Abelian duality for quantum difference equations

1 Hypertoric varieties

In this section, our gauge group is abelian, that is, a torus $T^k \cong (\mathbb{C}^\times)^k$ of dimension k. Let $V \cong \mathbb{C}^n$ be a faithful representation of T^k given by a collection of characters $\boldsymbol{\chi} = (\chi_1, \dots, \chi_n)$, that is, by a homomorphism

$$\boldsymbol{\chi} : T^k \to T^n \,. \tag{1}$$

It can be also recorded as a $n \times k$ integer matrix $\boldsymbol{\chi} = (\chi_{i,j})_{i \le n, j \le k}$ which describes the induced map of cocharacter lattices. If we want the resulting hypertoric variety to be smooth, we should require

$$k \times k \text{ minors of } \boldsymbol{\chi} \ \in \{-1, 0, 1\} \,. \tag{2}$$

While a orbifold generalization of the story most certainly exists, we will limit ourselves here to smooth hypertoric varieties.

In particular, this means we can choose a basis in which

$$\boldsymbol{\chi} = \begin{bmatrix} 1_{k\times k} \\ W \end{bmatrix} \tag{3}$$

that is, in which the first k rows of $\boldsymbol{\chi}$ form a $k \times k$ identity matrix and $W = (w_{ij})$ is a $d \times k$ matrix with

$$|w_{i,j}| \le 1 \tag{4}$$

for all i and j, where $d = n - k$.

We denote by $A = T^n/T^k$ be the cokernel of the map (1). It acts naturally on X preserving the holomorphic symplectic form. Its fixed points are labelled by k-element subsets of $\{1, \dots, n\}$ such that the corresponding minor of $\boldsymbol{\chi}$ is nonzero. The presentation (3) is thus adapted to one particular fixed point.

2 The integral

2.1

We denote

$$\phi(x) = (x)_\infty = \prod_{n \ge 0} (1 - xq^n)$$

which converges for $|q| < 1$ and defines an entire function of x known as quantum dilogarithm or the reciprocal of the q-analog of the Γ-function.

We work with integrals of the form

$$I = \int_\gamma \mu_z(dt) \prod_{i=1}^n \frac{\phi(\hbar a_i \chi_i(t))}{\phi(a_i \chi_i(t))} \tag{5}$$

where

$$\mu_z(dt) = e^{-\dfrac{\langle \ln z, \ln t\rangle}{\ln q}} \prod \frac{dt_i}{2\pi i t_i}\,,$$

the n-tuple $(a_1, \dots, a_n)$ defines an element of T^n and hence of A, $\hbar$ is an element of the group $\mathbb{C}^\times_\hbar$ that acts by dilating the cotangent direction of T^*V, and

$$\gamma \in H_k(T^k, \{\text{poles}\}, \mathbb{Z})$$

is a cycle of integration invariant under the shift of variables by q.

2.2

With the normalization (3), the integral (5) takes the following form

$$I = \int \mu_z(dt)\, F \tag{6}$$

where

$$F = \prod_{i=1}^{k} \frac{\phi(\hbar t_i)}{\phi(t_i)} \prod_{i=1}^{d} \frac{\phi(\hbar a_i \mathbf{w}_i)}{\phi(a_i \mathbf{w}_i)}\,. \tag{7}$$

Here $\mathbf{w}_i = \prod t_j^{w_{ij}}$ and we relabeled the coordinates a_i in A to correspond to the last d coordinates in T^n.

3 The difference equation

3.1

The difference equations satisfied by the function F involve shift operators

$$\big[\nabla_x f\big](x, \dots) = f(qx, \dots)$$

in the variables $x \in \{z_i, a_i\}$. Clearly

$$\begin{aligned} z_i\, I &= \int \mu_z(dt)\, \nabla_{t_i} F\,, & a_i\, I &= \int \mu_z(dt)\, a_i F\,, \\ \nabla_{z_i} I &= \int \mu_z(dt)\, t_i^{-1} F\,, & \nabla_{a_i} I &= \int \mu_z(dt)\, \nabla_{a_i} F\,, \end{aligned} \tag{8}$$

which means that the difference module generated by the function I is a partial Fourier transform of the difference module generated by the function F.

3.2

Since

$$\nabla_x \phi(x) = \frac{\phi(x)}{1-x}\,,$$

the function F satisfies the following difference equations

$$\left[\nabla_{t_i} - \frac{1-t_i}{1-\hbar t_i}\prod_{j=1}^{d}\frac{(a_j\mathbf{w}_j)_{w_{ji}}}{(\hbar a_j\mathbf{w}_j)_{w_{ji}}}\right]F=0\,,\quad i=1,\dots,k\,, \tag{9}$$

$$\left[\nabla_{a_i} - \frac{1-a_i\mathbf{w}_i}{1-\hbar a_i\mathbf{w}_i}\right]F=0\,,\quad i=1,\dots,d\,, \tag{10}$$

where

$$(x)_k = \prod_{i=0}^{k-1}(1-q^i x) = \frac{\phi(x)}{\phi(q^k x)}\,.$$

A better way to write the (9) is

$$\left[\nabla_{t_i} - \frac{1-t_i}{1-\hbar\, t_i}\prod_{j=1}^{d}\nabla_{a_j}^{w_{ij}}\right]F=0\,,\quad i=1,\dots,k\,, \tag{11}$$

while (10) is equivalent to

$$\left[\frac{a_i}{\widetilde{\hbar}} - \frac{1-\nabla_{a_i}}{\widetilde{\hbar}-\nabla_{a_i}}\prod_{j=1}^{k}t^{-w_{ij}}\right]F=0\,,\quad i=1,\dots,d\,, \tag{12}$$

where $\widetilde{\hbar} = q/\hbar$.

3.3

After partial Fourier transform, these become the following equations

$$\left[z_i - \frac{1-\nabla_{z_i}}{\hbar-\nabla_{z_i}}\prod_{j=1}^{d}\nabla_{a_j}^{w_{ij}}\right]I=0\,,\quad i=1,\dots,k\,, \tag{13}$$

$$\left[\frac{a_i}{\widetilde{\hbar}} - \frac{1-\nabla_{a_i}}{\widetilde{\hbar}-\nabla_{a_i}}\prod_{j=1}^{k}\nabla_{z_i}^{w_{ij}}\right]I=0\,,\quad i=1,\dots,d\,, \tag{14}$$

which are symmetric in a and z after

1. transposing the matrix W, which means going to the dual hypertoric variety
2. exchanging $\hbar$ with $\widetilde{\hbar}$, which we always do,
3. rescaling the variables a_i .