed the relative version of such local invariants using the moduli of relative stable morphisms and proved a fusion rule for such invariants using essentially the degeneration formula proved in [23]. In the remainder of this section, we give a brief sketch of this application.

Let $X^{\mathrm{rel}}=(X,b_1,\cdots,b_r)$ be a fixed non-singular genus g curve with r distinct marked points and let $\mathfrak{X}^{\mathrm{rel}}$ be the stack of expanded relative pairs of X^{rel}. ($\mathfrak{X}^{\mathrm{rel}}$ will be constructed explicitly later.) We let $\alpha=(\alpha_1,\cdots,\alpha_r)$ be r partitions of d and let $\mathfrak{M}(\mathfrak{X}^{\mathrm{rel}},\alpha)^\bullet$ be the moduli stack of relative stable morphisms to $\mathfrak{X}^{\mathrm{rel}}$ of degree d, with possibly disconnected domains, and branched over the i-th relative divisor with monodromy given by α_i. Clearly, $\mathfrak{M}(\mathfrak{X}^{\mathrm{rel}},\alpha)^\bullet$ is a union of $\mathfrak{M}(\mathfrak{X}^{\mathrm{rel}},\Gamma)$ for a set of graphs Γ in the notation of the previous sections. In particular, each connected component is proper and separated.

We let

$$\begin{tikzcd}
\mathcal{R} \arrow[r] & \mathcal{C} \arrow[r, "\mathcal{F}"] \arrow[d, "\pi"] & \mathcal{X} \arrow[d] & \mathcal{B} \arrow[l] \\
 & \mathfrak{M}(\mathfrak{X}^{\mathrm{rel}},\alpha)^\bullet \arrow[r, equal] & \mathfrak{M}(\mathfrak{X}^{\mathrm{rel}},\alpha)^\bullet &
\end{tikzcd}$$

be the universal family, where $\mathcal{C}$ is the universal domain, $\mathcal{X}$ is the universal target, $\mathcal{B}$ is the universal branch divisor and $\mathcal{R}$ is the universal ramification divisor. Let ω be the relative dualizing sheaf of $\mathcal{X}/\mathfrak{M}(\mathfrak{X}^{\mathrm{rel}},\alpha)^\bullet$. We then look at the perfect complex

$$\mathcal{I}=R^1\pi_*\big(\mathcal{F}^*(\omega(\mathcal{B}))\oplus\mathcal{O}_{\mathcal{C}}(-\mathcal{R})\big)-R^0\pi_*\big(\mathcal{F}^*(\omega(\mathcal{B}))\oplus\mathcal{O}_{\mathcal{C}}(-\mathcal{R})\big). \tag{12.1}$$

The *relative local invariants* are

$$Z_d^b(g;\alpha)=\int_{[\mathfrak{M}(\mathfrak{X}^{\mathrm{rel}},\alpha)^\bullet]^{\mathrm{vir}}} c_b(\mathcal{I})$$

and their corresponding generating series is

$$Z_d(g,\alpha)=\sum_{b=0}^{\infty} Z_d^b(g;\alpha)t^b.$$

In [5], Bryan-Pandharipande showed how to use such local invariants to form a TQFT over $\mathbb{Q}[\![t]\!]$. For this, they need the following gluing theorem.

Theorem 21. *For any choice $g_1+g_2+l-1=g$ and any splitting $\alpha=\alpha^-\cup\alpha^+$ we have*

$$Z_d(g;\alpha)=\sum_\beta \mathfrak{z}(\beta)Z_d(g_1,\alpha^-\cup\beta)Z_d(g_2,\alpha^+\cup\beta).$$

Here $\beta=(\beta_1,\cdots,\beta_l)$ run through the set of all possible l-tuples of partitions of d. Also, $\alpha=\{\alpha_1,\cdots,\alpha_r\}$, $\alpha^-=\{\alpha_1,\cdots,\alpha_k\}$ and $\alpha^+=\{\alpha_{k+1},\cdots,\alpha_r\}$ is a splitting of α.

We now sketch the proof of this theorem. The proof essentially consists of two parts. The first part is a direct generalization of the degeneration studied in [23]. We begin with a family of curves W/T over a smooth l-dimensional variety $0\in T$ so that its general fiber W_t is a smooth genus g curve while its central fiber W_0 is a nodal curve with l nodes $o_1,\cdots,o_l$ and having two smooth components Y_1 and Y_2 of genus g_1 and g_2. We require that W/T has r disjoint sections $q_1,\cdots,q_r$ so that the first k of them passing through Y_1 and the remainder passing through Y_2. We call W with r ordered divisors $q_1(T),\cdots,q_r(T)$ the relative pair and denote it by W^{rel}. To establish the degeneration formula, our first step is to form the stack of expanded degenerations $\mathfrak{W}^{\mathrm{rel}}/T$ of the relative pair W^{rel}. In our previous discussion, we only looked at the case where the singular loci of W_0 is connected or the relative divisor is connected. In the current case we need to study when the singular loci of W_0 has l different components and at the same time W has a relative divisor which has r connected components. In the following, we will briefly outline the modification one needed to prove the gluing formula.

We let $N\subset W$ be the loci of singular points of the fibers of W/T. By an étale base change of T, we can assume that N has l connected components $N_1,\cdots,N_l$ that map injectively into T. By shrinking T if necessary, we can assume that there is an étale morphism $\varphi:T\to\mathbf{A}^l$ so that each $\varphi|_{N_i}:N_i\to\mathbf{A}^l$ factors through the i-th coordinate hyperplane $\mathbf{H}_i\subset\mathbf{A}^l$ ($\mathbf{H}_i$ is defined by $z_i=0$) and $N_i\to\mathbf{H}_i$ is an étale morphism. We let $\mathbf{A}^1_i$ be the i-th coordinate axis and let $W\to\mathbf{A}^1_i$ be the composite of $W\to\mathbf{A}^l$ with the projection $\mathbf{A}^1\to\mathbf{A}^1_i$. Then the family $W/\mathbf{A}^1_i$ is a degeneration of smooth varieties to a scheme with normal crossing singularities and smooth singular loci N_i. Following the construction in the previous section, we can form the expanded degeneration (of $W/\mathbf{A}^1_1$)

$$W[m_1]/\mathbf{A}^{m_1+1}$$

with the equivariant $G[m_1]$-action on them. Note that fibers of $W[m_1]$ over points in the coordinate hyperplanes in $\mathbf{A}^{m_1}$ are $W \times_{\mathbf{A}^1_1} 0$ with certain chain of ruled varieties inserted into N_1. We then consider the induced family $W[m_1]/\mathbf{A}^1_2$ induced by $W[m_1] \to W \to \mathbf{A}^1_2$. Similarly, we form the expanded degeneration of this family:

$$W[m_1][m_2]/\mathbf{A}^{m_2+1}.$$

We denote the resulting family by $W[m_1, m_2]$, viewed as a $G[m_1] \times G[m_2]$-equivariant family over $\mathbf{A}^{m_1+1} \times \mathbf{A}^{m_2+1}$. Inductively, we can construct the family

$$W[m_1, \cdots, m_l]/\mathbf{A}^{m_1+1} \times \cdots \times \mathbf{A}^{m_l+1}$$

with an equivariant $\times^l G[m_i]$ action. For convenience, we let $\mathbf{m} = (m_1, \cdots, m_l)$ be the multi-index and denote by $W[\mathbf{m}]$, $\mathbf{A}^{\mathbf{m}+1}$ and $G[\mathbf{m}]$ the obvious schemes and group just mentioned.

Our next step is to build the expanded relative pairs $W[\mathbf{m}; \mathbf{m}']^{\mathrm{rel}}$. This can be achieved inductively. Let $D_1 \subset W[\mathbf{m}]$ be the preimage of the divisor $q_1(T)$ in W. By viewing $D_1 \subset W[\mathbf{m}]$ as a relative pair we can form its expanded relative pair $W[\mathbf{m}; m_1']^{\mathrm{rel}}/\mathbf{A}^{m_1'}$. We then let $D_2 \subset W[\mathbf{m}; m_1']$ be the preimage of $q_2(T) \subset W$ and form its expanded relative pair $W[\mathbf{m}; m_1', m_2']^{\mathrm{rel}}/\mathbf{A}^{m_2'}$. Inductively, we can form the relative pair

$$W[\mathbf{m}; \mathbf{m}']^{\mathrm{rel}}/\mathbf{A}^{\mathbf{m}+1} \times \mathbf{A}^{\mathbf{m}'}$$

with $G[\mathbf{m}] \times G[\mathbf{m}']$-equivariant action, where $\mathbf{m}' = (m_1', \cdots, m_r')$ and $\mathbf{A}^{\mathbf{m}'} = \mathbf{A}^{m_1'} \times \cdots \times \mathbf{A}^{m_r'}$, with the relative divisors $\mathbf{D}_1, \cdots, \mathbf{D}_r$ that are the proper transforms of $q_1(T), \cdots, q_r(T) \subset W$.

With the standard expanded pairs $W[\mathbf{m}; \mathbf{m}']^{\mathrm{rel}}$, we can construct the stack of expanded pairs $\mathfrak{W}^{\mathrm{rel}}$ as we did before. It is clear that closed points in $\mathfrak{W}^{\mathrm{rel}}$ over $0 \in T$ are the central fiber W_0 with each node replaced by a chain of rational curves and each marked points attached with a chain of rational curves. We denote such scheme by

$$W[\mathbf{m}; \mathbf{m}']^{\mathrm{rel}}_0 \triangleq W[m_1, \cdots, m_l; m_1', \cdots, m_r']^{\mathrm{rel}}_0.$$

Also, closed points of $\mathfrak{W}^{\mathrm{rel}}$ are deformations of $W_0[\mathbf{m}; \mathbf{m}']^{\mathrm{rel}}$ which are also semistable models of W_t^{rel} for some $t \in T$.

With α fixed, we can now form the moduli space of relative stable morphisms, with possibly disconnected domains, to $\mathfrak{W}^{\mathrm{rel}}$ that are branched at the i-th relative divisor with monodromy α_i.

Proposition 22. *Such moduli space $\mathfrak{M}(\mathfrak{W}^{rel},\alpha)^\bullet$ is a countable union of T-proper and separated DM-stacks* [2].

Based on the universal family of $\mathfrak{M}(\mathfrak{W}^{\mathrm{rel}},\alpha)^\bullet$, we can form a similar sheaf $\mathfrak{I}$ on $\mathfrak{M}(\mathfrak{W}^{\mathrm{rel}},\alpha)^\bullet$ as defined in (12.1). Let $\eta\in T$ be a general closed point and let

$$\eta^! : A_*\mathfrak{M}(\mathfrak{W}^{\mathrm{rel}},\alpha)^\bullet \to A_*\mathfrak{M}(\mathfrak{W}^{\mathrm{rel}}_\eta,\alpha)^\bullet$$

be the Gysin map. Then

$$Z^b_d(g;\alpha)=\int_{\eta^![\mathfrak{M}(\mathfrak{W}^{\mathrm{rel}},\alpha)^\bullet]^{\mathrm{vir}}} c_b(\mathfrak{I}).$$

Since

$$\int_{\eta^![\mathfrak{M}(\mathfrak{W}^{\mathrm{rel}},\alpha)^\bullet]^{\mathrm{vir}}} c_b(\mathfrak{I})=\int_{0^![\mathfrak{M}(\mathfrak{W}^{\mathrm{rel}},\alpha)^\bullet]^{\mathrm{vir}}} c_b(\mathfrak{I})=\int_{[\mathfrak{M}(\mathfrak{W}^{\mathrm{rel}}_0,\alpha)^\bullet]^{\mathrm{vir}}} c_b(\mathfrak{I}),$$

to prove the gluing formula we need to study in details the central fiber

$$\mathfrak{M}(\mathfrak{W}^{\mathrm{rel}},\alpha)^\bullet\times_T 0=\mathfrak{M}(\mathfrak{W}^{\mathrm{rel}}_0,\alpha)^\bullet$$

and its decomposition into fiber product of moduli of relative stable morphisms.

To this end, we consider the variety $W\times_{\mathbf{A}^l}\mathbf{H}_i$, which has nodal singularity along N_i and its desingularization W_i. We view it as a family over $T_i\triangleq T\times_{\mathbf{A}^l}\mathbf{H}_i$ and with $r+2$ relative divisors: the $q_j(T_i)$'s with two additional sections of W_i/T_i that are the preimages of $N_i\subset W\times_{\mathbf{A}^l}\mathbf{H}_i$. (Here we fix an ordering of these two sections.) Then as before, for any tuples of integers $\mathbf{m}=(m_1,\cdots,m_{l-1})$ and $\mathbf{m}'=(m'_1,\cdots,m'_{r+2})$, we can form the standard expanded relative pair

$$W_i[\mathbf{m};\mathbf{m}']^{\mathrm{rel}}/\mathbf{A}^{\mathbf{m}+1}\times\mathbf{A}^{\mathbf{m}'}$$

with the relative divisors $\mathbf{D}_1,\cdots,\mathbf{D}_{r+2}$. We let $\mathfrak{W}^{\mathrm{rel}}_i$ be the associated stack of the expanded relative pairs of W^{rel}_i/T_i.

Now given any r-tuple of partitions α and a single partition β of d, we let $\alpha\cup 2\beta$ be the associated $(r+2)$-tuple $(\alpha;\beta,\beta)$. We then form the moduli stack

$$\mathfrak{M}(\mathfrak{W}^{\mathrm{rel}}_i,\alpha\cup 2\beta)^\bullet.$$

[2] In [5], this moduli space is noted by $\overline{M}^\bullet(W)_\alpha$.

We now show that up to an étale cover, we can define a morphism from $\mathfrak{M}(\mathfrak{W}_i^{\mathrm{rel}}, \alpha \cup 2\beta)^\bullet$ to $\mathfrak{M}(\mathfrak{W}, \alpha)^\bullet \times_T T_i$, as follows. First of all, gluing the last two relative divisors D_{r+1} and D_{r+2} of any $\mathcal{W} \in \mathfrak{W}_i^{\mathrm{rel}}$ gives rise to an element $\tilde{\mathcal{W}} \in \mathfrak{W}^{\mathrm{rel}} \times_T T_i$. This defines a morphism

$$\mathfrak{W}_i^{\mathrm{rel}} \longrightarrow \mathfrak{W}^{\mathrm{rel}} \times_T T_i.$$

Next, let $f : X \to \mathcal{W}$ be any morphism in $\mathfrak{M}(\mathfrak{W}_i^{\mathrm{rel}}, \alpha \cup 2\beta)^\bullet$. We let Λ be the set of isomorphisms

$$\psi : f^{-1}(D_{r+1}) \longrightarrow f^{-1}(D_{r+2})$$

that preserve the branching orders in the respective sets. Then for each $\psi \in \Lambda$ we can glue $f^{-1}(D_{r+1})$ with $f^{-1}(D_{r+2})$ according to ψ to obtain a new curve $\tilde{X}$ and the associated morphism

$$\tilde{f} : \tilde{X} \longrightarrow \tilde{\mathcal{W}}.$$

We call $\tilde{f}$ the self-gluing of f according to ψ. Clearly, the set Λ only depend on the partition β. Hence by forming all possible self-gluing indexed by $\psi \in \Lambda$ we obtain a proper étale covering of uniform degree $|\Lambda|$ and a morphism:

$$\mathfrak{M}(\mathfrak{W}_i^{\mathrm{rel}}, \alpha \cup 2\beta)^\bullet \longleftarrow \mathfrak{M}^\Lambda(\mathfrak{W}_i^{\mathrm{rel}}, \alpha \cup 2\beta)^\bullet \xrightarrow{\Phi_\beta^i} \mathfrak{M}(\mathfrak{W}^{\mathrm{rel}}, \alpha)^\bullet \times_T T_i.$$

The next step is to show that the image of Φ_β^i forms a Cartier divisor. First, imitating the construction in [23] for each β we can construct a line bundle $\mathbf{L}_\beta^i$ on $\mathfrak{M}(\mathfrak{W}^{\mathrm{rel}}, \alpha)^\bullet$ with a section $\mathbf{s}_\beta^i$ so that its vanishing locus is $\mathrm{Im}(\Phi_\beta^i)$, possibly with different multiplicity. Furthermore, if we let $\mathbf{s}^i$ and $\mathbf{L}^i$ be the pair of a section of a line bundle that is the pull back of 1 and $\mathcal{O}_{\mathbf{A}^l}(\mathbf{H}_i)$, then we have

$$(\mathbf{s}^i, \mathbf{L}^i) \cong (\prod_\beta \mathbf{s}_\beta^i, \otimes_\beta \mathbf{L}_\beta^i),$$

which gives rise to

$$c_1(\mathbf{L}^i, \mathbf{s}^i)[\mathfrak{M}(\mathfrak{W}^{\mathrm{rel}}, \alpha)^\bullet]^{\mathrm{vir}} = \sum_\beta c_1(\mathbf{L}_\beta^i, \mathbf{s}_\beta^i)[\mathfrak{M}(\mathfrak{W}^{\mathrm{rel}}, \alpha)^\bullet]^{\mathrm{vir}}.$$

Next, it is clear that the special fiber $\mathfrak{M}(\mathfrak{W}^{\mathrm{rel}}, \alpha)^\bullet \times_T 0$ is the intersection of the vanishing locus of all $\mathbf{s}^i$ for $i = 1, \cdots, l$. Hence

$$[\mathfrak{M}(\mathfrak{W}_0^{\mathrm{rel}}, \alpha)^\bullet]^{\mathrm{vir}} = \sum_{\beta = (\beta_1, \cdots, \beta_l)} \prod_{i=1}^l c_1(\mathbf{L}_{\beta_i}^i, \mathbf{s}_{\beta_i}^i)[\mathfrak{M}(\mathfrak{W}^{\mathrm{rel}}, \alpha)^\bullet]^{\mathrm{vir}}.$$

To prove Theorem 21, it remains to relate the summands above to the virtual cycles of the moduli of relative stable morphisms to $\mathfrak{Y}_1^{\mathrm{rel}}$ and $\mathfrak{Y}_2^{\mathrm{rel}}$. Here Y_1^{rel} and Y_2^{rel} are the two pairs resulting from breaking the nodal curve W_0 into two irreducible components Y_1 and Y_2 with relative divisors the original ones $q_i(0)$ and the preimages of the nodes of W_0. Since we assume the first k of q_i passing through Y_1 and the remainder passing through Y_2, Y_1^{rel} has $(k+l)$-relative divisors while Y_2^{rel} has $(r-k+l)$-relative divisors. The $\mathfrak{Y}_i^{\mathrm{rel}}$ are then the stacks of expanded relative pairs of Y_i^{rel}, respectively. Now let α be an r-tuple of partitions of d. We split α into α^+ and α^-, consisting of the first k partitions in α and the remainder ones. We let β be an l-tuple of partitions of d. Then up to an étale covering, we have a canonical gluing morphism

$$\mathfrak{M}(\mathfrak{Y}_1^{\mathrm{rel}}, \alpha^+ \cup \beta)^\bullet \times \mathfrak{M}(\mathfrak{Y}_1^{\mathrm{rel}}, \alpha^- \cup \beta)^\bullet \xleftarrow{\psi_-} \mathbf{M}(\alpha,\beta) \xrightarrow{\psi_+} \mathfrak{M}(\mathfrak{W}^{\mathrm{rel}}, \alpha)^\bullet.$$

At the cycle level, we have

$$c(\beta)[\mathfrak{M}(\mathfrak{Y}_1^{\mathrm{rel}}, \alpha^+ \cup \beta)^\bullet]^{\mathrm{vir}} \times [\mathfrak{M}(\mathfrak{Y}_1^{\mathrm{rel}}, \alpha^- \cup \beta)^\bullet]^{\mathrm{vir}} = \psi_{-*}[\mathbf{M}(\alpha,\beta)]^{\mathrm{vir}}$$

and

$$\psi_{+*}[\mathbf{M}(\alpha,\beta)]^{\mathrm{vir}} = c'(\beta)\prod_{i=1}^{l} c_1(\mathbf{L}^i_{\beta_i}, \mathbf{s}^i_{\beta_i})[\mathfrak{M}(\mathfrak{W}^{\mathrm{rel}}, \alpha)^\bullet]^{\mathrm{vir}}.$$

Here $c(\beta)$ and $c'(\beta)$ are constants depending only on β.

The second part of the proof is to show that for each α^+, α^- and β as above, there are K-classes $\mathcal{I}_{\alpha^+,\beta}$ and $\mathfrak{I}_{\alpha^-,\beta}$ on $\mathfrak{M}(\mathfrak{Y}_1^{\mathrm{rel}}, \alpha^- \cup \beta)^\bullet$ and on $\mathfrak{M}(\mathfrak{Y}_2^{\mathrm{rel}}, \alpha^+ \cup \beta)^\bullet$, respectively, so that

$$\psi_-^*(\mathcal{I}_{\alpha^+,\beta} \oplus \mathfrak{I}_{\alpha^-,\beta}) = \psi_+^*(\mathfrak{I}).$$

Here we will skip the sketch of this formula and refer the proof to [5]. With this established, the cycle decompositions above will yield the desired decomposition theorem stated in Theorem 21.

13 Further developments

To the author, the degeneration scheme developed in this project provides a satisfactory machinery to study moduli spaces via degenerating the base variety X to a singular scheme X_0 with normal crossing singularity *and* having smooth singular loci. The next question is how to generalize this to degeneration so that X_0 can have arbitrary normal crossing singularities. By Hironaka's theorem on resolution, given any family we can resolve its degeneracy so that the central fiber of the new family (possibly after a base change) will have only normal crossing singularities. In this sense, the generalization of such a theory will be very useful in studying moduli spaces by degeneration.

Another interesting feature touched on in this project, but that has yet to be fully exploited, is the notion of log-morphisms. It is imperative that the notion of the moduli of log-morphisms can be used to replace the "geometric degeneration" developed in this project, to achieve the same goal. The author as well as [28] believe that the moduli space $\mathfrak{M}(\mathfrak{W}, \Gamma)$ constructed here can be reconstructed as the moduli of log-stable morphisms.

In [23] and here, the author concentrated on the degeneration of the moduli of stable morphisms. Another important class of moduli spaces is the moduli of stable sheaves on varieties. Recently, YH. Kim and the author applied this method to study the degeneration of the moduli of stable vector bundles over curves, proving a vanishing theorem as an application. The new technical difficulty in doing this is that the stability condition for sheaves is more subtle than that in stable morphisms. It will be interesting to see how such a theory can be worked out for high dimensional base varieties.

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